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# $(p,q)$-LAPLACIAN ELLIPTIC SYSTEMS AT RESONANCE

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Abstract. We show the existence of weak solutions for a class of $(p,q)$-Laplacian elliptic systems at resonance, under certain Landesman-Lazer-type conditions by using critical point theorem.

## 1. Introduction and statement of main results

Let $\Omega$ be a bounded domain with smooth boundary in $\mathbb{R}^N$ and $\Delta_p$ be the $p$-Laplacian operator. In this paper, we study the existence of solutions for the problem

$$\begin{gathered} -\Delta_p u = \lambda_1 |u|^{p-2}u + \frac{\lambda_1}{\beta+1}|u|^{\alpha}|v|^{\beta}v + G_s(x,u,v) - h_1(x) \quad \text{in } \Omega, \\ -\Delta_q v = \lambda_1 |v|^{q-2}v + \frac{\lambda_1}{\alpha+1}|u|^{\alpha}|v|^{\beta}u + G_t(x,u,v) - h_2(x) \quad \text{in } \Omega, \\ u = v = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where $1 < p, q < +\infty$ and $\alpha \geq 0, \beta \geq 0$ satisfy

$$\frac{\alpha+1}{p} + \frac{\beta+1}{q} = 1. \tag{1.2}$$

The nonlinearity $G : \Omega \times \mathbb{R}^2 \to R$ is a Caratheodory function which has continuous derivatives $G_s(x,s,t), G_t(x,s,t)$ with respect to $s$ and $t$ for almost any $x \in \Omega$, and $h_1 \in L^{p/(p-1)}(\Omega), h_2 \in L^{q/(q-1)}(\Omega)$.

Let $W = W_0^{1,p}(\Omega) \times W_0^{1,q}(\Omega)$ with the norm $\|(u,v)\| = \|u\|_p + \|v\|_q$ for all $(u,v) \in W$, where $W_0^{1,p}(\Omega)$ is the usual Banach space with the norm $\|u\|_p = \big(\int_\Omega |\nabla u|^p dx\big)^{1/p}$ for any $u \in W_0^{1,p}(\Omega)$. From Sobolev embedding Theorem, the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is continuous and compact, and there is constant $C > 0$ such that

$$\|u\|_{L^p} \leq C\|u\|_p,\ \forall u \in W_0^{1,p}(\Omega), \quad \text{and} \quad \|v\|_{L^q} \leq C\|v\|_q,\ \forall v \in W_0^{1,q}(\Omega), \tag{1.3}$$

where $\|\cdot\|_{L^p}$ denotes the norm of $L^p(\Omega)$ and throughout this paper, let $C$ always denote an embedding constant with relation to (1.3). For the following nonlinear

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eigenvalue problem

$$\begin{gathered}-\Delta_p u=\lambda|u|^{p-2}u+\frac{\lambda}{\beta+1}|u|^{\alpha}|v|^{\beta}v \quad \text{in } \Omega,\\ -\Delta_q v=\lambda|v|^{q-2}v+\frac{\lambda}{\alpha+1}|u|^{\alpha}|v|^{\beta}u \quad \text{in } \Omega,\\ u=v=0 \quad \text{on } \partial\Omega,\end{gathered}\tag{1.4}$$

consider the functionals $\phi,\varphi$ on $W$ defined by

$$\phi(u,v)=\frac{1}{p}\int_\Omega|\nabla u|^p dx+\frac{1}{q}\int_\Omega|\nabla v|^q dx,$$

$$\varphi(u,v)=\frac{1}{p}\int_\Omega|u|^p dx+\frac{1}{q}\int_\Omega|v|^q dx+\frac{1}{(\alpha+1)(\beta+1)}\int_\Omega|u|^{\alpha}|v|^{\beta}uv dx,$$

and the manifold

$$\Sigma=\{(u,v)\in W:\varphi(u,v)=1\}.$$

It is easy to prove that $\phi(u,v),\varphi(u,v)$ are $(p,q)$-homogeneous, namely

$$\phi(t^{1/p}u,t^{1/q}v)=t\phi(u,v),\quad \varphi(t^{1/p}u,t^{1/q}v)=t\varphi(u,v)$$

for any $t>0$ and $(u,v)\in W$, and $\Sigma$ is a symmetric nonempty manifold in $W$. By an argument similar to the ones in [3, 7], problem (1.4) has a sequence of eigenvalues with the variational characterization

$$\lambda_k=\inf_{\Lambda\in\Sigma_k}\sup_{(u,v)\in\Lambda}\phi(u,v),$$

where $\Sigma_k=\{\Lambda\subset\Sigma:$ there is an odd, continuous and surjective $\gamma:S^{k-1}\to\Lambda\}$ and $S^{k-1}$ denotes the unit sphere in $\mathbb{R}^k$.

On the other hand, let

$$\lambda_1'=\inf_{(u,v)\in\Sigma}\phi(u,v),$$

we can see that $\lambda_1=\lambda_1'$. Moreover, $\lambda_1$ is a simple, isolated and positive principal eigenvalue of (1.4) and has a positive normalized eigenvalue $(\mu_0,\nu_0)$, namely, $\|\mu_0\|_p+\|\nu_0\|_q=1$. By a simple computation, there exists a positive constant $t_0$ such that

$$\|t_0^{1/p}\mu_0\|_p^p+\|t_0^{1/q}\nu_0\|_q^q=1.$$

Let $\mu_1=t_0^{1/p}\mu_0,\nu_1=t_0^{1/q}\nu_0$, since $\phi,\varphi$ are $(p,q)$-homogeneous, hence the set of all eigenfunctions corresponding to $\lambda_1$ is

$$E_1:=\{(t^{1/p}\mu_1,t^{1/q}\nu_1):t\ge 0\}\cup\{(-t^{1/p}\mu_1,-t^{1/q}\nu_1):t\ge 0\}.$$

The set $E_1$ is not an one-dimensional linear subspace of $W$ and the corresponding orthogonal decomposition on $W$ does not hold with respect to the the first eigenvalue $\lambda_1$.

In many papers, existence of weak solutions for the resonant elliptic problems were investigated under the well-known Landesman-Lazer-type conditions, which were introduced by Landesman and Lazer in [5] and were extended by Tang in [12]. Since then they were used widely for the different types of equations, for example, in [1, 3, 9] for the quasilinear elliptic equations, in [4] for asymptotically linear noncooperative elliptic systems, in [13] for the forced duffing equations, in [11] for Kirchhoff type equations. Especially, in [2] the case $p=q=2$(the semilinear elliptic systems) was considered and the case $p=q\ge 2$(the quasilinear elliptic systems) was discussed in [6, 7, 14] where $G_s(x,s,t) = g_1(x,s)$ and $G_t(x,s,t) = g_2(x,t)$. As far as we know, when $p \neq q > 1$, the similar results are not discussed under the Landesman-Lazer-type conditions due to Landesman and Lazer. Motivated by these finding, we consider the existence of solutions for problem (1.1) at resonance with the first eigenvalue under the Landesman-Lazer-type conditions. We first state the following fundamental hypotheses.

- (H1) There is $h \in C(\bar{\Omega}, \mathbb{R}^+)$ such that $|G_s(x,s,t)| \le h(x)$ and $|G_t(x,s,t)| \le h(x)$ for all $(x,s,t) \in \Omega \times \mathbb{R}^2$.
- (H2) There exist two functions $g_1^{++}, g_1^{--} \in C(\Omega, R)$ such that
$$g_1^{++}(x) = \liminf_{s\to+\infty,\, t\to+\infty} G_s(x,s,t), \quad g_1^{--}(x) = \limsup_{s\to-\infty,\, t\to-\infty} G_s(x,s,t)$$
uniformly a.e. $x \in \Omega$.
- (H3) There is two functions $g_2^{++}, g_2^{--} \in C(\Omega, R)$ such that
$$g_2^{++}(x) = \liminf_{s\to+\infty,\, t\to+\infty} G_t(x,s,t), \quad g_2^{--}(x) = \limsup_{s\to-\infty,\, t\to-\infty} G_t(x,s,t)$$
uniformly a.e. $x \in \Omega$.

The Landesman-Lazer-type conditions for problem (1.1) are read either
$$\begin{aligned}\int_\Omega g_1^{--}\mu_1 dx + \int_\Omega g_2^{--}\nu_1 dx &< \int_\Omega h_1\mu_1 dx + \int_\Omega h_2\nu_1 dx \\ &< \int_\Omega g_1^{++}\mu_1 dx + \int_\Omega g_2^{++}\nu_1 dx;\end{aligned} \tag{1.5}$$
or
$$\begin{aligned}\int_\Omega g_1^{++}\mu_1 dx + \int_\Omega g_2^{++}\nu_1 dx &< \int_\Omega h_1\mu_1 dx + \int_\Omega h_2\nu_1 dx \\ &< \int_\Omega g_1^{--}\mu_1 dx + \int_\Omega g_2^{--}\nu_1 dx.\end{aligned} \tag{1.6}$$
We are ready to state the main results.

**Theorem 1.1.** *Let $h_1 \in L^{p/(p-1)}(\Omega)$, $h_2 \in L^{q/(q-1)}(\Omega)$, and* (1.2), (H1), (H2), (H3) *and* (1.5) *be satisfied. If $1 < p < q$ and the following inequalities hold:*
$$\int_\Omega h_1\mu_1 dx - \int_\Omega g_1^{++}\mu_1 dx < 0, \quad \int_\Omega h_1\mu_1 dx - \int_\Omega g_1^{--}\mu_1 dx > 0, \tag{1.7}$$
*then problem* (1.1) *has at least one solution.*

In the other case $1 < q < p$, the following result holds.

**Theorem 1.2.** *Let $h_1 \in L^{p/(p-1)}(\Omega)$, $h_2 \in L^{q/(q-1)}(\Omega)$, and* (1.2), (H1), (H2), (H3) *and* (1.5) *be satisfied. If $1 < q < p$ and the following inequalities hold:*
$$\int_\Omega h_2\nu_1 dx - \int_\Omega g_2^{++}\nu_1 dx < 0, \quad \int_\Omega h_2\nu_1 dx - \int_\Omega g_2^{--}\nu_1 dx > 0, \tag{1.8}$$
*then problem* (1.1) *has at least one solution.*

**Theorem 1.3.** *Let $h_1 \in L^{p/(p-1)}(\Omega), h_2 \in L^{q/(q-1)}(\Omega)$. If* (1.2), (H1), (H2), (H3) *and* (1.6) *are satisfied, then problem* (1.1) *has at least one solution.*

Our results extends the ones in [2] from the semilinear elliptic systems to $(p, q)$-Laplacian elliptic systems, and are also the generalizations of [14], where they considered the case $p = q \geq 2$ and $G_s(x, s, t) = g_1(s), G_t(x, s, t) = g_2(t)$. Moreover, the conditions (1.7) and (1.8) are the technical assumptions. Theorem 1.2 is similar to Theorem 1.1, and we will prove Theorem 1.1 and Theorem 1.3.

## 2. Proofs of Theorems

Now consider the functionals $J, J_1, J_2$ on $W$ defined by

$$\begin{aligned}J(u,v) = &\frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{q}\int_\Omega |\nabla v|^q dx - \frac{\lambda_1}{p}\int_\Omega |u|^p dx \\ &- \frac{\lambda_1}{q}\int_\Omega |v|^q dx - \frac{\lambda_1}{(\alpha+1)(\beta+1)}\int_\Omega |u|^\alpha |v|^\beta uv dx \\ &- \int_\Omega G(x,u,v)dx + \int_\Omega h_1 u dx + \int_\Omega h_2 v dx,\end{aligned}$$

$$\begin{aligned}J_1(u,v) = &\frac{1}{p}\int_\Omega |\nabla u|^p dx - \frac{\lambda_1}{p}\int_\Omega |u|^p dx - \frac{\lambda_1}{p(\beta+1)}\int_\Omega |u|^\alpha |v|^\beta uv dx \\ &- \int_\Omega \int_0^1 G_s(x, ru, rv)u \, dr \, dx + \int_\Omega h_1 u dx,\end{aligned}$$

$$\begin{aligned}J_2(u,v) = &\frac{1}{q}\int_\Omega |\nabla v|^q dx - \frac{\lambda_1}{q}\int_\Omega |v|^q dx - \frac{\lambda_1}{q(\alpha+1)}\int_\Omega |u|^\alpha |v|^\beta uv dx \\ &- \int_\Omega \int_0^1 G_t(x, ru, rv)v \, dr \, dx + \int_\Omega h_2 v dx.\end{aligned}$$

Noting that

$$G(x,s,t) = \int_0^1 G_s(x, rs, rt) s dr + \int_0^1 G_t(x, rs, rt) t dr, \tag{2.1}$$

from (1.2) and (2.1), it follows that

$$J(u,v) = J_1(u,v) + J_2(u,v) \quad \text{for all } (u,v) \in W.$$

From (H1), it is easy to prove that the functional $J$ is well defined and $J \in C^1(W, R)$. Moreover, from the variational view of point, a weak solution of problem (1.1) is equivalent to a critical point of the functional $J$ in $W$. In this paper, we will prove Theorem 1.1 and Theorem 1.2 by using the following G-linking Theorem due to Drábek and Robinson (see [3, 9]) and Theorem 1.3 by using Ekeland's Variational Principle (see [8, 10]). In these abstract theorems, a compact condition, i.e., $(PS)$ condition, is needed.

**Definition 2.1.** Let $X$ be a real Banach space, if for any sequence $\{u_n\} \subset X$ such that $f(u_n)$ is bounded and $f'(u_n) \to 0$ as $n \to \infty$, $\{u_n\}$ has a convergent subsequence, the functional $f$ satisfies the $(PS)$ condition.

**Definition 2.2** ([3, 9])**.** Let $Q$ be a submanifold of a Banach space $X$ with relative boundary $\partial Q, S$ be a closed subset of a Banach space $Y$ and $G$ be a subset of $C(\partial Q, Y \backslash S)$. $S$ and $\partial Q$ are G-linking if for any map $h \in C(Q, Y)$ such that $h|_{\partial Q} \in G$ there holds $h(Q) \cap S \neq \emptyset$.

**Theorem 2.3** ([3, 9]). *Let $X, Y$ be Banach spaces, $S$ be a closed subset of $Y$, $Q$ be a submanifold of $X$ with relative boundary $\partial Q$ and $G$ be a subset of $C(\partial Q, Y\backslash S)$. Let $\Gamma = \{h \in C(Q,Y) : h|_{\partial Q} \in G\}$, assume that $S$ and $\partial Q$ are $G$-linking and $f \in C^1(Y,R)$ satisfies*

(a) *There is $\tilde{h} \in \Gamma$ such that $\sup_{x\in Q} f(\tilde{h}(x)) < +\infty$;*
(b) *There is $\beta_0 > \alpha_0$ such that*
$$\inf_{y\in S} f(y) \geq \beta_0 \quad \text{and} \quad \sup_{x\in\partial Q} f(h(x)) \leq \alpha_0, \ \forall h \in \Gamma;$$
(c) *The $(PS)$ condition holds.*

*Then, the number*
$$c = \inf_{h\in\Gamma} \sup_{x\in Q} f(h(x))$$
*is a critical value of $f$ with $c \geq \beta_0$.*

*Proof.* The proof is divided into two steps.

**Step 1.** The $(PS)$ condition for the functional $J$ is satisfied. Let $(u_n, v_n)$ be a $(PS)$ sequence for the functional $J$; that is,
$$J(u_n, v_n) \text{ is bouned and } J'(u_n, v_n) \to 0 \text{ as } n \to \infty. \tag{2.2}$$
From (H1) and by a standard argument, it is sufficient to prove that $(u_n, v_n)$ is bounded in $W$. If this does not hold, assume that $\|(u_n, v_n)\| = \|u_n\|_p + \|v_n\|_q \to \infty$ as $n \to \infty$. Define $K_n := \|u_n\|_p^p + \|v_n\|_q^q$, hence it follows that $K_n \to \infty$ as $n \to \infty$. Let $\bar{u}_n = u_n \setminus K_n^{1/p}, \bar{v}_n = v_n \setminus K_n^{1/q}$, then $(\bar{u}_n, \bar{v}_n)$ is bounded in $W$, i.e.,
$$\|\bar{u}_n\|_p^p + \|\bar{v}_n\|_q^q = 1 \quad \text{for all } n.$$
Extracting subsequences if necessary, we can assume that there exists $(\bar{u}, \bar{v}) \in W$ such that
$$(\bar{u}_n, \bar{v}_n) \rightharpoonup (\bar{u}, \bar{v}) \quad \text{weakly in } W, \tag{2.3}$$
$$(\bar{u}_n, \bar{v}_n) \to (\bar{u}, \bar{v}) \quad \text{strongly in } L^p(\Omega) \times L^q(\Omega), \tag{2.4}$$
$$(\bar{u}_n(x), \bar{v}_n(x)) \to (\bar{u}(x), \bar{v}(x)) \quad \text{for a.e. } x \in \Omega. \tag{2.5}$$
From (2.2), it follows that
$$\limsup_{n\to\infty} \frac{J(u_n, v_n)}{K_n} \leq 0, \tag{2.6}$$
From (2.1), (H1), the Hölder's inequality and (1.3), we have
$$\begin{aligned} \Big|\int_\Omega G(x,u,v)dx\Big| &\leq \int_\Omega \Big|\int_0^1 (G_s(x,\tau u,\tau v)u + G_t(x,\tau u,\tau v)v)d\tau\Big| dx \\ &\leq \int_\Omega h(x)(|u| + |v|)dx \\ &\leq \|h\|_{L^\infty}(|\Omega|^{\frac{p-1}{p}}\|u\|_{L^p} + |\Omega|^{\frac{q-1}{q}}\|v\|_{L^q}) \\ &\leq C_1(\|u\|_p + \|v\|_q) \end{aligned} \tag{2.7}$$
for all $(u,v) \in W$, where $C_1$ is a positive constant, hence it follows that
$$\frac{1}{K_n}\int_\Omega G(x,u_n,v_n)dx \to 0 \quad \text{as } n \to \infty. \tag{2.8}$$

From $h_1 \in L^{p/(p-1)}(\Omega), h_2 \in L^{q/(q-1)}(\Omega)$ and the Hölder's inequality, we obtain

$$\frac{1}{K_n}\int_\Omega (h_1 u_n + h_2 v_n)dx \to 0 \quad \text{as } n \to \infty. \tag{2.9}$$

From (2.4) and (2.5), it follows that $|\bar{u}_n|^\alpha \bar{u}_n \to |\bar{u}|^\alpha \bar{u}$ strongly in $L^{\frac{p}{\alpha+1}}(\Omega)$ and $|\bar{v}_n|^\beta \bar{v}_n \to |\bar{v}|^\alpha \bar{v}$ strongly in $L^{\frac{q}{\beta+1}}(\Omega)$, hence from Hölder's inequality, we obtain

$$\begin{aligned}
&\Big|\int_\Omega (|\bar{u}_n|^\alpha |\bar{v}_n|^\beta \bar{u}_n \bar{v}_n - |\bar{u}|^\alpha |\bar{v}|^\beta \bar{u}\bar{v})dx\Big| \\
&\le \int_\Omega ||\bar{u}_n|^\alpha |\bar{v}_n|^\beta \bar{u}_n \bar{v}_n - |\bar{u}_n|^\alpha |\bar{v}|^\beta \bar{u}_n \bar{v}|dx + \int_\Omega ||\bar{u}_n|^\alpha |\bar{v}|^\beta \bar{u}_n \bar{v} - |\bar{u}|^\alpha |\bar{v}|^\beta \bar{u}\bar{v}|dx \\
&\le \int_\Omega |\bar{u}_n|^{\alpha+1} \cdot ||\bar{v}_n|^\beta \bar{v}_n - |\bar{v}|^\beta \bar{v}|dx + \int_\Omega ||\bar{u}_n|^\alpha \bar{u}_n - |\bar{u}|^\alpha \bar{u}| \cdot |\bar{v}|^{\beta+1}dx \\
&\le \|\bar{u}_n\|_{L^p}^{\alpha+1} \cdot \||\bar{v}_n|^\beta \bar{v}_n - |\bar{v}|^\beta \bar{v}\|_{L^{\frac{q}{\beta+1}}} + \|\bar{v}_n\|_{L^q}^{\beta+1} \cdot \||\bar{u}_n|^\alpha \bar{u}_n - |\bar{u}|^\alpha \bar{u}\|_{L^{\frac{p}{\alpha+1}}} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{2.10}$$

From the definition of $J$, (2.4), (2.6), (2.8), (2.9) and (2.10), we have

$$\begin{aligned}
&\limsup_{n\to\infty} \Big(\frac{1}{p}\int_\Omega |\nabla \bar{u}_n|^p dx + \frac{1}{q}\int_\Omega |\nabla \bar{v}_n|^q dx\Big) \\
&\le \lambda_1 \Big(\frac{1}{p}\int_\Omega |\bar{u}|^p dx + \frac{1}{q}\int_\Omega |\bar{v}|^q dx + \frac{1}{(\alpha+1)(\beta+1)}\int_\Omega |\bar{u}|^\alpha |\bar{v}|^\beta \bar{u}\bar{v}dx\Big).
\end{aligned}$$

From (2.3), it follows that

$$\int_\Omega |\nabla \bar{u}|^p dx \le \liminf_{n\to\infty} \int_\Omega |\nabla \bar{u}_n|^p dx, \quad \int_\Omega |\nabla \bar{v}|^q dx \le \liminf_{n\to\infty} \int_\Omega |\nabla \bar{v}_n|^q dx,$$

hence, combining this with the definition of $\lambda_1$, we obtain

$$\begin{aligned}
&\lambda_1 \Big(\frac{1}{p}\int_\Omega |\bar{u}|^p dx + \frac{1}{q}\int_\Omega |\bar{v}|^q dx + \frac{1}{(\alpha+1)(\beta+1)}\int_\Omega |\bar{u}|^\alpha |\bar{v}|^\beta \bar{u}\bar{v}dx\Big) \\
&\le \frac{1}{p}\int_\Omega |\nabla \bar{u}|^p dx + \frac{1}{q}\int_\Omega |\nabla \bar{v}|^q dx \\
&\le \liminf_{n\to\infty} \Big(\frac{1}{p}\int_\Omega |\nabla \bar{u}_n|^p dx + \frac{1}{q}\int_\Omega |\nabla \bar{v}_n|^q dx\Big) \\
&\le \limsup_{n\to\infty} \Big(\frac{1}{p}\int_\Omega |\nabla \bar{u}_n|^p dx + \frac{1}{q}\int_\Omega |\nabla \bar{v}_n|^q dx\Big) \\
&\le \lambda_1 \Big(\frac{1}{p}\int_\Omega |\bar{u}|^p dx + \frac{1}{q}\int_\Omega |\bar{v}|^q dx + \frac{1}{(\alpha+1)(\beta+1)}\int_\Omega |\bar{u}|^\alpha |\bar{v}|^\beta \bar{u}\bar{v}dx\Big),
\end{aligned}$$

hence it follows that

$$\begin{aligned}
&\frac{1}{p}\int_\Omega |\nabla \bar{u}|^p dx + \frac{1}{q}\int_\Omega |\nabla \bar{v}|^q dx \\
&= \lambda_1 \Big(\frac{1}{p}\int_\Omega |\bar{u}|^p dx + \frac{1}{q}\int_\Omega |\bar{v}|^q dx + \frac{1}{(\alpha+1)(\beta+1)}\int_\Omega |\bar{u}|^\alpha |\bar{v}|^\beta \bar{u}\bar{v}dx\Big),
\end{aligned}$$

and by the uniform convexity of $W$, we have that $(\bar{u}_n, \bar{v}_n)$ converges strongly to $(\bar{u}, \bar{v})$ in $W$, and from the definition of $(\mu_1, \nu_1)$, it follows that $(\bar{u}, \bar{v}) = \pm(\mu_1, \nu_1)$.

In the following, let $(\bar{u}, \bar{v}) = (\mu_1, \nu_1)$, and the other case where $(\bar{u}, \bar{v}) = -(\mu_1, \nu_1)$ may be considered similarly. Hence from the definition of $J$, we have

$$\begin{aligned} &\frac{pJ_1(u_n, v_n)}{(p-1)K_n^{1/p}} + \frac{qJ_2(u_n, v_n)}{(q-1)K_n^{1/q}} - \langle J'(u_n, v_n), (\frac{\bar{u}_n}{p-1}, \frac{\bar{v}_n}{q-1})\rangle \\ &= \frac{1}{p-1}\Big(\int_\Omega G_s(x, u_n, v_n)\bar{u}_n dx - \frac{p}{K_n^{1/p}}\int_\Omega\int_0^1 G_s(x, ru_n, rv_n)u_n\,dr\,dx\Big) \\ &\quad + \frac{1}{q-1}\Big(\int_\Omega G_t(x, u_n, v_n)\bar{v}_n dx - \frac{q}{K_n^{1/q}}\int_\Omega\int_0^1 G_t(x, ru_n, rv_n)v_n\,dr\,dx\Big) \\ &\quad + \int_\Omega h_1\bar{u}_n dx + \int_\Omega h_2\bar{v}_n dx. \end{aligned} \tag{2.11}$$

From $h_1 \in L^{p/(p-1)}(\Omega), h_2 \in L^{q/(q-1)}(\Omega)$, we have

$$\int_\Omega h_1\bar{u}_n dx \to \int_\Omega h_1\mu_1 dx \quad \text{and} \quad \int_\Omega h_2\bar{v}_n dx \to \int_\Omega h_2\nu_1 dx \quad \text{as} \quad n \to \infty. \tag{2.12}$$

From (H2) and (H3), it is easy to know that

$$\begin{gathered} \int_\Omega G_s(x, u_n, v_n)\bar{u}_n dx \to \int_\Omega g_1^{++}\mu_1 dx, \\ \int_\Omega G_t(x, u_n, v_n)\bar{v}_n dx \to \int_\Omega g_2^{++}\nu_1 dx \end{gathered} \tag{2.13}$$

as $n \to \infty$. Finally, from (H2) and Lebesgue dominated convergence theorem, we have

$$\begin{aligned} \frac{1}{K_n^{1/p}}\int_\Omega\int_0^1 G_s(x, ru_n, rv_n)u_n\,dr\,dx &= \int_\Omega\int_0^1 G_s(x, ru_n, rv_n)\frac{u_n}{K_n^{1/p}}\,dr\,dx \\ &\to \int_\Omega g_1^{++}\mu_1 dx \quad \text{as } n \to \infty. \end{aligned} \tag{2.14}$$

Similarly, we obtain

$$\frac{1}{K_n^{1/q}}\int_\Omega\int_0^1 G_t(x, ru_n, rv_n)v_n\,dr\,dx \to \int_\Omega g_2^{++}\nu_1 dx \quad \text{as } n \to \infty. \tag{2.15}$$

Therefore, letting $n \to \infty$ in (2.11) and from (2.2), (2.12), (2.13), (2.14) and (2.15), we obtain

$$\int_\Omega h_1\mu_1 dx + \int_\Omega h_2\nu_1 dx = \int_\Omega g_1^{++}\mu_1 dx + \int_\Omega g_2^{++}\nu_1 dx,$$

which contradicts with (1.5). Hence, $(u_n, v_n)$ is bounded in $W$.

**Step 2.** The functional $J$ satisfies the geometries of Theorem 2.3. For any

$$(u, v) \in E_1 = \{(t^{1/p}\mu_1, t^{1/q}\nu_1) : t \ge 0\} \cup \{(-t^{1/p}\mu_1, -t^{1/q}\nu_1) : t \ge 0\},$$

we have

$$\begin{aligned} &\frac{1}{p}\int_\Omega |\nabla u|^p dx + \frac{1}{q}\int_\Omega |\nabla v|^q\,dx \\ &= \lambda_1\Big(\frac{1}{p}\int_\Omega |u|^p dx + \frac{1}{q}\int_\Omega |v|^q dx + \frac{1}{(\alpha+1)(\beta+1)}\int_\Omega |u|^\alpha|v|^\beta uv\,dx\Big). \end{aligned}$$

From the above equality and the definition of $J$, for any $(t^{1/p}\mu_1, t^{1/q}\nu_1) \in E_1$, we obtain

$$
\begin{aligned}
&J(t^{1/p}\mu_1, t^{1/q}\nu_1)\\
&= t^{1/p}\int_\Omega h_1\mu_1 dx + t^{1/q}\int_\Omega h_2\nu_1 dx - \int_\Omega G(x, t^{1/p}\mu_1, t^{1/q}\nu_1)dx\\
&= t^{1/p}\Big(\int_\Omega h_1\mu_1 dx - \int_\Omega\int_0^1 G_s(x, rt^{1/p}\mu_1, rt^{1/q}\nu_1)\mu_1\, dr\, dx\Big)\\
&\quad + t^{1/q}\Big(\int_\Omega h_2\nu_1 dx - \int_\Omega\int_0^1 G_t(x, rt^{1/p}\mu_1, rt^{1/q}\nu_1)\nu_1\, dr\, dx\Big).
\end{aligned} \tag{2.16}
$$

From (H1), (H2) and Lebesgue dominated convergence theorem, it follows that

$$
\lim_{t\to+\infty}\int_\Omega\int_0^1 G_s(x, rt^{1/p}\mu_1, rt^{1/q}\nu_1)\mu_1\, dr\, dx = \int_\Omega g_1^{++}\mu_1 dx, \tag{2.17}
$$

$$
\lim_{t\to+\infty}\int_\Omega\int_0^1 G_t(x, rt^{1/q}\mu_1, rt^{1/q}\nu_1)\nu_1\, dr\, dx = \int_\Omega g_2^{++}\nu_1 dx. \tag{2.18}
$$

Hence, from (1.7), (2.16), (2.17) and (2.18), we obtain

$$
J(t^{1/p}\mu_1, t^{1/q}\nu_1) \to -\infty \quad \text{as } t \to +\infty.
$$

Similarly, the following result can be obtained with $g_1^{++}$ and $g_2^{++}$ exchanged with $g_1^{--}$ and $g_2^{--}$ respectively,

$$
J(-t^{1/p}\mu_1, -t^{1/q}\nu_1) \to -\infty \quad \text{as } t \to +\infty.
$$

Finally, it follows that

$$
\lim_{|t|\to\infty} J(\pm t^{1/p}\mu_1, \pm t^{1/q}\nu_1) = -\infty. \tag{2.19}
$$

On the other hand, letting $\Lambda_2 := \{(u, v) \in W : \phi(u, v) \geq \lambda_2\varphi(u, v)\}$, from (1.3), (2.7) and the Hölder's inequality, for any $(u, v) \in \Lambda_2$, we obtain

$$
\begin{aligned}
J(u, v) &\geq \frac{\lambda_2 - \lambda_1}{p\lambda_2}\|u\|_p^p + \frac{\lambda_2 - \lambda_1}{q\lambda_2}\|v\|_q^q - C_1(\|u\|_p + \|v\|_q)\\
&\quad - (\|h_1\|_{L^{\frac{p}{p-1}}}\|u\|_{L^p} + \|h_2\|_{L^{\frac{q}{q-1}}}\|v\|_{L^q})\\
&\geq \frac{\lambda_2 - \lambda_1}{p\lambda_2}(\|u\|_p^p + \|v\|_q^q) - C_2(\|u\|_p + \|v\|_q),
\end{aligned}
$$

where $C_2 = C_1 + C\max\{\|h_1\|_{L^{\frac{p}{p-1}}}, \|h_2\|_{L^{\frac{q}{q-1}}}\}$. Combining the above expression with (2.19), we obtain that there exists a positive constant $T$ such that

$$
\alpha_0 := \sup_{t\geq T} J(\pm t^{1/p}\mu_1, \pm t^{1/q}\nu_1) < \beta_0 := \inf_{(u,v)\in\Lambda_2} J(u, v). \tag{2.20}
$$

Let $M = \{(\pm t^{1/p}\mu_1, \pm t^{1/q}\nu_1) : t \geq T\}$ and

$$
G = \{h \in C(S^0, W) : h \text{ is odd and } h(S^0) \subset M\},
$$

where $S^0$ is the boundary of the closed unit ball $B^1$ in $\mathbb{R}^1$, i.e., $S^0 = \partial B^1$. For any $h \in G$, by (2.20), we have $h(S^0) \cap \Lambda_2 = \emptyset$, which implies that $G$ is a subset of $C(S^0, W \setminus \Lambda_2)$. Let

$$
\Gamma = \{h \in C(B^1, W) : h|_{S^0} \in G\},
$$

we can claim: $\Gamma$ is nonempty and $\Lambda_2$ and $S^0$ are G-linking, that is $h(B^1)\cap\Lambda_2\neq\emptyset$ for any $h\in\Gamma$. The similar proof of the conclusion may be found in [7, 3, 9], but for the readers convenience and completeness, we write it.

In fact, define $\bar h : B^1\to W$ by

$$\begin{gathered}\bar h(t)=((tT)^{1/p}\mu_1,(tT)^{1/q}\nu_1)\quad\text{for all } t\in[0,1],\\ \bar h(-t)=(-(tT)^{1/p}\mu_1,-(tT)^{1/q}\nu_1)\quad\text{for all } t\in[0,1].\end{gathered}$$

Hence, $\bar h\in\Gamma$ and $\Gamma$ is nonempty. Now let $h\in\Gamma$, if there is $(u,v)\in h(B^1)$ such that $\varphi(u,v)=0$, we get $h(B^1)\cap\Lambda_2\neq\emptyset$. If not, we consider the map $\hat h : S^1\to\Sigma$ defined by

$$\hat h(x_1,x_2)=\begin{cases}\pi\circ h(x_1), & \text{if } x_2\geq 0,\\ -\pi\circ h(-x_1), & \text{if } x_2\leq 0,\end{cases}$$

where $\pi(u,v)=(u\backslash(\varphi(u,v))^{1/p},v\backslash(\varphi(u,v))^{1/q})$. It is easy to know that $\hat h(S^1)\subset\Sigma_2$. Therefore, $\phi(u_0,v_0)\geq\lambda_2$ for some $(u_0,v_0)\in\hat h(S^1)$, namely, $(u_0,v_0)\in\Lambda_2$. From $\pi\circ h(x)\in\Lambda_2$, we have implies $h(x)\in\Lambda_2$, which implies that $h(B^1)\cap\Lambda_2\neq\emptyset$. Hence $\Lambda_2$ and $S^0$ are G-linking.

Now, from the compactness of $B^1$, (a) of Theorem 2.3 holds, (b) of Theorem 2.3 is satisfied from (2.20), (c) of Theorem 2.3 comes from (i). Accordingly, Theorem 1.1 holds from the G-linking Theorem with the critical value

$$c=\inf_{h\in\Gamma}\sup_{x\in B^1}J(h(x)).\quad\square$$

$\square$

*Proof of Theorem 1.3.* (i) The functional $J$ satisfies the $(PS)$ condition. From (1.6), the claim can be proved with similar to step 1 of Theorem 1.1.

(ii) Now we will prove that the functional $J$ is coercive, that is,

$$J(u,v)\to+\infty\quad\text{as}\quad\|(u,v)\|\to\infty.$$

If the claim does not hold, there is a constant $c$ and a sequence $(u_n,v_n)$ such that $J(u_n,v_n)\leq c$ and $\|(u_n,v_n)\|\to\infty$. From the proof of the (PS) condition of Theorem 1.1, $(\bar u_n,\bar v_n)$ converges strongly to $\pm(\mu_1,\nu_1)$, where $\bar u_n=u_n\setminus K_n^{1/p}$, $\bar v_n=v_n\setminus K_n^{1/q}$. Assume that $(\bar u_n,\bar v_n)$ converges strongly to $(\mu_1,\nu_1)$ (the case $(\bar u_n,\bar v_n)$ converges strongly to $(-\mu_1,-\nu_1)$ may be treated similarly) and $p\geq q>1$ (the case $q\geq p>1$ may also be treated similarly). From the definitions of $J,J_1,J_2$ and $J(u_n,v_n)\leq c$ for all $n$, (2.14) and (2.15), we have

$$\begin{aligned}0&\geq\limsup_{n\to\infty}\frac{J(u_n,v_n)}{K_n^{1/p}}=\limsup_{n\to\infty}\Big(\frac{J_1(u_n,v_n)}{K_n^{1/p}}+\frac{J_2(u_n,v_n)}{K_n^{1/p}}\Big)\\ &\geq\limsup_{n\to\infty}\Big(\frac{J_1(u_n,v_n)}{K_n^{1/p}}+\frac{J_2(u_n,v_n)}{K_n^{1/q}}\Big)\\ &\geq\lim_{n\to\infty}\Big(\int_\Omega(h_1\bar u_n-\frac{1}{K_n^{1/p}}\int_0^1G_s(x,ru_n,rv_n)u_ndr)dx\\ &\quad+\int_\Omega(h_2\bar v_n-\frac{1}{K_n^{1/q}}\int_0^1G_t(x,ru_n,rv_n)v_ndr)dx\Big)\\ &=\int_\Omega(h_1\mu_1-g_1^{++}\mu_1)dx+\int_\Omega(h_2\nu_1-g_2^{++}\nu_1)dx,\end{aligned}$$

which is a contradiction to (1.6). By Ekeland's Variational Principle, the proof is complete. □

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