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SOLVABILITY OF BOUNDARY-VALUE PROBLEMS FOR POISSON EQUATIONS WITH HADAMARD TYPE BOUNDARY OPERATOR

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Abstract. In this article we study properties of some integro-differential operators of fractional order. As an application of the properties of these operators for Poisson equation we examine questions on solvability of a fractional analogue of Neumann problem and analogues of periodic boundary-value problems for circular domains. The exact conditions for solvability of these problems are found.

1. Introduction

Let Q be a bounded domain from $\mathbb{R}^n$ with a smooth boundary S. It is known that classical problems for the Poisson equation.

$$\Delta u(x) = f(x), x \in Q, \tag{1.1}$$

are Dirichlet and Neumann problems. Let ν be a normal vector to S, and $D_\nu = \frac{d}{d\nu}$ be an operator of differentiation along the normal, $D_\nu^0 = I$ be a unit vector. Then Dirichlet and Neumann boundary conditions can be given in the form

$$D_\nu^\alpha u(x) = g_\alpha(x), x \in S, \tag{1.2}$$

where $\alpha = 0$ or $\alpha = 1$, $D_\nu^0 u(x) = u(x)$. It is known that the Dirichlet problem is unconditionally solvable, and for solvability of the Neumann condition the following condition is necessary [7]:

$$\int_Q f(x)dx = \int_S g_1(x)dx. \tag{1.3}$$

In this article, we introduce fractional analogues of the boundary operators D_ν^α, and for the equation (1.1) we study the boundary-value problem with the boundary condition (1.2) for all values of the parameter $\alpha \in (0, \infty)$. Moreover, we investigate solvability of some analogues of periodic boundary-value problems for circular domains.

The structure of this paper is as follows. In Introduction we provide an overview of some papers published on the subject. Further, we give concepts of Hadamard type integral-differential operators of fractional order. In the second section we study properties of integral-differential operators of fractional order in the class

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of smooth functions. Properties of these operators in Hölder class are studied. Propositions about reversibility of the operators are proved. In the third section we present some auxiliary statements related to the properties of solutions of the Dirichlet problem for the Poisson equation. In the fourth and fifth sections we consider applications of these integral - differential operators of fractional order to examine questions on solvability of some boundary-value problems with boundary operators of fractional order. In the fourth section we study questions about solvability of a fractional analogue of the Neumann problem. The problem is solved by reducing it to an equivalent Dirichlet problem with the additional condition at the point $x = 0$. In the fifth section we also study analogues of periodic problems for circular domains. The problem is reduced to two auxiliary problems: Dirichlet problem and an analogue of Neumann problem.

Note that the local and nonlocal boundary-value problems with boundary operators of fractional order for the second order elliptic equations were studied in [4, 9, 10, 12, 13, 14, 15, 16, 20, 21, 22, 24, 25, 26] and for higher-order equations in [3, 5, 6, 23]. As the boundary operators in [9, 10, 12, 13, 14, 15, 20, 21, 22, 24, 25, 26] operators with Riemann-Liouville and Caputo type derivatives, and in [3, 4, 16] the Hadamard - Marchaud type operators were considered. We also note that applications of boundary-value problems for elliptic equations with boundary operators of fractional order have been considered in [1, 2, 27]. Now let us turn to the definitions of integration and differentiation operator of fractional order.

Let $\Omega = \{x \in \mathbb{R}^n : |x| < 1\}$ be a unit ball, $n \geq 2$, $\partial\Omega = \{x \in \mathbb{R}^n : |x| = 1\}$ - unit sphere. Suppose further that, $u(x)$ is a smooth function in the domain Ω, $r = |x|, \theta = x/r$, $\delta = r\frac{d}{dr}$- Dirac operator, where

$$r\frac{d}{dr} = \sum_{j=1}^{n} x_j \frac{\partial}{\partial x_j}, \quad \alpha > 0.$$

Further, let $0 < \alpha < \infty$. The expression

$$J^\alpha[u](x) = \frac{1}{\Gamma(\alpha)} \int_0^r \left(\ln\frac{r}{s}\right)^{\alpha-1} u(s\theta)\frac{ds}{s}$$

is called integration operator of the α order in the Hadamard sense (see e.g. [11]). Furthermore, we assume that $J^0[u](x) = u(x)$.

Note that, if $u(0) \neq 0$, then in the class of continuous functions the operator J^α is not defined, since the integral $\int_0^1 (\ln\frac{1}{s})^{\alpha-1} s^{-1} ds$ diverges. Therefore, as the differentiation operator we consider the Hadamard - Caputo type operator. Namely, differentiation operator of the $\alpha > 0$ order is the expression:

$$D^\alpha[u](x) = \frac{1}{\Gamma(\ell-\alpha)} \int_0^r (\ln\frac{r}{s})^{\ell-1-\alpha} (s\frac{d}{ds})^\ell u(s\theta)\frac{ds}{s}, \ell-1 < \alpha \leq \ell,\ \ell \geq 1.$$

2. Properties of J^α and D^α operators

In this section we study properties of J^α and D^α operators. Further, by the symbol C we denote the constant whose value can be different.

Lemma 2.1. *Let $\alpha > 0$, $\quad 0 < \lambda < 1$ and $u(x) \in C^{\lambda+p}(\overline{\Omega}), p \geq 0$. If the condition $u(0) = 0$ holds, then $J^\alpha[u](x) \in C^{\lambda+p}(\overline{\Omega})$ and $J^\alpha[u](0) = 0$.*

Proof. If $u(0) = 0$, then

$$|J^\alpha[u](x)| \le \frac{1}{\Gamma(\alpha)} \int_0^1 (\ln\frac{1}{s})^{\alpha-1} \frac{|u(s\theta)|}{s} \le \frac{C}{\Gamma(\alpha)} \int_0^1 (\ln\frac{1}{s})^{\alpha-1} s^{\lambda-1} ds.$$

Since the last integral converges, the function $J^\alpha[u](x)$ is defined in the domain $\overline{\Omega}$. Let $x^{(1)}$, $x^{(2)}$ be arbitrary points of the domain $\overline{\Omega}$. Denote $h(x) = J^\alpha[u](x)$. Then

$$\begin{aligned} |h(x^{(1)}) - h(x^{(2)})| &\le \frac{1}{\Gamma(\alpha)} \int_0^1 (\ln\frac{1}{s}) s^{\mu-1} |u(sx^{(1)}) - u(sx^{(2)})| ds \\ &\le \frac{C|x^{(1)} - x^{(2)}|^\lambda}{\Gamma(\alpha)} \int_0^1 (\ln\frac{1}{s})^{\alpha-1} s^{\mu+\lambda-1} ds \\ &\le C|x^{(1)} - x^{(2)}|^\lambda, \end{aligned}$$

i.e. $h(x) \in C^\lambda(\overline{\Omega})$. Further, if $\beta = (\beta_1, \beta_2, \dots \beta_n)$ is a multi-index and $\partial_x^\beta = \frac{\partial^{|\beta|}}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n}}$, then for all β with length $|\beta| \le p$ and $x^{(1)}$, $x^{(2)} \in \overline{\Omega}$ we have

$$\begin{aligned} |\partial_x^\beta h(x^{(1)}) - \partial_x^\beta h(x^{(2)})| &\le \frac{1}{\Gamma(\alpha)} \int_0^1 (\ln\frac{1}{s})^{\alpha-1} s^{\mu+|\beta|-1} |\partial_x^\beta u(y_1) - \partial_x^\beta h(y_2)| ds \\ &\le C|x^{(1)} - x^{(2)}|^\lambda, \end{aligned}$$

where $y = sx = (sx_1, sx_2, \dots sx_n)$, and, consequently $J^\alpha[u](x) \in C^{\lambda+p}(\overline{\Omega})$. Lastly,

$$\lim_{x\to 0} |J_0^\alpha[u](x)| \le C \lim_{x\to 0} |x|^\lambda = 0.$$

Then $J_0^\alpha[u](0) = 0$. □

Similarly we can prove the following statement.

Lemma 2.2. *Let $\ell - 1 < \alpha \le \ell$, $\ell = 1, 2, \dots$, $0 < \lambda < 1$ and $u(x) \in C^{\lambda+p}(\bar{\Omega}), p \ge \ell$. Then $D^\alpha[u](x) \in C^{\lambda+p-\ell}(\bar{\Omega})$ and the equality $D^\alpha[u](0) = 0$. holds.*

Lemma 2.3. *Let $\ell - 1 < \alpha \le \ell$, $\ell = 1, 2, \dots$, $0 < \lambda < 1$ and $u(x) \in C^{\lambda+p}(\bar{\Omega})$, $p \ge \ell$, $p = 1, 2, \dots$. Then for any $x \in \bar{\Omega}$:*

$$J^\alpha[D^\alpha[u]](x) = u(x) - u(0), \tag{2.1}$$

and if $u(0) = 0$, then we obtain

$$D^\alpha[J^\alpha[u]](x) = u(x). \tag{2.2}$$

Proof. If $u(x) \in C^{\lambda+p}(\bar{\Omega})$, $p \ge \ell$, then by Lemma 2.2 we obtain $D^\alpha[u](x) \in C^{\lambda+p-\ell}(\bar{\Omega})$ and $D^\alpha[u](0) = 0$. Let us prove equality (2.1) for the case $\alpha = \ell$ - integer. Since $D^\alpha[u](0) = 0$, then in the class of these functions the operator J^α is defined, and in this case:

$$J^\ell[D^\ell[u]](x) = \frac{1}{(\ell-1)!} \int_0^r s^{-1} (\ln\frac{r}{s})^{\ell-1} (s\frac{d}{ds})^\ell u(s\theta) ds.$$

Integrating by parts the last integral $\ell - 1$ times, we obtain

$$J^\ell[D^\ell[u]](x) = \int_0^r \frac{d}{ds}[u(s\theta)] ds = u(s\theta)|_{s=0}^{s=r} = u(x) - u(0).$$

Let now $\ell - 1 < \alpha < \ell$, $\ell = 1, 2, \dots$. Then

$$J^\alpha[D^\alpha[u]](x) = J^\alpha[J^{\ell-\alpha}[\delta^\ell[u]]](x).$$

Further, since $J^\alpha \cdot J^{\ell-\alpha} = J^\ell$ (see e.g. [11], page 114), it follows that

$$J^\alpha[D^\alpha[u]](x) = J^\ell[\delta^\ell[u]](x) = u(x) - u(0).$$

The equality (2.1) is proved. Let us turn to the proof of the equality (2.2). If $\alpha = \ell$, then

$$\begin{aligned} D^\ell[J^\ell[u]](x) &= \delta^\ell\Big\{\frac{1}{(\ell-1)!}\int_0^r s^{-1}(\ln\frac{r}{s})^{\ell-1}u(s\theta)ds\Big\} \\ &= \delta^{\ell-1}\Big\{\frac{1}{(\ell-2)!}\int_0^r s^{-1}(\ln\frac{r}{s})^{\ell-2}u(s\theta)ds\Big\} \\ &= r\frac{d}{dr}\{\int_0^r s^{-1}u(s\theta)ds\} = u(x). \end{aligned}$$

Further, in the case $\ell-1 < \alpha < \ell$, $\ell = 1, 2, \dots$ by the definitions of D^α and J^α operators, we obtain

$$\begin{aligned} D^\alpha[J^\alpha[u]](x) &= \frac{1}{\Gamma(\ell-\alpha)}\int_0^r \frac{1}{\ell-\alpha}(\ln\frac{r}{s})^{\ell-\alpha-1}\delta^\ell[J^\alpha[u]](sx)\frac{ds}{s} \\ &= \frac{1}{\Gamma(\ell-\alpha)}r\frac{d}{dr}\int_0^r \frac{1}{\ell-\alpha}(\ln\frac{r}{s})^{\ell-\alpha}\frac{d}{ds}[\delta^{\ell-1}[J^\alpha[u]]]ds \\ &= \frac{1}{\Gamma(\ell-\alpha)}r\frac{d}{dr}\int_0^r (\ln\frac{r}{s})^{\ell-1-\alpha}\delta^{\ell-1}[J^\alpha[u]]\frac{ds}{s}. \end{aligned}$$

Performing this operation again $(\ell-1)$ times, we have

$$D^\alpha[J^\alpha[u]](x) = \delta^\ell[J^{\ell-\alpha}J^\alpha[u]](x) = \delta^\ell[J^\ell[u]](x) = u(x).$$

□

Lemma 2.4. *Let $\ell-1 < \alpha \le \ell$, $\ell = 1, 2, \dots$, $0 < \lambda < 1$, $f(x)$ be a smooth function in the domain $\bar\Omega$ and $\Delta u(x) = f(x)$, $x \in \Omega$. Then*

$$\Delta D^\alpha[u](x) = |x|^{-2}D^\alpha[|x|^2 f](x),\ x \in \Omega. \tag{2.3}$$

Proof. We represent the function $D^\alpha[u](x)$ in the following form:

$$D^\alpha[u](x) = \frac{1}{\Gamma(\ell-\alpha)}\int_0^1 (\ln\frac{1}{\xi})^{\ell-1-\alpha}(\xi\frac{d}{d\xi})^\ell[u(\xi x)]\frac{d\xi}{\xi}.$$

Further, since

$$\Delta(\xi\frac{d}{d\xi})^\ell[u](\xi x) = \xi^2(\xi\frac{d}{d\xi}+2)^\ell f(\xi x) = \xi^2(\delta+2)^\ell[f](\xi x),$$

it follows that

$$\begin{aligned} \Delta D^\alpha[u](x) &= \frac{1}{\Gamma(\ell-\alpha)}\int_0^1 (\ln\frac{1}{\xi})^{\ell-1-\alpha}\xi^2(\delta+2)^\ell[f](\xi x)\frac{d\xi}{\xi} \\ &= r^{-2}J^{\ell-\alpha}[\delta^\ell[r^2 f]](x) \\ &= r^{-2}D^\alpha[r^2 f](x). \end{aligned}$$

□

Remark 2.5. It is easy to prove that for the function $F(x) = |x|^{-2}D^\alpha[|x|^2 f](x)$ the following representation holds:

$$F(x) = \big(r\frac{d}{dr}+2\big)f_{\ell-\alpha}(x), \tag{2.4}$$

where

$$f_{\ell-\alpha}(x) = r^{-2} J^{\ell-\alpha}[r^2\delta^{\ell-1}[f]](x). \tag{2.5}$$

3. A property of the Dirichlet problem solution

In the domain Ω we consider the Dirichlet problem

$$\begin{aligned} \Delta v(x) &= F(x), \quad x \in \Omega, \\ v(x) &= g(x), \quad x \in \partial\Omega. \end{aligned} \tag{3.1}$$

It is known [8] that if $0 < \lambda < 1$, $F(x) \in C^{\lambda+p}(\overline{\Omega})$, $g(x) \in C^{\lambda+p+2}(\partial\Omega)$, $p \geq 1$, then a solution of the problem exists, is unique, belongs to the class $C^{\lambda+2}$ and can be represented in the form:

$$v(x) = -\frac{1}{\omega_n}\int_\Omega G(x,y)F(y)dy + \frac{1}{\omega_n}\int_{\partial\Omega} \frac{1-|x|^2}{|x-y|^n} g(y)ds_y, \tag{3.2}$$

where ω_n is a square of the unit sphere, $G(x,y)$ is the Green function of the problem (3.1). Moreover, $G(x,y)$ is represented in the form [7]:

$$G(x,y) = \begin{cases} \frac{1}{n-2}[|x-y|^{2-n} - |x|y| - \frac{y}{|y|}|^{2-n}], & n \geq 3 \\ \ln\frac{1}{|x-y|}, & n = 2. \end{cases}$$

Let $\rho = |y|$.

Lemma 3.1. *Let $F(y), g(y)$ be smooth enough functions and $F(y)$ be represented in the form $F(y) = (\rho\frac{\partial}{\partial\rho} + 2)f_1(y), v(x)$ be a solution of the problem (3.1). Then the condition $v(0) = 0$ holds if and only if*

$$\int_\Omega f_1(y)dy = \int_{\partial\Omega} g(y)ds_y. \tag{3.3}$$

Proof. Since $F(y)$ and $g(y)$ are smooth enough functions, then solution of the problem (3.1) exists and can be represented as (3.2). Then in the case $n \geq 3$ we have

$$v(0) = -\frac{1}{\omega_n}\int_\Omega \frac{1}{n-2}[|y|^{2-n} - 1]F(y)dy + \frac{1}{\omega_n}\int_{\partial\Omega} g(y)dy. \tag{3.4}$$

We consider the following two integrals:

$$\begin{aligned} I_1(\rho,\xi) &= \int_0^1 \rho^{n-1}[\rho^{2-n} - 1]\rho\frac{\partial}{\partial\rho} f_1(\rho,\xi)d\rho, \\ I_2(\rho,\xi) &= 2\int_0^1 \rho^{n-1}[\rho^{2-n} - 1] f_1(\rho,\xi)d\rho. \end{aligned}$$

Integrating I_1 by parts, we obtain

$$\begin{aligned} I_1(\rho,\xi) &= \int_0^1 [\rho^2 - \rho^n]\frac{\partial}{\partial\rho} f_1(\rho,\xi)d\rho \\ &= -\int_0^1 [2\rho - n\rho^{n-1}] f_1(\rho,\xi)d\rho \\ &= \int_0^1 \rho^{n-1}[n - 2\rho^{2-n}] f_1(\rho,\xi)d\rho. \end{aligned}$$

Since $F(y)$ has the form $(\rho\frac{\partial}{\partial\rho}+2)f_1(y)$, moving to spherical coordinates for the first integral in the right-hand side of (3.4), we have

$$
\begin{aligned}
&-\frac{1}{\omega_n}\int_\Omega \frac{1}{n-2}[|y|^{2-n}-1](\rho\frac{\partial}{\partial\rho}+2)f_1(y)dy\\
&=-\frac{1}{(n-2)\omega_n}\int_{|\xi|=1}\int_0^1 \rho^{n-1}[\rho^{2-n}-1](\rho\frac{\partial}{\partial\rho}+2)f_1(\rho,\xi)d\rho d\xi\\
&=-\frac{1}{(n-2)\omega_n}\int_{|\xi|=1}[I_1(\rho,\xi)+I_2(\rho,\xi)]d\xi\\
&=-\frac{1}{\omega_n}\int_{|\xi|=1}\int_0^1 \rho^{n-1}f_1(\rho,\xi)d\rho d\xi=-\frac{1}{\omega_n}\int_\Omega f_1(y)dy.
\end{aligned}
$$

Consequently, if $v(0)=0$, then the equality (3.3) holds. Hence, necessity of the condition (3.3) is proved. Sufficiency is proved in reverse order. □

4. Neumann type problem

In this section we consider a fractional analogue of the Neumann problem with the boundary operator D^α.

Problem 4.1. Let $0<\alpha$. Find a function $u(x)\in C^2(\Omega)\cap C(\overline{\Omega})$ such that $D^\alpha[u](x)\in C(\overline{\Omega})$, and satisfying the equation

$$\Delta u(x)=f(x), x\in\Omega, \tag{4.1}$$

and the boundary value condition

$$D^\alpha[u](x)=g(x), x\in\partial\Omega. \tag{4.2}$$

Since $J^0=I$, when $\alpha=1$ we have

$$D^1u(x)\big|_{\partial\Omega}=J^0[\delta[u]](x)\big|_{\partial\Omega}=r\frac{du(x)}{dr}\big|_{\partial\Omega}=\frac{\partial u(x)}{\partial\nu}\big|_{\partial\Omega}.$$

Therefore, when $\alpha=1$ the problem (4.1) - (4.2) coincides with the classical Neumann problem.

Theorem 4.2. *Let $\ell-1<\alpha\le\ell$, $\ell=1,2,\dots$, $0<\lambda<1$, $f(x)\in C^{\lambda+2\ell-1}(\overline{\Omega})$, $g(x)\in C^{\lambda+\ell+1}(\partial\Omega)$. Then for solvability of the problem 4.1 it is necessary and sufficient the condition*

$$\int_\Omega f_{\ell-\alpha}(y)dy=\int_{\partial\Omega}g(y)dy. \tag{4.3}$$

where the function $f_{\ell-\alpha}(x)$ is defined by the equality (2.5)*.*

If a solution of the problem exists, then it is unique up to a constant term, belongs to the class $C^{\lambda+\ell+1}(\overline{\Omega})$ and can be represented in the form

$$u(x)=C+J^\alpha[v](x), \tag{4.4}$$

where $v(x)$ is a solution of problem (3.1) *with the function $F(x)=r^{-2}D^\alpha[r^2f](x)$ and satisfies the condition $v(0)=0$.*

Proof. Let $u(x)$ be a solution of problem 4.1. Apply the operator D^α to the function $u(x)$, and denote $v(x)=D^\alpha[u](x)$. Find conditions, which the function $v(x)$

satisfies. It is obvious that $v(x)|_{\partial\Omega} = D^\alpha[u](x)|_{\partial\Omega} = g(x)$. Applying the operator Δ to the equality $v(x) = D^\alpha[u](x)$, due to (2.3), we obtain

$$\Delta v(x) = r^{-2}D^\alpha[r^2 f](x).$$

Therefore, if $u(x)$ is a solution of the problem 4.1, then $v(x) = D^\alpha[u](x)$ will be a solution of (3.1) with the function $F(x) = r^{-2}D^\alpha[r^2 f]$. Moreover, according to Lemma 2.2, the function $v(x)$ satisfies the condition $v(0) = 0$. By (2.4), the function $F(x)$ can be represented in the form

$$F(x) = \Big(r\frac{d}{dr} + 2\Big)f_{\ell-\alpha}(x),$$

where $f_{\ell-\alpha}(x)$ is defined by the equality (2.5). Then, by Lemma 3.1 for the equality $v(0) = 0$ the following condition is necessary:

$$\int_\Omega f_{\ell-\alpha}(y)dy = \int_{\partial\Omega} g(y)dS_y.$$

Therefore, necessity of the condition (4.3) is proved. Applying the operator J^α to the equality $v(x) = D^\alpha[u](x)$, because of (2.1), we obtain

$$u(x) - u(0) = J^\alpha[v](x).$$

Hence, if a solution of the problem 4.1 exists, then it can be represented as (4.4). We show that the condition (4.3) is sufficient for the existence of any solution of the problem 4.1.

Indeed, let $v(x)$ be a solution of the problem (3.1) with $F(x) = r^{-2}D^\alpha[r^2 f](x)$. If $f(x) \in C^{\lambda+2\ell-1}(\overline{\Omega})$, then $F(x) \in C^{\lambda+\ell-1}(\overline{\Omega})$, and since $g(x) \in C^{\lambda+\ell+1}(\partial\Omega)$, a solution of (3.1) exists, is unique and belongs to the class $C^{\lambda+p+1}(\bar{\Omega})$ (see e.g. [8]). We represent the function $F(x) = |x|^{-2}D^\alpha[|x|^2 f](x)$ as $F(x) = (r\frac{d}{dr} + 2)f_{\ell-\alpha}(x)$. If for the function $f_{\ell-\alpha}(x)$ the condition (4.3) holds, then corresponding solution of the problem (3.1) satisfies the condition $v(0) = 0$. Then we should to consider the function $u(x) = C + J^\alpha[v](x)$, which satisfies all conditions of problem 4.1. By Lemma 2.1 this function belongs to the class $C^{\lambda+p+1}(\bar{\Omega})$. Further, using (2.2), we obtain

$$D^\alpha[u](x)|_{\partial\Omega} = D^\alpha[C] + D^\alpha[J^\alpha[v]](x)|_{\partial\Omega} = v(x)|_{\partial\Omega} = g(x).$$

Moreover,

$$\begin{aligned}
\Delta u(x) &= \Delta\Big[\frac{1}{\Gamma(1-\alpha)}\int_0^r \big(\ln\frac{r}{s}\big)^{1-\alpha} v(s\theta)\frac{ds}{s}\Big] \\
&= \Delta\Big[\frac{1}{\Gamma(1-\alpha)}\int_0^1 \big(\ln\frac{1}{\xi}\big)^{1-\alpha} v(\xi x)\frac{d\xi}{\xi}\Big] \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^1 \big(\ln\frac{1}{\xi}\big)^{1-\alpha} \xi^2 F(\xi x)\frac{d\xi}{\xi} \\
&= \frac{1}{\Gamma(1-\alpha)}\int_0^1 \big(\ln\frac{1}{\xi}\big)^{1-\alpha} \xi^2 |\xi x|^{-2} D^\alpha[|\xi x|^2 f(\xi x)]\frac{d\xi}{\xi} \\
&= \frac{|x|^{-2}}{\Gamma(1-\alpha)}\int_0^r \big(\ln\frac{r}{s}\big)^{1-\alpha} D^\alpha[s^2 f(s\theta)]\frac{ds}{s} \\
&= r^{-2}J^\alpha[D^\alpha[r^2 f]](x) = r^{-2}\cdot r^2 f(x) = f(x).
\end{aligned}$$

Thus, the function $u(x) = C + J^\alpha[v](x)$ satisfies all conditions of problem 4.1. $\square$

Remark 4.3. If $\alpha = 1$, then $f_1(x) = r^{-2}J^0[r^2 f](x) = f(x)$ and (4.3) coincides with the condition of solvability of the Neumann problem (1.3).

5. Boundary-value problems with periodic conditions

In this section we study some analogues of periodic problems in Ω. Let $x = (x_1, \tilde{x}) \in \Omega, \tilde{x} = (x_2, \dots, x_n)$ For any $x = (x_1, \tilde{x}) \in \Omega$ we put "opposite" point $x^* = (-x_1, a\tilde{x}) \in \Omega$, where $a = (a_2, a_3, \dots, a_n)$ and $a_j, j = 2, \dots, n$ take one of the values ± 1. Denote

$$\partial\Omega_+ = \{x \in \partial\Omega : x_1 \geq 0\}, \quad \partial\Omega_- = \{x \in \partial\Omega : x_1 \leq 0\}, \quad I = \{x \in \partial\Omega : x_1 = 0\}.$$

Let $0 < \alpha \leq 1$. Consider in Ω the following problem:

Problem 5.1. Find a function $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$, such that $D^\alpha[u](x) \in C(\bar{\Omega})$ and

$$\Delta u(x) = f(x), \quad x \in \Omega, \tag{5.1}$$

$$u(x) - (-1)^k u(x^*) = g_0(x), \quad x \in \partial\Omega_+, \tag{5.2}$$

$$D^\alpha[u](x) + (-1)^k D^\alpha[u](x^*) = g_1(x), \quad x \in \partial\Omega_+, \tag{5.3}$$

where $k = 1, 2$.

The problem (5.1)–(5.3) in the case $\alpha = 1$ have been studied in [17],[18] and in the case $0 < \alpha < 1$ for the Riemann - Liouville and Caputo operators in [19].

If $x = (0, \tilde{x}) \in I$, then $x^* = (0, \alpha\tilde{x}) \in I$, therefore, a necessary condition for existence of a solution from the class $u(x) \in C^2(\Omega) \cap C(\bar{\Omega})$, $D^\alpha[u](x) \in C(\bar{\Omega})$ is the fulfillment of the conditions

$$g_0(0, \tilde{x}) = -(-1)^k g_0(0, a\tilde{x}), \tag{5.4}$$

$$\frac{\partial g_0(0, \tilde{x})}{\partial x_j} = -(-1)^k \frac{\partial g_0(0, a\tilde{x})}{\partial x_j}, \quad j = 1, \dots, n, (0, \tilde{x}) \in I, \tag{5.5}$$

$$g_1(0, \tilde{x}) = (-1)^k g_1(0, a\tilde{x}), \quad (0, \tilde{x}) \in I. \tag{5.6}$$

Theorem 5.2. *Let $0 < \lambda < 1$, $f(x) \in C^{\lambda+1}(\bar{\Omega})$, $g_0(x) \in C^{\lambda+2}(\partial\Omega_+)$, $g_1(x) \in C^{\lambda+2}(\partial\Omega_+)$ and the matching conditions* (5.4),(5.6) *hold. Then*

1. *if $k = 1$, then a solution of the problem* (5.1)–(5.3) *exists, is unique and belongs to the class $C^{\lambda+2}(\bar{\Omega})$;*
2. *if $k = 2$, then for solvability of the problem* (5.1) - (5.3) *the following condition is necessary and sufficient:*

$$\int_\Omega f_{1-\alpha}(y)dy = \int_{\partial\Omega_+} g_1(y)dS_y. \tag{5.7}$$

If a solution exists, then it is unique up to a constant term, and belongs to the class $C^{\lambda+2}(\bar{\Omega})$.

Proof. First we prove uniqueness. Let $u(x)$ be a solution of the homogenous problem (5.1) - (5.3). Putting the function $u(x)$ into the boundary value conditions of the problem (5.1)–(5.3), we have

$$u(x) = (-1)^k u(x^*), \quad x \in \partial\Omega_+, \tag{5.8}$$

$$D^\alpha[u](x) = -(-1)^k D^\alpha[u](x^*), \quad x \in \partial\Omega_+. \tag{5.9}$$

If $x \in \partial\Omega_-$, then $x^* \in \partial\Omega_+$. Then the condition (5.8) implies $u(x^*) = (-1)^k u(x)$, $x \in \partial\Omega_-$, and (5.9) yields $D^\alpha[u](x^*) = -(-1)^k D^\alpha[u](x), x \in \partial\Omega_-$ Consequently, the equalities (5.8) and (5.9) hold for all points $x \in \partial\Omega$, i.e.

$$u(x) = (-1)^k u(x^*), D^\alpha[u](x) = -(-1)^k D^\alpha[u](x^*), \quad x \in \partial\Omega.$$

Since $D^\alpha[u](x) \in C(\bar{\Omega})$, then from the equality $u(x) = (-1)^k u(x^*), x \in \partial\Omega$ it follows: $D^\alpha u(x) = (-1)^k D^\alpha u(x^*), x \in \partial\Omega$. Consequently, $D^\alpha u(x) = 0, x \in \partial\Omega$, i.e. solution of the homogeneous problem (5.1) - (5.3) is also solution of the homogeneous Problem 4.1. Then by Theorem 4.2: $u(x) \equiv C, x \in \bar{\Omega}$. Hence, putting $u(x) \equiv C$ into (5.8), when $k = 1$ we have $u(x) \equiv 0$. Therefore, solution of the problem (5.1)–(5.3) when $k = 1$ is unique, and when $k = 2$ it is unique up to a constant term. Uniqueness is proved. Now let us turn to study existence of a solution. Consider the auxiliary functions

$$v(x) = \frac{1}{2}(u(x) + u(x^*)), \quad w(x) = \frac{1}{2}(u(x) - u(x^*)).$$

It is obvious that $u(x) = v(x) + w(x)$. Moreover, $v(x) = v(x^*)$, $w(x) = -w(x^*)$.

We find problems, which these functions satisfy. Let $k = 1$. Applying the operator Δ to the functions $v(x)$ and $w(x)$, we have

$$\Delta v(x) = \frac{1}{2}[\Delta u(x) + \Delta u(x^*)] = \frac{1}{2}[f(x) + f(x^*)] \equiv f^+(x), \quad x \in \Omega,$$
$$\Delta w(x) = \frac{1}{2}[\Delta u(x) - \Delta u(x^*)] = \frac{1}{2}[f(x) - f(x^*)] \equiv f^-(x), x \in \Omega.$$

Further, from the boundary value conditions (5.2) and (5.3) we obtain

$$v(x)\big|_{\partial\Omega_+} = \frac{1}{2}[u(x) + u(x^*)]\big|_{\partial\Omega_+} = \frac{g_0(x)}{2},$$
$$D^\alpha w(x)\big|_{\partial\Omega} = \frac{1}{2}[D^\alpha u(x) - D^\alpha u(x^*)]\big|_{\partial\Omega_+} = \frac{g_1(x)}{2}.$$

If $x \in \partial\Omega_-$, then $x^* \in \partial\Omega_+$, so the following equalities hold:

$$v(x)\big|_{\partial\Omega_-} = \frac{1}{2}[u(x^*) + u(x)]\big|_{\partial\Omega_+} = \frac{g_0(x^*)}{2},$$
$$D^\alpha w(x)\big|_{\partial\Omega} = -\frac{1}{2}[D^\alpha u(x^*) - D^\alpha u(x)]\big|_{\partial\Omega_+} = -\frac{g_1(x^*)}{2}.$$

We introduce the functions

$$2\tilde{g}_0(x) = \begin{cases} g_0(x), x \in \partial\Omega_+ \\ g_0(x^*), x \in \partial\Omega_- \end{cases}, \quad 2\tilde{g}_1(x) = \begin{cases} g_1(x), x \in \partial\Omega_+ \\ -g_1(x^*), x \in \partial\Omega_- \end{cases}.$$

Therefore, functions $v(x)$ and $w(x)$ are solutions of the two problems:

$$\Delta v(x) = f^+(x), \quad x \in \Omega; \quad v(x)|_{\partial\Omega} = \tilde{g}_0(x), \tag{5.10}$$
$$\Delta w(x) = f^-(x), x \in \Omega; \quad D^\alpha w(x)|_{\partial\Omega} = \tilde{g}_1(x). \tag{5.11}$$

If for the functions $f(x), g_0(x)$ and $g_1(x)$ the conditions of the theorem hold, then $f^\pm(x) \in C^{\lambda+1}(\bar{\Omega})$, $\tilde{g}_0(x) \in C^{\lambda+2}(\partial\Omega)$, $\tilde{g}_1(x) \in C^{\lambda+2}(\partial\Omega)$. Then a solution of the Dirichlet problem (5.10) exists, is unique, belongs to the class $C^{\lambda+2}(\bar{\Omega})$. By Theorem 4.2, for solvability of the problem (5.11) it is necessary and sufficient the following condition:

$$\int_\Omega f^-_{1-\alpha}(y)dy = \int_{\partial\Omega} \tilde{g}_1(y)dy, \tag{5.12}$$

where $f^-_{1-\alpha}(y) = r^{-2}J^{1-\alpha}[r^2 f^-](x)$.

Since

$$\begin{aligned}\int_\Omega f^-_{1-\alpha}(y)dy &= \frac{1}{2}\int_\Omega f^-_{1-\alpha}(y)dy - \frac{1}{2}\int_\Omega f^-_{1-\alpha}(y*)dy = 0,\\ \int_{\partial\Omega} \tilde{g}_1(y)dS_y &= \frac{1}{2}\int_{\partial\Omega} g_1(y)dS_y - \frac{1}{2}\int_{\partial\Omega} g_1(y*)dS_y = 0,\end{aligned}$$

it follows that the condition for solvability of (5.12) always holds, and therefore, in this case $f^-(x) \in C^{\lambda+1}(\bar{\Omega})$, $\tilde{g}_1(x) \in C^{\lambda+2}(\partial\Omega)$ a solution of problem (5.11) exists and belongs to the class $C^{\lambda+1}(\bar{\Omega})$. Note that a solution of the problem (5.11) is unique up to a constant term C. Since the function $w(x)$ should have the property $w(x) = -w(x^*)$, we obtain $C \equiv 0$. Therefore, the existence of a solution of problem (5.1)–(5.3) for the case $k = 1$ is proved.

Let $k = 2$. In this case for auxiliary functions $v(x)$ and $w(x)$ we obtain the following problems:

$$\Delta w(x) = f^-(x), \quad x \in \Omega; \quad w(x)|_{\partial\Omega} = \tilde{g}_0(x), \tag{5.13}$$

$$\Delta v(x) = f^+(x), \quad x \in \Omega; \quad D^\alpha v(x)|_{\partial\Omega} = \tilde{g}_1(x). \tag{5.14}$$

Here

$$2\tilde{g}_0(x) = \begin{cases} g_0(x), & x \in \partial\Omega_+ \\ -g_0(x^*), & x \in \partial\Omega_- \end{cases}, \quad 2\tilde{g}_1(x) = \begin{cases} g_1(x), & x \in \partial\Omega_+ \\ g_1(x^*), & x \in \partial\Omega_-. \end{cases}$$

When the conditions of the theorem hold, a solution of problem (5.13) exists, it is unique and belongs to the class $C^{\lambda+1}(\bar{\Omega})$. And for solvability of the problem (5.14) it is necessary and sufficient the condition

$$\int_\Omega f^+_{1-\alpha}(y)dy = \int_{\partial\Omega} \tilde{g}_1(y)dS_y. \tag{5.15}$$

Since

$$\begin{aligned}\int_\Omega f^+_{1-\alpha}(y)dy &= \frac{1}{2}\int_\Omega f_{1-\alpha}(y)dy + \frac{1}{2}\int_\Omega f_{1-\alpha}(y*)dy = \int_\Omega f_{1-\alpha}(y)dy,\\ \int_{\partial\Omega} \tilde{g}_1(y)dS_y &= \frac{1}{2}\int_{\partial\Omega_+} g_1(y)dS_y + \frac{1}{2}\int_{\partial\Omega_-} g_1(y*)dS_y = \int_{\partial\Omega_+} g_1(y)dS_y,\end{aligned}$$

it follows that (5.15) can be rewritten as (5.7). Under this condition, a solution of problem (5.14) exists, is unique up to a constant term, and belongs to the class $C^{\lambda+1}(\bar{\Omega})$. □

Remark 5.3. When $\alpha = 1$ the propositions in Theorems 4.2 and 5.2 coincide with the results in [17, 18].

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