Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 16, pp. 1–18.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

OSCILLATORY BEHAVIOR OF N-TH-ORDER NEUTRAL DYNAMIC EQUATIONS WITH MIXED NONLINEARITIES ON TIME SCALES

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Abstract. In this article, several new oscillation theorems for n-th-order neutral dynamic equations with mixed nonlinearities are established. Our work extends some known results in the literature on second-order, third-order, and higher-order linear and half-linear dynamic equations. Two examples are provided to illustrate the relevance of the new theorems.

1. Introduction

The theory of time scales was introduced by Hilger (see [8]) in 1988 in order to unify continuous and discrete analysis. Not only can this theory of the so-called "dynamic equations" unify the theory of differential and difference equations, but it can also extend some classical cases to cases "in between", e.g., to the so-called q-difference equations. Oscillations of delay dynamic equations are common in applications, for example, in economics, where the demand depends on current price and the supply depends on the price at an earlier time, and in the study of population dynamic models (see [5]).

There are available sufficient conditions for the oscillation and nonoscillation of solutions of various neutral dynamic equations. For second order neutral dynamic equations on time scales, Wu et al [16] in 2006 studied the second order nonlinear neutral dynamic equation of the form

$$[r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\alpha}]^{\Delta}+f(t,\delta(t))=0, \tag{1.1}$$

where $\alpha \geq 1$ is a quotient of odd positive integers. Zhang and Wang [19] improved and complemented some results in [16] for $\alpha \geq 1$ and gave new results for $0<\alpha<1$. Sun et al [14] considered the second order quasilinear neutral dynamic equation

$$[r(t)((x(t)+p(t)x(\tau(t)))^{\Delta})^{\gamma}]^{\Delta}+q_1(t)x^{\alpha}(\delta_1(t))+q_2(t)x^{\beta}(\delta_2(t))=0, \tag{1.2}$$

where γ,α,β are quotients of odd positive integers with $0<\alpha<\gamma<\beta$. For more results on second order neutral dynamic equations, we refer the reader to the papers (see [1, 3, 4, 6, 11, 17]).

2010 *Mathematics Subject Classification.* 34N05, 34K40, 34K11.
Key words and phrases. Neutral dynamic equation; oscillation; mixed nonlinearities; generalized Riccati technique.

Submitted August 26, 2015. Published January 8, 2016.

Saker and Graef [13] and Zhang [18] considered the third order half-linear neutral dynamic equation of the form

$$[r_1(t)((r_2(t)(x(t)+a(t)x(\tau(t)))^{\Delta})^{\Delta})^{\gamma}]^{\Delta}+p(t)x^{\gamma}(\delta(t))=0. \tag{1.3}$$

Their results were further extended by Utku et al [15] to the equation

$$[r(t)((x(t)+p(t)x(\tau_0(t)))^{\Delta\Delta})^{\gamma}]^{\Delta}+q_1(t)x^{\alpha}(\tau_1(t))+q_2(t)x^{\beta}(\tau_2(t))=0, \tag{1.4}$$

where $0<\alpha<\gamma<\beta$.

Higher order dynamic equations have recently also been studied by many authors. For instance, in 2014, Hassan and Kong [9] established some oscillation criteria for nth-order half-linear dynamic equation

$$(x^{[n-1]})^{\Delta}(t)+p(t)\phi_{\alpha[1,n-1]}(x(g(t)))=0 \tag{1.5}$$

on time scale $\mathbb{T}$, where $n\geq 2$, $\phi_\beta(u):=|u|^\beta \operatorname{sgn} u$, $\alpha[i,j]:=\alpha_i\dots\alpha_j$, $x^{[i]}(t):=r_i(t)\phi_{\alpha_i}[(x^{[i-1]})^{\Delta}(t)]$, $i=1,2,\dots,n-1$, with $x^{[0]}=x$.

Chen and Qu [7] extended the work in [9] to even order advanced type delay dynamic equations on time scales containing mixed nonlinearities

$$[r(t)\Phi_\alpha(x^{\Delta^{n-1}}(t))]^{\Delta}+p(t)\Phi_\alpha(x(\delta(t)))+\sum_{i=1}^{k}p_i(t)\Phi_{\alpha_i}(x(\delta(t)))=0, \tag{1.6}$$

where $n\geq 2$ is even, $\alpha,\ \alpha_i>0,\ \delta(t)\geq t$, $\Phi_*(u)=|u|^{*-1}u$.

Our goal in this paper is to study the oscillation of nth-order neutral dynamic equations with mixed nonlinearities of the form

$$\begin{gathered}[r(t)|y^{\Delta^{n-1}}(t)|^{\alpha-1}y^{\Delta^{n-1}}(t)]^{\Delta}+q_0(t)|x(\delta_0(t))|^{\alpha-1}x(\delta_0(t))\\ +\sum_{i=1}^{m}q_i(t)|x(\delta_i(t)))|^{\beta_i-1}x(\delta_i(t))=0\end{gathered} \tag{1.7}$$

on an arbitrary time scale $\mathbb{T}$, where $y(t)=x(t)+p(t)x(\tau(t))$, under the following hypotheses.

- (A1) $r,q_i\in C_{rd}(\mathbb{T},\mathbb{R}^+)$ for $i=0,1,\dots,m$, where $\mathbb{R}^+=(0,\infty)$, and $p\in C_{rd}(\mathbb{T},[0,1))$, $\int_{t_0}^{\infty}r^{-\frac{1}{\alpha}}(s)\Delta s=\infty$;
- (A2) $n\geq 2$ is an integer, α,β_i $(i=1,2,\dots,m)$ are constants, $\beta_1>\dots>\beta_k>\alpha>\beta_{k+1}>\dots>\beta_m>0$;
- (A3) $\delta_i\in C_{rd}(\mathbb{T},\mathbb{T}),\delta_i(t)\geq t$ $(i=0,1,\dots,m)$, $\tau\in C_{rd}(\mathbb{T},\mathbb{R}^+)$, $\tau(t)\leq t$, $\lim_{t\to\infty}\tau(t)=\infty$.

For the study of oscillation purpose, we are only interested in the solutions that are extendable to ∞. Thus, we assume that the time scale $\mathbb{T}$ under consideration satisfies $\inf\mathbb{T}=t_0>0$ and $\sup\mathbb{T}=\infty$. For $T\in\mathbb{T}$, denote $[T,\infty)_{\mathbb{T}}:=\{t\in\mathbb{T}:t\geq T\}$, $\tau^*(t)=\min\{\tau(t),\delta_0(t),\delta_1(t),\dots,\delta_m(t)\}$, $T_0=\min\{\tau^*(t):t\geq t_0\}$ and $\tau^*_{-1}(t)=\sup\{s\geq t_0:\tau^*(s)\leq t\}$. Clearly $\tau^*_{-1}(t)\geq t$ for $t\geq T_0$, $\tau^*_{-1}(t)$ is nondecreasing and coincides with the inverse of $\tau^*(t)$ when the latter exists.

Definition 1.1. By a solution x of (1.7), we mean a nontrivial real-valued function in $C^1_{rd}([\tau^*_{-1}(t_0)_{\mathbb{T}},\infty),\mathbb{R})$ with $y\in C^1_{rd}([\tau^*_{-1}(t_0)_{\mathbb{T}},\infty),\mathbb{R})$ and $r|y^{\Delta^{n-1}}|^{\alpha-1}y^{\Delta^{n-1}}\in C^1_{rd}([\tau^*_{-1}(t_0)_{\mathbb{T}},\infty),\mathbb{R})$, and such that (1.7) is satisfied on the interval $[\tau^*_{-1}(t_0),\infty)_{\mathbb{T}}$.

Our attention is restricted to those solutions of (1.7) that exist on some half line $[\tau^*_{-1}(t_0),\infty)_{\mathbb{T}}$ and satisfy $\sup\{|x(t)|:t\geq t_x\}>0$ for any $t_x\geq\tau^*_{-1}(t_0)$. About the existence and uniqueness of solutions to dynamic equations, we refer the reader to [12]. A solution of (1.7) is called *nonoscillatory* if it is either eventually positive or eventually negative, otherwise it is called *oscillatory*. Equation (1.7) is said to be *oscillatory* if all its solutions are *oscillatory*.

Note that the results obtained in [1, 3, 4, 6, 7, 9, 11, 13, 14, 15, 16, 17, 18, 19] cannot be applied to nth-order neutral dynamic equation (1.7). Therefore, it is of interest to study the oscillation of (1.7). Motivated by the works mentioned above, by applying the generalized Riccati transformation and certain well-known techniques, we establish new sufficient conditions to guarantee that every solution of (1.7) is oscillatory or tends to zero eventually. The results obtained in this paper extend some known results in the literature on the oscillation for second and third order, and higher order linear and half-linear dynamic equations.

For convenience, throughout this article we use the notation:

$$x(\sigma(t)) = x^\sigma(t), \quad x^\Delta(\sigma(t)) = (x^\Delta(t))^\sigma.$$

The article is organized as follows. In Section 2, we give some basic lemmas which play a key in the subsequence. In Section 3, we establish several sufficient conditions to guarantee that every solution of (1.7) is oscillatory when n is even. The case when n is odd is discussed in Section 4. Finally, in Section 5, two examples are provided to illustrate the relevance of our results.

2. Basic lemmas

Lemma 2.1 ([2, Lemma 2.2])**.** *For any m-tuple $\{\beta_1, \beta_2, \dots, \beta_m\}$ satisfying*

$$\beta_1 > \dots > \beta_k > \alpha > \beta_{k+1} > \dots > \beta_m > 0,$$

there corresponds an m-tuple $\{\eta_1, \eta_2, \dots, \eta_m\}$ such that

$$\sum_{i=1}^m \beta_i \eta_i = \alpha, \quad \sum_{i=1}^m \eta_i = 1, \quad 0 < \eta_i < 1, \ i = 1, 2, \dots, m. \tag{2.1}$$

If $m = 2$ and $k = 1$, it turns out that

$$\eta_1 = \frac{\alpha - \beta_2}{\beta_1 - \beta_2}, \quad \eta_2 = \frac{\beta_1 - \alpha}{\beta_1 - \beta_2}.$$

Lemma 2.2 ([10, Young's Inequality])**.** *If X and Y are nonnegative, then for $\lambda > 1$,*

$$\lambda X Y^{\lambda-1} - X^\lambda \le (\lambda - 1) Y^\lambda,$$

where the equality holds if and only if $X = Y$.

Lemma 2.3. *If (1.7) has an eventually positive solution x. Then there exists an integer $l \in \{0, 1, \dots, n-1\}$ with $l + n$ odd such that*

$$y^{\Delta^j}(t) > 0, \quad j = 0, 1, \dots, l \tag{2.2}$$

and

$$(-1)^{l+j} y^{\Delta^j}(t) > 0, \quad j = l+1, l+2, \dots, n-1 \tag{2.3}$$

eventually, where $y^{\Delta^0}(t) := y(t) = x(t) + p(t)x(\tau(t))$.

The proof of the above lemma is similar to that of [9, Lemma 1].

3. Oscillation for even order equations

In this section, we establish several oscillation criteria for equation (1.7) when n is even. Throughout this section, we denote

$$\theta_1(t) = q_0(t)(1 - p(\delta_0(t)))^{\alpha} + \sum_{i=1}^{m} q_i(t)(1 - p(\delta_i(t)))^{\beta_i},$$

$$\theta_2(t) = q_0(t)(1 - p(\delta_0(t)))^{\alpha} + \prod_{i=1}^{m} \eta_i^{-\eta_i} q_i^{\eta_i}(t)(1 - p(\beta_i(t)))^{\beta_i \eta_i}.$$

The first theorem can be considered as the extension of Fite-Winter type oscillation criterion.

Theorem 3.1. *Assume that*

$$\int_{t_0}^{\infty} \theta_1(u)\Delta u = \infty. \tag{3.1}$$

Then (1.7) *is oscillatory.*

Proof. Assume, for the sake of contradiction, that (1.7) has a nonoscillatory solution x. We may assume that x is eventually positive by replacing x by $-x$, otherwise. By Lemma 2.3, there exist $t_1 \in [t_0, \infty)_{\mathbb{T}}$ and an odd integer $l \in \{1, 3, \dots, n-1\}$ such that (2.2) and (2.3) hold eventually. Note that odd $l \in \{1, 3, \dots, n-1\}$ implies that $y^{\Delta}(t) > 0$ and $y^{\Delta^{n-1}}(t) > 0$ for all $t \in [t_1, \infty)_{\mathbb{T}}$. This implies that $y(t)$ is strictly increasing on $[t_1, \infty)_{\mathbb{T}}$. By (A3), we conclude $y(\delta_i(t)) \ge y(t) \ge y(t_1) := a_2 > 0$ for $t \in [t_1, \infty)_{\mathbb{T}}$, and

$$x(t) = y(t) - p(t)x(\tau(t)) \ge y(t) - p(t)y(\tau(t)) \ge y(t) - p(t)y(t) = (1 - p(t))y(t),$$

then, for $i = 0, 1, \dots, m$, $t \in [t_1, \infty)_{\mathbb{T}}$, we have

$$x(\delta_i(t)) \ge (1 - p(\delta_i(t)))y(\delta_i(t)) \ge (1 - p(\delta_i(t)))y(t) \ge a_2(1 - p(\delta_i(t))). \tag{3.2}$$

From (1.7) and (3.2), it follows that for $t \in [t_1, \infty)_{\mathbb{T}}$,

$$\begin{aligned}
[r(t)(y^{\Delta^{n-1}}(t))^{\alpha}]^{\Delta} &= -q_0(t)x^{\alpha}(\delta_0(t)) - \sum_{i=1}^{m} q_i(t)x^{\beta_i}(\delta_i(t)) \\
&\le -a_2^{\alpha}(1 - p(\delta_0(t)))^{\alpha} q_0(t) - \sum_{i=1}^{m} a_2^{\beta_i}(1 - p(\delta_i(t)))^{\beta_i} q_i(t) \\
&\le -a_3\theta_1(t),
\end{aligned}$$

where

$$a_3 := \min\{a_2^{\alpha}, a_2^{\beta_1}, a_2^{\beta_2}, \dots, a_2^{\beta_m}\} > 0,$$

$$\theta_1(t) = q_0(t)(1 - p(\delta_0(t)))^{\alpha} + \sum_{i=1}^{m} q_i(t)(1 - p(\delta_i(t)))^{\beta_i}.$$

Integrating the above inequality from $t \ge t_1$ to $u \ge t$, we obtain

$$r(t)(y^{\Delta^{n-1}}(t))^{\alpha} \ge r(u)(y^{\Delta^{n-1}}(u))^{\alpha} + a_3 \int_t^u \theta_1(u)\Delta u > a_3 \int_t^u \theta_1(u)\Delta u. \tag{3.3}$$

Letting $u \to \infty$, we have

$$\int_t^{\infty} \theta_1(u)\Delta < \infty,$$

which contradicts the assumption (3.1) and so the proof is complete. $\square$

For the next lemma, we define the functions $\{\Theta_i\}_{i=0}^{\infty}$ by

$$\Theta_0(t,u)=\frac{1}{r(t)},\quad \Theta_i(t,u)=\int_u^t \Theta_{i-1}(s,u)\Delta s,\quad t,u\in[t_0,\infty)_{\mathbb{T}},\ i\in\mathbb{N}. \tag{3.4}$$

Lemma 3.2. *Assume that either*

$$\begin{gathered}\int_{t_0}^{\infty}\Big(\frac{1}{r(s)}\int_s^{\infty}\theta_1(u)\Delta u\Big)^{1/\alpha}\Delta s=\infty,\\ \textit{or}\\ \int_{t_0}^{\infty}\Big[\int_v^{\infty}\Big(\frac{1}{r(s)}\int_s^{\infty}\theta_1(u)\Delta u\Big)^{1/\alpha}\Delta s\Big]\Delta v=\infty.\end{gathered} \tag{3.5}$$

If (1.7) *has an eventually positive solution x, then there exists a sufficiently large $t_*\in[t_0,\infty)_{\mathbb{T}}$ such that for $t\in[t_*,\infty)_{\mathbb{T}}$,*

$$y^{\Delta^j}(t)>0,\quad j=0,1,\dots,n-1, \tag{3.6}$$

$$y^{\Delta}(t)>r^{1/\alpha}(t)y^{\Delta^{n-1}}(t)\Theta_{n-2}(t,t_*), \tag{3.7}$$

$$y(t)>r^{1/\alpha}(t)y^{\Delta^{n-1}}(t)\Theta_{n-1}(t,t_*). \tag{3.8}$$

Proof. Since x is an eventually positive solution of (1.7). By Lemma 2.3, there exist $t_*\in[t_0,\infty)_{\mathbb{T}}$ and an odd integer $l\in\{1,3,\dots,n-1\}$ such that (2.2) and (2.3) hold eventually. Therefore, one concludes that for $t\in[t_*,\infty)_{\mathbb{T}}$

$$y^{\Delta}(t)>0. \tag{3.9}$$

So (3.6) holds for $n=2$.

If $n\ge 4$, we claim that (3.5) implies that $l=n-1$, hence (3.6) holds. In fact, if $1\le l\le n-3$, then for $t\ge t_*$

$$[r(t)|y^{\Delta^{n-1}}(t)|^{\alpha-1}y^{\Delta^{n-1}}(t)]^{\Delta}<0,\quad y^{\Delta^{n-1}}(t)>0,\quad y^{\Delta^{n-2}}(t)<0,\quad y^{\Delta^{n-3}}(t)>0.$$

Proceeding as in the proof of Theorem 3.1, we see that (3.3) holds for all $t\in[t_*,\infty)_{\mathbb{T}}$. Taking limits as $u\to\infty$ in (3.3), we have for $t\in[t_*,\infty)_{\mathbb{T}}$,

$$r(t)(y^{\Delta^{n-1}}(t))^{\alpha}\ge a_3\int_t^{\infty}\theta_1(u)\Delta u.$$

It is known from Theorem 3.1 that $\int_t^{\infty}\theta_1(u)\Delta u<\infty$. Thus, one concludes that for $t\in[t_*,\infty)_{\mathbb{T}}$,

$$y^{\Delta^{n-1}}(t)\ge a_3^{1/\alpha}(\frac{1}{r(t)}\int_s^{\infty}\theta_1(u)\Delta u)^{1/\alpha}. \tag{3.10}$$

Assume

$$\int_{t_0}^{\infty}(\frac{1}{r(s)}\int_s^{\infty}\theta_1(u)\Delta u)^{1/\alpha}\Delta s=\infty.$$

By integrating both sides of (3.10) from t_* to $t\in[t_*,\infty)_{\mathbb{T}}$ we obtain

$$y^{\Delta^{n-2}}(t)-y^{\Delta^{n-2}}(t_*)\ge a_3^{1/\alpha}\int_{t_*}^{t}(\frac{1}{r(s)}\int_s^{\infty}\theta_1(u)\Delta u)^{1/\alpha}\Delta s.$$

Letting $t\to\infty$, we have

$$\lim_{t\to\infty}y^{\Delta^{n-2}}(t)=\infty,$$

which contradicts the fact that $y^{\Delta^{n-2}}(t) < 0$ on $[t_*, \infty)_{\mathbb{T}}$.

Assume

$$\int_{t_0}^{\infty} [\int_{v}^{\infty} (\frac{1}{r(s)} \int_{s}^{\infty} \theta_1(u)\Delta u)^{1/\alpha} \Delta s] \Delta v = \infty.$$

By integrating both sides of (3.10) from v to $u \in [t_*, \infty)_{\mathbb{T}}$ and then taking limits as $u \to \infty$ and using the fact $y^{\Delta^{n-2}}(u) < 0$ eventually, we obtain

$$-y^{\Delta^{n-2}}(t) > a_3^{1/\alpha} \int_{v}^{\infty} \Big(\frac{1}{r(s)} \int_{s}^{\infty} \theta_1(u)\Delta u\Big)^{1/\alpha} \Delta s. \tag{3.11}$$

Again, by integrating both sides of (3.11) from t_* to $t \in [t_*, \infty)_{\mathbb{T}}$ and noting $y^{\Delta^{n-3}}(t_*) > 0$ eventually, we have

$$-y^{\Delta^{n-3}}(t) + y^{\Delta^{n-3}}(t_*) \geq a_3^{1/\alpha} \int_{t_*}^{t} \Big[\int_{v}^{\infty} \Big(\frac{1}{r(s)} \int_{s}^{\infty} \theta_1(u)\Delta u\Big)^{1/\alpha} \Delta s\Big] \Delta v.$$

Letting $t \to \infty$, for $t \in [t_*, \infty)_{\mathbb{T}}$, we find that

$$\lim_{t\to\infty} y^{\Delta^{n-3}}(t) = -\infty,$$

which contradicts the fact that $y^{\Delta^{n-3}}(t) > 0$ on $[t_*, \infty)_{\mathbb{T}}$. Thus we have $l = n-1$, and then (3.6) holds. It follows from (1.7) that $r^{1/\alpha}(t)y^{\Delta^{n-1}}(t)$ is strictly decreasing on $[t_*, \infty)_{\mathbb{T}}$. Therefore, one concludes that for $t \in [t_*, \infty)_{\mathbb{T}}$,

$$\begin{aligned} y^{\Delta^{n-2}}(t) &= y^{\Delta^{n-2}}(t_*) + \int_{t_*}^{t} r^{1/\alpha}(s) y^{\Delta^{n-1}}(s) r^{-\frac{1}{\alpha}}(s)\Delta s \\ &> r^{1/\alpha}(t) y^{\Delta^{n-1}}(t) \int_{t_*}^{t} r^{-\frac{1}{\alpha}}(s)\Delta s \\ &:= r^{1/\alpha}(t) y^{\Delta^{n-1}}(t) \Theta_1(t, t_*). \end{aligned}$$

Integrating the above inequality from t_* to t, for $t \in [t_*, \infty)_{\mathbb{T}}$, we have

$$\begin{aligned} y^{\Delta^{n-3}}(t) &\geq y^{\Delta^{n-3}}(t_*) + \int_{t_*}^{t} r^{1/\alpha}(s) y^{\Delta^{n-1}}(s) \Theta_1(s, t_*)\Delta s \\ &> r^{1/\alpha}(t) y^{\Delta^{n-1}}(t) \int_{t_*}^{t} \Theta_1(s, t_*)\Delta s \\ &:= r^{1/\alpha}(t) y^{\Delta^{n-1}}(t) \Theta_2(t, t_*). \end{aligned}$$

Analogously, for $t \in [t_*, \infty)_{\mathbb{T}}$, we obtain

$$\begin{gathered} y^{\Delta}(t) > r^{1/\alpha}(t) y^{\Delta^{n-1}}(t) \Theta_{n-2}(t, t_*), \\ y(t) > r^{1/\alpha}(t) y^{\Delta^{n-1}}(t) \Theta_{n-1}(t, t_*), \end{gathered}$$

which are (3.7) and (3.8), respectively. This completes the proof. □

Remark 3.3. For the even order delay dynamic equation (1.6), Chen and Qu [7] obtained a similar lemma (see [7, Lemma 2.2]).

By Lemma 3.2, we have the following criterion.

Theorem 3.4. *Let (3.5) be satisfied. Assume in addition that there exist a function $\varphi \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ and m-tuple $\{\eta_1,\eta_2,\dots,\eta_m\}$ satisfying Lemma 2.1 such that for all $T_1,T_2 \in [t_0,\infty)_{\mathbb{T}}$ with $T_2 > T_1$,*

$$\limsup_{t\to\infty} \int_{T_2}^{t} \big\{\varphi(s)\theta_2(s) - \varphi_+^{\Delta}(s)\Theta_{n-1}^{-\alpha}(s,T_1)\big\}\Delta s = \infty, \tag{3.12}$$

where $\varphi_+^{\Delta}(s) := \max\{0,\varphi^{\Delta}(s)\}$. Then (1.7) is oscillatory.

Proof. Let x be a nonoscillatory solution of (1.7). Without loss of generality, we may assume that x is eventually positive. In view of (3.5), by Lemma 3.2, there exists a $t_* \ge t_0$ such that (3.2) and (3.6), (3.8) hold.

Consider the Riccati substitution

$$Z(t) = \varphi(t)\frac{r(t)(y^{\Delta^{n-1}}(t))^{\alpha}}{y^{\alpha}(t)}, \quad t \in [t_*,\infty)_{\mathbb{T}}, \tag{3.13}$$

then $Z(t) > 0$ for $t \in [t_*,\infty)_{\mathbb{T}}$. By the product and quotient rules (see [5, Theorem 1.20]), in view of (1.7), (A1)–(A3), (3.2), (3.6) and (3.13), one concludes that for $t \in [t_*,\infty)_{\mathbb{T}}$,

$$\begin{aligned}
&Z^{\Delta}(t)\\
&= [r(t)(y^{\Delta^{n-1}}(t))^{\alpha}]^{\Delta}\frac{\varphi(t)}{y^{\alpha}(t)} + (r(t)(y^{\Delta^{n-1}}(t))^{\alpha})^{\sigma}(\frac{\varphi(t)}{y^{\alpha}(t)})^{\Delta}\\
&= -\varphi(t)\frac{q_0(t)x^{\alpha}(\delta_0(t)) + \sum_{i=1}^m q_i(t)x^{\beta_i}(\delta_i(t))}{y^{\alpha}(t)} + (r(t)(y^{\Delta^{n-1}}(t))^{\alpha})^{\sigma}(\frac{\varphi(t)}{y^{\alpha}(t)})^{\Delta}\\
&\le -\varphi(t)\frac{q_0(t)(1-p(\delta_0(t)))^{\alpha}y^{\alpha}(\delta_0(t))}{y^{\alpha}(t)} - \varphi(t)\frac{\sum_{i=1}^m q_i(t)(1-p(\delta_i(t)))^{\beta_i}y^{\beta_i}(\delta_i(t))}{y^{\alpha}(t)}\\
&\quad + (r(t)(y^{\Delta^{n-1}}(t))^{\alpha})^{\sigma}(\frac{\varphi^{\Delta}(t)}{(y^{\sigma}(t))^{\alpha}} - \frac{\varphi(t)(y^{\alpha}(t))^{\Delta}}{y^{\alpha}(t)(y^{\sigma}(t))^{\alpha}})\\
&\le -\varphi(t)\big[q_0(t)(1-p(\delta_0(t)))^{\alpha} + \sum_{i=1}^m q_i(t)(1-p(\delta_i(t)))^{\beta_i}y^{\beta_i-\alpha}(t)\big]\\
&\quad + (r(t)(y^{\Delta^{n-1}}(t))^{\alpha})^{\sigma}(\frac{\varphi_+^{\Delta}(t)}{(y^{\sigma}(t))^{\alpha}} - \frac{\varphi(t)(y^{\alpha}(t))^{\Delta}}{y^{\alpha}(t)(y^{\sigma}(t))^{\alpha}}),
\end{aligned} \tag{3.14}$$

where σ is the forward jump operator on time scale $\mathbb{T}$.

In view of the arithmetic-geometric mean inequality, see [10],

$$\sum_{i=1}^m \eta_i u_i \ge \prod_{i=1}^m u_i^{\eta_i}, \quad u_i \ge 0,$$

where $\eta_1,\eta_2,\dots,\eta_m$ are chosen to satisfy Lemma 2.1. Now returning to (3.14) and substituting

$$u_i = \eta_i^{-1}q_i(t)(1-p(\delta_i(t)))^{\beta_i}y^{\beta_i-\alpha}(t), \quad i = 1,2,\dots,m$$

into (3.14), we obtain

$$\begin{aligned} Z^{\Delta}(t) &\le -\varphi(t)\big[q_0(t)(1-p(\delta_0(t)))^{\alpha} + \prod_{i=1}^{m} \eta_i^{-\eta_i} q_i^{\eta_i}(t)(1-p(\delta_i(t)))^{\beta_i\eta_i}\big] \\ &\quad + (r(t)(y^{\Delta^{n-1}}(t))^{\alpha})^{\sigma}\Big(\frac{\varphi_+^{\Delta}(t)}{(y^{\sigma}(t))^{\alpha}} - \frac{\varphi(t)(y^{\alpha}(t))^{\Delta}}{y^{\alpha}(t)(y^{\sigma}(t))^{\alpha}}\Big) \\ &:= -\varphi(t)\theta_2(t) + (r(t)(y^{\Delta^{n-1}}(t))^{\alpha})^{\sigma}\Big(\frac{\varphi_+^{\Delta}(t)}{(y^{\sigma}(t))^{\alpha}} - \frac{\varphi(t)(y^{\alpha}(t))^{\Delta}}{y^{\alpha}(t)(y^{\sigma}(t))^{\alpha}}\Big). \end{aligned} \tag{3.15}$$

Employing the Pätzsche chain rule ([5, Theorem 1.87]) and the fact that y is strictly increasing on $t \in [t_*, \infty)_{\mathbb{T}}$, we have for $t \in [t_*, \infty)_{\mathbb{T}}$

$$\begin{aligned} (y^{\alpha}(t))^{\Delta} &= \alpha\Big\{\int_0^1 [y(t) + h\mu(t)y^{\Delta}(t)]^{\alpha-1} dh\Big\} y^{\Delta}(t) \\ &= \alpha\Big\{\int_0^1 [(1-h)y(t) + hy^{\sigma}(t)]^{\alpha-1} dh\Big\} y^{\Delta}(t) \\ &\ge \begin{cases} \alpha(y^{\sigma}(t))^{\alpha-1} y^{\Delta}(t), & 0 < \alpha \le 1, \\ \alpha(y(t))^{\alpha-1} y^{\Delta}(t), & \alpha \ge 1. \end{cases} \end{aligned}$$

Noting that y is increasing on $[t_*, \infty)_{\mathbb{T}}$, we obtain $y(t) \le y^{\sigma}(t)$ for $t \in [t_*, \infty)_{\mathbb{T}}$, and then

$$\frac{(y^{\alpha}(t))^{\Delta}}{y^{\alpha}(t)} \ge \alpha \frac{y^{\Delta}(t)}{y^{\sigma}(t)} > 0. \tag{3.16}$$

Since $\sigma(t) \ge t$ on $\mathbb{T}$, from (3.6) and the fact that $r(t)(y^{\Delta^{n-1}}(t))^{\alpha}$ is decreasing on $[t_*, \infty)_{\mathbb{T}}$, one concludes that for $t \in [t_*, \infty)_{\mathbb{T}}$

$$0 \le (r(t)(y^{\Delta^{n-1}}(t))^{\alpha})^{\sigma} \le r(t)(y^{\Delta^{n-1}}(t))^{\alpha}, \quad y^{\sigma}(t) \ge y(t). \tag{3.17}$$

From (3.15), (3.16) and (3.17), it follows that

$$\begin{aligned} Z^{\Delta}(t) &\le -\varphi(t)\theta_2(t) + (r(t)(y^{\Delta^{n-1}}(t))^{\alpha})^{\sigma}\Big(\frac{\varphi_+^{\Delta}(t)}{(y^{\sigma}(t))^{\alpha}} - \frac{\varphi(t)(y^{\alpha}(t))^{\Delta}}{y^{\alpha}(t)(y^{\sigma}(t))^{\alpha}}\Big) \\ &\le -\varphi(t)\theta_2(t) + (r(t)(y^{\Delta^{n-1}}(t))^{\alpha})^{\sigma} \frac{\varphi_+^{\Delta}(t)}{(y^{\sigma}(t))^{\alpha}} \\ &\le -\varphi(t)\theta_2(t) + (r(t)(y^{\Delta^{n-1}}(t))^{\alpha} \frac{\varphi_+^{\Delta}(t)}{y^{\alpha}(t)}. \end{aligned} \tag{3.18}$$

In view of (3.8), we obtain

$$\begin{aligned} Z^{\Delta}(t) &\le -\varphi(t)\theta_2(t) + (r(t)(y^{\Delta^{n-1}}(t)))^{\alpha} \frac{\varphi_+^{\Delta}(t)}{y^{\alpha}(t)} \\ &\le -\varphi(t)\theta_2(t) + y^{\alpha}(t)\Theta_{n-1}^{-\alpha}(t, t_*) \frac{\varphi_+^{\Delta}(t)}{y^{\alpha}(t)} \\ &= -\varphi(t)\theta_2(t) + \varphi_+^{\Delta}(t)\Theta_{n-1}^{-\alpha}(t, t_*). \end{aligned}$$

Integrating both sides of (3.18) from $T > t_*$ to $t \ge T$ leads to

$$0 < Z(t) \le Z(T) - \int_T^t \big\{\varphi(s)\theta_2(s) - \varphi_+^{\Delta}(s)\Theta_{n-1}^{-\alpha}(s, t_*)\big\}\Delta s.$$

Taking the limit superior on both sides, the result contradicts (3.12). This completes the proof. □

According to Theorem 3.4, by further applying Young's inequality and noting that (3.7) we have the following theorem.

Theorem 3.5. *Let* (3.5) *be satisfied. Assume in addition that there exist a function* $\varphi \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$ *and* m*-tuple* $\{\eta_1,\eta_2,\dots,\eta_m\}$ *satisfying Lemma 2.1 such that for all* $T_1, T_2 \in [t_0,\infty)_{\mathbb{T}}$ *with* $T_2 > T_1$,

$$\limsup_{t\to\infty}\int_{T_2}^{t}\big\{\varphi(s)\theta_2(s) - \frac{(\varphi^\Delta_+(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}[\varphi(s)\Theta_{n-2}(s,T_1)]^\alpha}\big\}\Delta s = \infty. \tag{3.19}$$

Then (1.7) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.7). Without loss of generality, we may assume that x is eventually positive. In view of (3.5), by Lemma 3.2, there is a $t_* \ge t_0$ such that (3.6), (3.7) hold. Define the function Z as in (3.13), proceeding as in the proof of Theorem 3.4, we see that (3.14)-(3.17) hold. From (3.13) and (3.15), one concludes that for $t \in [t_*,\infty)_{\mathbb{T}}$

$$Z^\Delta(t) \le -\varphi(t)\theta_2(t) + \varphi^\Delta_+(t)\frac{Z^\sigma(t)}{\varphi^\sigma(t)} - \varphi(t)\frac{Z^\sigma(t)}{\varphi^\sigma(t)}\frac{(y^\alpha(t))^\Delta}{y^\alpha(t)}. \tag{3.20}$$

From $[r(t)|y^{\Delta^{n-1}}(t)|^{\alpha-1}y^{\Delta^{n-1}}(t)]^\Delta < 0$, we have

$$(r(t)(y^{\Delta^{n-1}}(t))^\alpha)^\sigma \le r(t)(y^{\Delta^{n-1}}(t))^\alpha.$$

In view of (3.7) and (3.16), we have

$$\begin{aligned}\frac{(y^\alpha(t))^\Delta}{y^\alpha(t)} &\ge \alpha\frac{y^\Delta(t)}{y^\sigma(t)}\\ &\ge \alpha\frac{r^{1/\alpha}(t)y^{\Delta^{n-1}}(t)\Theta_{n-2}(t,t_*)}{y^\sigma(t)}\\ &\ge \alpha\frac{(r^{1/\alpha}(t))^\sigma(y^{\Delta^{n-1}}(t))^\sigma\Theta_{n-2}(t,t_*)}{y^\sigma(t)}\\ &= \alpha\Theta_{n-2}(t,t_*)\Big(\frac{Z^\sigma(t)}{\varphi^\sigma(t)}\Big)^{1/\alpha}\end{aligned} \tag{3.21}$$

on $[t_*,\infty)_{\mathbb{T}}$. Substituting (3.21) into (3.20), we obtain for $t \in [t_*,\infty)_{\mathbb{T}}$,

$$Z^\Delta(t) \le -\varphi(t)\theta_2(t) + \varphi^\Delta_+(t)\frac{Z^\sigma(t)}{\varphi^\sigma(t)} - \alpha\varphi(t)\Theta_{n-2}(t,t_*)\Big(\frac{Z^\sigma(t)}{\varphi^\sigma(t)}\Big)^{1+\frac{1}{\alpha}}. \tag{3.22}$$

Taking

$$X = [\alpha\varphi(t)\Theta_{n-2}(t,t_*)]^{\frac{\alpha}{\alpha+1}}\frac{Z^\sigma(t)}{\varphi^\sigma(t)}, \quad Y = (\frac{\alpha}{\alpha+1})^\alpha(\varphi^\Delta_+(t))^\alpha[\alpha\varphi(t)\Theta_{n-2}(t,t_*)]^{\frac{-\alpha^2}{\alpha+1}},$$
$$\lambda = \frac{\alpha+1}{\alpha} = 1 + \frac{1}{\alpha} > 1,$$

by Lemma 2.2 and (3.22), for $t \in [t_*,\infty)_{\mathbb{T}}$, we obtain

$$Z^\Delta(t) \le -\varphi(t)\theta_2(t) + \frac{(\varphi^\Delta_+(t))^{\alpha+1}}{(\alpha+1)^{\alpha+1}[\varphi(t)\Theta_{n-2}(t,t_*)]^\alpha}.$$

Integrating both sides of the above inequality from $T > t_*$ to $t \geq T$ leads to

$$0 < Z(t) \leq Z(T) - \int_T^t \big\{\varphi(s)\theta_2(s) - \frac{(\varphi_+^\Delta(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}[\varphi(s)\Theta_{n-2}(s,t_*)]^\alpha}\big\}\Delta s.$$

Taking the limit superior on both sides, the result obtained contradicts (3.19). This completes the proof. $\square$

Similarly, using the equality $bx - ax^2 \leq \frac{b^2}{4a}$ for x, a, $b \in \mathbb{R}$, we have the following theorem.

Theorem 3.6. *Let* (3.5) *be satisfied and* $\alpha \geq 1$. *Assume in addition that there exist a function* $\varphi \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ *and* m*-tuple* $\{\eta_1, \eta_2, \dots, \eta_m\}$ *satisfying Lemma 2.1 such that for all* $T_1, T_2 \in [t_0,\infty)_{\mathbb{T}}$ *with* $T_2 > T_1$,

$$\limsup_{t\to\infty} \int_{T_2}^t \big\{\varphi(s)\theta_2(s) - \frac{(\varphi_+^\Delta(s))^2}{4\alpha\varphi(s)\Theta_{n-2}(s,T_1)\Theta_{n-1}^{\alpha-1}(\sigma(s),T_1)}\big\}\Delta s = \infty. \tag{3.23}$$

Then (1.7) *is oscillatory.*

The following two theorems give new oscillation criteria for (1.7) which can be considered as the extension of Philos-type oscillation criterion. Define $D = \{(t,s) \in \mathbb{T}\times\mathbb{T} : t \geq s > 0\}$ and

$$\Omega = \big\{H \in C^1(D,\mathbb{R}^+) : H(t,t) = 0,\ H(t,s) > 0,\ H_s^\Delta(t,s) \leq 0, \quad for \quad t > s \geq 0\big\}.$$

Theorem 3.7. *Let* (3.5) *be satisfied. Assume in addition that there exist function* $\varphi \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$, $h \in C_{rd}(D,\mathbb{R})$, $H \in \Omega$ *and* m*-tuple* $\{\eta_1, \eta_2, \dots, \eta_m\}$ *satisfying the assumptions in Lemma 2.1, such that for all* $T_1, T_2 \in [t_0,\infty)_{\mathbb{T}}$ *with* $T_2 > T_1$,

$$H_s^\Delta(t,s) + H(t,s)\frac{\varphi_+^\Delta(s)}{\varphi^\sigma(s)} = \frac{h(t,s)}{\varphi^\sigma(s)} H^{\frac{\alpha}{\alpha+1}}(t,s), \quad for\ (t,s) \in D, \tag{3.24}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T_2)} \int_{T_2}^t \big\{H(t,s)\varphi(s)\theta_2(s) - \frac{h_+^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}\big[\varphi(s)\Theta_{n-2}(s,T_1)\big]^\alpha}\big\}\Delta s = \infty, \tag{3.25}$$

where $h_+(t,s) := \max\{0, h(t,s)\}$. *Then* (1.7) *is oscillatory.*

Proof. Let x be a nonoscillatory solution of (1.7). Without loss of generality, we may assume that x is eventually positive. Proceeding as in the proof of Theorem 3.5, we see that (3.22) holds. Multiplying (3.22) by $H(t,s)$ and integrating it from $T > t_*$ to $t \geq T$, one concludes that for $t \in [T,\infty)_{\mathbb{T}}$,

$$\begin{aligned}\int_T^t H(t,s)Z^\Delta(s)\Delta s &\leq -\int_T^t H(t,s)\varphi(s)\theta_2(s)\Delta s + \int_T^t H(t,s)\varphi_+^\Delta(s)\frac{Z^\sigma(s)}{\varphi^\sigma(s)}\Delta s \\ &\quad - \int_T^t H(t,s)\alpha\varphi(s)\Theta_{n-2}(s,t_*)(\frac{Z^\sigma(s)}{\varphi^\sigma(s)})^{1+\frac{1}{\alpha}}\Delta s \\ &:= -\int_T^t H(t,s)\varphi(s)\theta_2(s)\Delta s + \int_T^t H(t,s)\frac{\varphi_+^\Delta(s)}{\varphi^\sigma(s)}Z^\sigma(s)\Delta s \\ &\quad - \int_T^t H(t,s)U_1(s,t_*)\Big(\frac{Z^\sigma(s)}{\varphi^\sigma(s)}\Big)^{1+\frac{1}{\alpha}}\Delta s,\end{aligned} \tag{3.26}$$

where $U_1(s,t_*) := \alpha\varphi(s)\Theta_{n-2}(s,t_*)$. Integrating by parts, we obtain

$$\int_T^t H(t,s)Z^\Delta(s)\Delta s = -H(t,T)Z(T) - \int_T^t H_s^\Delta(t,s)Z^\sigma(s)\Delta s. \tag{3.27}$$

From (3.26) and (3.27), one concludes that for $t \in [T,\infty)_\mathbb{T}$,

$$\begin{aligned}
&\int_T^t H(t,s)\varphi(s)\theta_2(s)\Delta s \\
&\le -\int_T^t H(t,s)Z^\Delta(s)\Delta s + \int_T^t H(t,s)\frac{\varphi_+^\Delta(s)}{\varphi^\sigma(s)}Z^\sigma(s)\Delta s \\
&\quad - \int_T^t H(t,s)U_1(s,t_*)(\frac{Z^\sigma(s)}{\varphi^\sigma(s)})^{1+\frac{1}{\alpha}}\Delta s \\
&= H(t,T)Z(T) + \int_T^t \big\{\big[H_s^\Delta(t,s) + H(t,s)\frac{\varphi_+^\Delta(s)}{\varphi^\sigma(s)}\big]Z^\sigma(s) \\
&\quad - H(t,s)U_1(s,t_*)(\frac{Z^\sigma(s)}{\varphi^\sigma(s)})^{1+\frac{1}{\alpha}}\big\}\Delta s.
\end{aligned}$$

From $H \in \Omega$ and (3.24), we find for $t \in [T,\infty)_\mathbb{T}$,

$$\begin{aligned}
&\int_T^t H(t,s)\varphi(s)\theta_2(s)\Delta s \\
&\le H(t,T)Z(T) + \int_T^t \Big[\frac{h_+(t,s)}{\varphi^\sigma(s)}H^{\frac{\alpha}{\alpha+1}}(t,s)Z^\sigma(s) \\
&\quad - H(t,s)U_1(s,t_*)\Big(\frac{Z^\sigma(s)}{\varphi^\sigma(s)}\Big)^{1+\frac{1}{\alpha}}\Big]\Delta s,
\end{aligned} \tag{3.28}$$

where h_+ is defined as in Theorem 3.7. Taking $\lambda = \frac{\alpha+1}{\alpha} = 1 + \frac{1}{\alpha} > 1$,

$$X = \big[H(t,s)U_1(s,t_*)\big]^{\frac{\alpha}{\alpha+1}}\frac{Z^\sigma(s)}{\varphi^\sigma(s)}, \quad Y = (\frac{\alpha}{\alpha+1})^\alpha (h_+(t,s))^\alpha U_1^{\frac{-\alpha^2}{\alpha+1}}(s,t_*),$$

then by Lemma 2.2, (3.28), and $U_1(s,t_*) := \alpha\varphi(s)\Theta_{n-2}(s,t_*)$, we obtain

$$\int_T^t H(t,s)\varphi(s)\theta_2(s)\Delta s \le H(t,T)Z(T) + \int_T^t \frac{h_+^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}[\varphi(s)\Theta_{n-2}(s,t_*)]^\alpha}\Delta s.$$

Thus, we obtain

$$\begin{aligned}
&\limsup_{t\to\infty}\frac{1}{H(t,T)}\int_T^t \big\{H(t,s)\varphi(s)\theta_2(s) - \frac{h_+^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}[\varphi(s)\Theta_{n-2}(s,t_*)]^\alpha}\big\}\Delta s \\
&\le Z(T) < \infty,
\end{aligned}$$

which is a contradiction to (3.25). This completes the proof. □

In view of Theorems 3.6 and 3.7, applying the equality $bx - ax^2 \le \frac{b^2}{4a}$ for , x, a, $b \in \mathbb{R}$, we have the following theorem.

Theorem 3.8. *Let* (3.5) *be satisfied and* $\alpha \ge 1$. *Assume in addition that there exist function* $\varphi \in C_{rd}^1([t_0,\infty)_\mathbb{T},\mathbb{R}^+)$, $h \in C_{rd}(D,\mathbb{R})$, $H \in \Omega$ *and* m*-tuple* $\{\eta_1,\eta_2,\dots,\eta_m\}$ *satisfying Lemma 2.1 such that for all* $T_1, T_2 \in [t_0,\infty)_\mathbb{T}$ *with* $T_2 > T_1$,

$$H_s^\Delta(t,s) + H(t,s)\frac{\varphi_+^\Delta(s)}{\varphi^\sigma(s)} = \frac{h(t,s)}{\varphi^\sigma(s)}\sqrt{H(t,s)}, \quad \textit{for } (t,s) \in D, \tag{3.29}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T_2)} \int_{T_2}^{t} \big\{ H(t,s)\varphi(s)\theta_2(s) - \frac{h_+^2(t,s)}{4\alpha\varphi(s)\Theta_{n-2}(s,T_1)\Theta_{n-1}^{\alpha-1}(\sigma(s),T_1)} \big\} \Delta s = \infty. \tag{3.30}$$

Then (1.7) *is oscillatory.*

Remark 3.9. Let $p(t) = 0$, $\beta_i = \alpha_i$ and $\delta_i(t) = \delta(t)$, $i = 0, 1, \dots, k$, Theorems 3.4-3.8 reduce to [7, Theorems 3.1-3.5].

4. Oscillation for odd order equations

We establish oscillation criteria for (1.7) when n is odd. In this section, we assume that there exists a p such that $0 \le p(t) \le p < 1$ and use the following notation for simplicity:

$$\theta_1^*(t) = q_0(t) + \sum_{i=1}^{m} q_i(t), \quad \theta_2^*(t) = q_0(t) + \prod_{i=1}^{m} \eta_i^{-\eta_i} q_i^{\eta_i}(t).$$

Theorem 4.1. *Assume* (3.1) *with* θ_1^* *instead of* θ_1 *holds. Then every solution of* (1.7) *is either oscillatory or tends to zero eventually.*

Proof. Let x be a nonoscillatory solution of (1.7). Without loss of generality, we may assume that x is eventually positive. By Lemma 2.3, there exist $t_* \in [t_0, \infty)_{\mathbb{T}}$ and an even $l \in \{0, 2, \dots, n-1\}$ such that (2.2) and (2.3) hold for $t \in [t_*, \infty)_{\mathbb{T}}$.

(a) If $l \ge 2$. Then we use the same argument as in the proof of Theorem 3.1.

(b) We show that if $l = 0$, then $\lim_{t\to\infty} x(t) = 0$. Since $0 < x(t) \le y(t)$ for $t \ge t_*$, it suffices to show that $\lim_{t\to\infty} y(t) = 0$. From Lemma 2.3 with $l = 0$, one concludes that for $t \in [t_*, \infty)_{\mathbb{T}}$

$$(-1)^j y^{\Delta^j}(t) > 0, \quad \text{for } j = 0, 1, \dots, n-1.$$

Since $y^{\Delta}(t) < 0$ on $[t_*, \infty)_{\mathbb{T}}$, then $\lim_{t\to\infty} y(t) = l_1 \ge 0$. Assume $l_1 > 0$, then there exists a $t_x \in [t_*, \infty)_{\mathbb{T}}$ and choose $0 < \varepsilon < \frac{l_1(1-p)}{p}$ such that

$$l_1 < y(t) < l_1 + \varepsilon, \quad \text{for } t \ge t_x,$$

and for $t \ge t_x \ge t_*$ we obtain

$$\begin{aligned} x(t) &= y(t) - p(t)x(\tau(t)) \ge y(t) - p(t)y(\tau(t)) \\ &\ge y(t) - py(\tau(t)) > l_1 - p(l_1 + \varepsilon) > Ky(t), \end{aligned}$$

where $K := \frac{l_1 - p(l_1+\varepsilon)}{l_1+\varepsilon} > 0$. Thus, we have

$$x(t) > Ky(t), \quad \text{for } t \ge t_x.$$

For $i = 0, 1, \dots, m$, for $t \in [t_x, \infty)_{\mathbb{T}}$, we have

$$x(\delta_i(t)) > Ky(\delta_i(t)) > Kl_1. \tag{4.1}$$

From (1.7) and (4.1), one concludes that for $t \in [t_x, \infty)_{\mathbb{T}}$,

$$[r(t)(y^{\Delta^{n-1}}(t))^{\alpha}]^{\Delta} < -q_0(t)(Kl_1)^{\alpha} - \sum_{i=1}^{m} q_i(t)(Kl_1)^{\beta_i} \le -a_4\theta_1^*(t),$$

where

$$\theta_1^*(t) = q_0(t) + \sum_{i=1}^{m} q_i(t), \quad a_4 = \min\{(Kl_1)^\alpha, (Kl_1)^{\beta_1}, \dots, (Kl_1)^{\beta_m}\} > 0.$$

Integrating the above inequality from $t \geq t_x$ to $u \geq t$, we obtain

$$r(t)(y^{\Delta^{n-1}}(t))^\alpha \geq r(u)(y^{\Delta^{n-1}}(u))^\alpha + a_4 \int_t^u \theta_1^*(u)\Delta u > a_4 \int_t^u \theta_1^*(u)\Delta u. \tag{4.2}$$

By taking limits as $u \to \infty$ in the above inequality, which contradicts the assumption (3.1) with θ_1^* instead of θ_1. This completes the proof. □

Furthermore, we assume that either

$$\begin{gathered}\int_{t_0}^{\infty} \Big(\frac{1}{r(s)} \int_s^{\infty} \theta_1^*(u)\Delta u\Big)^{1/\alpha} \Delta s = \infty \\ \text{or} \\ \int_{t_0}^{\infty} \Big[\int_v^{\infty} \Big(\frac{1}{r(s)} \int_s^{\infty} \theta_1^*(u)\Delta u\Big)^{1/\alpha} \Delta s\Big] \Delta v = \infty.\end{gathered} \tag{4.3}$$

Theorem 4.2. *Let (4.3) be satisfied. Every solution of (1.7) is either oscillatory or tends to zero eventually provided that one of the following conditions is satisfied:*

(1) There exist a function $\varphi \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ and m-tuple $\{\eta_1, \eta_2, \dots, \eta_m\}$ satisfying Lemma 2.1 such that for all $T_1, T_2 \in [t_0,\infty)_{\mathbb{T}}$ with $T_2 > T_1$,

$$\limsup_{t\to\infty} \int_{T_2}^t \big\{\varphi(s)\theta_2^*(s) - \varphi_+^\Delta(s)\Theta_{n-1}^{-\alpha}(s,T_1)\big\}\Delta s = \infty. \tag{4.4}$$

(2) There exist a function $\varphi \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ and m-tuple $\{\eta_1, \eta_2, \dots, \eta_m\}$ satisfying Lemma 2.1 such that for all $T_1, T_2 \in [t_0,\infty)_{\mathbb{T}}$ with $T_2 > T_1$,

$$\limsup_{t\to\infty} \int_{T_2}^t \big\{\varphi(s)\theta_2^*(s) - \frac{(\varphi_+^\Delta(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}[\varphi(s)\Theta_{n-2}(s,T_1)]^\alpha}\big\}\Delta s = \infty. \tag{4.5}$$

(3) There exist a function $\varphi \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$ and m-tuple $\{\eta_1, \eta_2, \dots, \eta_m\}$ satisfying Lemma 2.1 such that for all $T_1, T_2 \in [t_0,\infty)_{\mathbb{T}}$ with $T_2 > T_1$,

$$\limsup_{t\to\infty} \int_{T_2}^t \big\{\varphi(s)\theta^*(s) - \frac{(\varphi_+^\Delta(s))^2}{4\alpha\varphi(s)\Theta_{n-2}(s,T_1)\Theta_{n-1}^{\alpha-1}(\sigma(s),T_1)}\big\}\Delta s = \infty, \tag{4.6}$$

for some $\alpha \geq 1$.

(4) There exist functions $\varphi \in C_{rd}^1([t_0,\infty)_{\mathbb{T}}, \mathbb{R}^+)$, $h \in C_{rd}(D,\mathbb{R})$, $H \in \Omega$ and m-tuple $\{\eta_1, \eta_2, \dots, \eta_m\}$ satisfying Lemma 2.1 such that for all $T_1, T_2 \in [t_0,\infty)_{\mathbb{T}}$ with $T_2 > T_1$,

$$H_s^\Delta(t,s) + H(t,s)\frac{\varphi_+^\Delta(s)}{\varphi^\sigma(s)} = \frac{h(t,s)}{\varphi^\sigma(s)} H^{\frac{\alpha}{\alpha+1}}(t,s), \quad \text{for } (t,s) \in D, \tag{4.7}$$

and

$$\limsup_{t\to\infty} \frac{1}{H(t,T_2)} \int_{T_2}^t \big\{H(t,s)\varphi(s)\theta_2^*(s) - \frac{h_+^{\alpha+1}(t,s)}{(\alpha+1)^{\alpha+1}\big[\varphi(s)\Theta_{n-2}(s,T_1)\big]^\alpha}\big\}\Delta s = \infty. \tag{4.8}$$

(5) There exist functions $\varphi \in C^1_{rd}([t_0,\infty)_{\mathbb{T}},\mathbb{R}^+)$, $h \in C_{rd}(D,\mathbb{R})$, $H \in \Omega$ and m-tuple $\{\eta_1,\eta_2,\dots,\eta_m\}$ satisfying Lemma 2.1 such that for all $T_1,T_2 \in [t_0,\infty)_{\mathbb{T}}$ with $T_2 > T_1$,

$$H^{\Delta}_s(t,s) + H(t,s)\frac{\varphi^{\Delta}_+(s)}{\varphi^{\sigma}(s)} = \frac{h(t,s)}{\varphi^{\sigma}(s)}\sqrt{H(t,s)}, \quad \text{for } (t,s) \in D, \tag{4.9}$$

and

$$\begin{aligned}&\limsup_{t\to\infty}\frac{1}{H(t,T_2)}\int_{T_2}^{t}\big\{H(t,s)\varphi(s)\theta_2^*(s)\\&-\frac{h_+^2(t,s)}{4\alpha\varphi(s)\Theta_{n-2}(s,T_1)\Theta_{n-1}^{\alpha-1}(\sigma(s),T_1)}\big\}\Delta s = \infty,\end{aligned} \tag{4.10}$$

for some $\alpha \ge 1$.

Proof. We only prove the case (1) here. For other cases the proofs are similar. Let x be a nonoscillatory solution of (1.7). Without loss of generality, we may assume that x is eventually positive. By Lemma 2.3, there exist $t_* \in [t_1,\infty)_{\mathbb{T}}$ and an even $l \in \{0,2,\dots,n-1\}$ such that (2.2) and (2.3) hold for $t \in [t_*,\infty)_{\mathbb{T}}$.

(1) Assume $l \ge 2$. The arguments are similar to the proofs of Theorem 3.4.

(2) We show that if $l = 0$, then $\lim_{t\to\infty} x(t) = 0$. Since $0 < x(t) \le y(t)$ for $t \ge t_*$, it suffices to show that $\lim_{t\to\infty} y(t) = 0$. Proceeding as in the proof of Theorem 4.1, we see that (4.2) hold for all $t \in [t_x,\infty)_{\mathbb{T}}$. Taking limits as $u \to \infty$ in (4.2), we have for $t \in [t_x,\infty)_{\mathbb{T}}$

$$r(t)(y^{\Delta^{n-1}}(t))^{\alpha} \ge a_4\int_t^{\infty}\theta_1^*(u)\Delta u.$$

It is known from Theorem 4.1 that $\int_t^{\infty}\theta_1^*(u)\Delta u < \infty$. Therefore, one concludes that for $t \in [t_x,\infty)_{\mathbb{T}}$

$$y^{\Delta^{n-1}}(t) \ge a_4^{1/\alpha}\Big(\frac{1}{r(t)}\int_s^{\infty}\theta_1^*(u)\Delta u\Big)^{1/\alpha}\Delta s. \tag{4.11}$$

Assume

$$\int_{t_0}^{\infty}(\frac{1}{r(s)}\int_s^{\infty}\theta_1^*(u)\Delta u)^{1/\alpha}\Delta s = \infty.$$

By integrating both sides of (4.11) from t_x to $t \in [t_x,\infty)_{\mathbb{T}}$ we obtain

$$y^{\Delta^{n-2}}(t) - y^{\Delta^{n-2}}(t_x) \ge a_4^{1/\alpha}\int_{t_x}^{t}\Big(\frac{1}{r(s)}\int_s^{\infty}\theta_1^*(u)\Delta u\Big)^{1/\alpha}\Delta s.$$

Letting $t \to \infty$, as a result for $t \in [t_x,\infty)_{\mathbb{T}}$,

$$\lim_{t\to\infty} y^{\Delta^{n-2}}(t) = \infty,$$

which contradicts the fact that $y^{\Delta^{n-2}}(t) < 0$ on $[t_x,\infty)_{\mathbb{T}}$.

Assume

$$\int_{t_0}^{\infty}\Big[\int_v^{\infty}\Big(\frac{1}{r(s)}\int_s^{\infty}\theta_1^*(u)\Delta u\Big)^{1/\alpha}\Delta s\Big]\Delta v = \infty.$$

By integrating both sides of (4.11) from v to $u \in [t_x,\infty)_{\mathbb{T}}$ and then taking limits as $u \to \infty$ and using the fact $y^{\Delta^{n-2}}(u) < 0$ eventually, we obtain

$$-y^{\Delta^{n-2}}(t) > a_4^{1/\alpha}\int_v^{\infty}\Big(\frac{1}{r(s)}\int_s^{\infty}\theta_1^*(u)\Delta u\Big)^{1/\alpha}\Delta s. \tag{4.12}$$

Again, by integrating both sides of (4.12) from t_x to $t \in [t_x, \infty)_{\mathbb{T}}$ and noting $y^{\Delta^{n-3}}(t_x) > 0$ eventually, we obtain

$$-y^{\Delta^{n-3}}(t) + y^{\Delta^{n-3}}(t_x) \geq a_3^{1/\alpha} \int_{t_x}^{t} \Big[\int_{v}^{\infty} \Big(\frac{1}{r(s)} \int_{s}^{\infty} \theta_1^*(u)\Delta u \Big)^{1/\alpha} \Delta s \Big] \Delta v.$$

Letting $t \to \infty$, for $t \in [t_x, \infty)_{\mathbb{T}}$, we find that $\lim_{t\to\infty} y^{\Delta^{n-3}}(t) = -\infty$, which contradicts the fact that $y^{\Delta^{n-3}}(t) > 0$ on $[t_x, \infty)_{\mathbb{T}}$. This shows that $l = 0$, then $\lim_{t\to\infty} y(t) = 0$. This completes the proof. □

Remark 4.3. Letting $q_0(t) = 0$, $n = 3$ and $m = 2$, Theorem 4.2 with condition (1) reduces to [15, Theorem 3.1].

5. Examples

In this section, we provide two examples to illustrate our main results.

Example 5.1. Consider the equation

$$\begin{gathered} \big[r(t)|y^{\Delta^{n-1}}(t)|^{\frac{1}{4}-1} y^{\Delta^{n-1}}(t)\big]^{\Delta} + q_0(t)|x(\delta_0(t))|^{\frac{1}{4}-1} x(\delta_0(t)) \\ + \sum_{i=1}^{3} q_i(t)|x(\delta_i(t)))|^{\beta_i - 1} x(\delta_i(t)) = 0 \end{gathered} \tag{5.1}$$

for $t \in [t_0, \infty)_{\mathbb{T}}$, where $q > 1$ is a constant, $\mathbb{T} = \overline{q^{\mathbb{Z}}} = q^{\mathbb{Z}} \bigcup \{0\} = \{q^d, d \in \mathbb{N}\} \cup \{0\}$, $t_0 = q$, $n \geq 6$ is even, $y(t) = x(t) + p(t)x(\tau(t))$, and $m = 3$, $r(t) = t^{-1}$, $p(t) = 2/3$, $\tau(t) \leq t$, $\alpha = 1/4$, $\beta_1 = 17/8$, $\beta_2 = 13/8$, $\beta_3 = 1/8$, $q_0(t) = q_0(t\sigma(t))^{-1}$, $q_1(t) = q_1 t^{-3}$, $q_2(t) = q_2 t^{-4}$, $q_3(t) = q_3 t^{-5}$, $q_i > 0$ $(i = 0, 1, 2, 3)$.

We choose $\delta_i(t) = \delta_i \times (qt)$, $\delta_i \geq 1$ $(i = 0, 1, 2, 3)$ and $\eta_1 = 1/64$, $\eta_2 = 1/16$, $\eta_3 = 59/64$. Noting

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)\Delta s = \int_{q}^{\infty} s^4 \Delta s = \infty$$

and taking $k = 2$, we find that (A1)–(A3) are satisfied. By direct computation, we have

$$\begin{gathered} \theta_1(t) = \big(\frac{1}{3}\big)^{1/4} q_0(t\sigma(t))^{-1} + \big(\frac{1}{3}\big)^{17/8} q_1 t^{-3} + \big(\frac{1}{3}\big)^{13/8} q_2 t^{-4} + \big(\frac{1}{3}\big)^{1/8} q_3 t^{-5}, \\ \theta_2(t) = \big(\frac{1}{3}\big)^{1/4} q_0(t\sigma(t))^{-1} + \big(\frac{1}{3}\big)^{1/4} (\frac{1}{64})^{-1/64} (\frac{1}{16})^{-1/16} \\ \times (\frac{59}{64})^{-59/64} q_1^{1/64} q_2^{1/16} q_3^{59/64} t^{-157/32}. \end{gathered}$$

Since $\int_s^{\infty} u^{-\alpha}\Delta s < \infty$, if $\alpha > 1$ for $s \geq q$, we obtain

$$\begin{aligned} \int_s^{\infty} \theta_1(u)\Delta u = \int_s^{\infty} \Big\{ \big(\frac{1}{3}\big)^{1/4} q_0(u\sigma(u))^{-1} + \big(\frac{1}{3}\big)^{17/8} q_1 u^{-3} + \big(\frac{1}{3}\big)^{13/8} q_2 u^{-4} \\ + \big(\frac{1}{3}\big)^{1/8} q_3 u^{-5} \Big\} \Delta u < \infty. \end{aligned}$$

Hence, the assumption (3.1) is not satisfied, we can not obtain the oscillation of (5.1) by Theorem 3.1.

However, it follows that for $s \geq q$,

$$\int_{t_0}^{\infty} \Big(\frac{1}{r(s)} \int_s^{\infty} \theta_1(u)\Delta u \Big)^{1/\alpha} \Delta s$$

$$
\begin{aligned}
&= \int_q^\infty \Big(s \int_s^\infty \Big\{ \big(\frac{1}{3}\big)^{1/4} q_0 (u\sigma(u))^{-1} + \big(\frac{1}{3}\big)^{17/8} q_1 u^{-3} + \big(\frac{1}{3}\big)^{13/8} q_2 u^{-4} \\
&\quad + \big(\frac{1}{3}\big)^{1/8} q_3 u^{-5} \Big\} \Delta u \Big)^4 \Delta s \\
&\geq \int_q^\infty (s \int_s^\infty \big(\frac{1}{3}\big)^{1/4} q_0 (u\sigma(u))^{-1} \Delta u)^4 \Delta s \\
&= \int_q^\infty ((\frac{1}{3})^{1/4} q_0 s \cdot \frac{1}{s})^4 \Delta s = \infty.
\end{aligned}
$$

Define recursively the Taylor monomials $\{g_i\}_{i=0}^\infty$ (see [5, Sect. 1.6]) as follows

$$
g_0(t,s) = 1, \quad g_i(t,s) = \int_s^t g_{i-1}(u,s)\Delta u, \quad \text{for } t, s \in \mathbb{T},\ i \in \mathbb{N}, \tag{5.2}
$$

then

$$
g_i(t,s) = \prod_{v=0}^{i-1} \frac{t - q^v s}{\sum_{\mu=0}^{v} q^\mu}, \quad \text{for } t, s \in \mathbb{T} = \overline{q^{\mathbb{Z}}},\ i \in \mathbb{N}. \tag{5.3}
$$

From (3.4), for $t \geq s \geq t_0 = q > 1$ and $i \in \mathbb{N}$, we obtain

$$
\begin{gathered}
\Theta_0(t,s) = r^{-1}(t) = t \geq 1 = g_0(t,s), \\
\Theta_1(t,s) = \int_s^t \Theta_0(u,s)\Delta u \geq \int_s^t g_0(u,s)\Delta u = g_1(t,s), \\
\dots \\
\Theta_i(t,s) = \int_s^t \Theta_{i-1}(u,s)\Delta u \geq \int_s^t g_{i-1}(u,s)\Delta u = g_i(t,s).
\end{gathered} \tag{5.4}
$$

In view of (5.3) and (5.4), we obtain

$$
\Theta_{n-2}(s,T_1) \geq g_{n-2}(s,T_1) = \prod_{v=0}^{n-3} \frac{t - q^v s}{\sum_{\mu=0}^{v} q^\mu}, \quad \text{for } s \geq T_1 \geq q. \tag{5.5}
$$

Take $\varphi(s) = s$ and define φ_+^Δ as in Theorem 3.5, then we conclude that for $s \geq T_1 \geq q$,

$$
\lim_{s\to\infty} \Big\{ \frac{(\varphi_+^\Delta(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}[\varphi(s) g_{n-2}(s,T_1)]^\alpha} \times \frac{1}{s^{\frac{-(n-1)}{4}}} \Big\} = \frac{(\prod_{v=0}^{n-3} \sum_{\mu=0}^{v} q^\mu)^\alpha}{(\alpha+1)^{\alpha+1}} > 0. \tag{5.6}
$$

Since $n \geq 6$, we ahve $\frac{n-1}{4} > 1$, and $\alpha = 1/4$, we have $\int_{t_0}^\infty s^{-\frac{n-1}{4}} \Delta s < \infty$. From (5.6) we obtain for $T_2 > T_1 \geq q$

$$
\int_{T_2}^\infty \frac{(\varphi_+^\Delta(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}[\varphi(s) g_{n-2}(s,T_1)]^\alpha} \Delta s < \infty.
$$

Therefore, from (5.5) we have that for $T_2 > T_1 \geq q$,

$$
\begin{aligned}
&\int_{T_2}^\infty \frac{(\varphi_+^\Delta(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}[\varphi(s) \Theta_{n-2}(s,T_1)]^\alpha} \Delta s \\
&\leq \int_{T_2}^\infty \frac{(\varphi_+^\Delta(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}[\varphi(s) g_{n-2}(s,T_1)]^\alpha} \Delta s < \infty.
\end{aligned} \tag{5.7}
$$

On the other hand, for $T_2 > T_1 \geq q$, we obtain

$$\begin{aligned}\int_{T_2}^{\infty} \varphi(s)\theta_2(s)\Delta s &= \int_{T_2}^{\infty} s\Big[\big(\frac{1}{3}\big)^{1/4} q_0(s\sigma(s))^{-1} + \big(\frac{1}{3}\big)^{1/4}\big(\frac{1}{64}\big)^{-1/64} \\ &\quad \times \big(\frac{1}{16}\big)^{-\frac{1}{16}}\big(\frac{59}{64}\big)^{-\frac{59}{64}} q_1^{1/64} q_2^{1/16} q_3^{59/64} s^{-157/32}\Big]\Delta s \\ &\geq \int_{T_2}^{\infty} s\big(\frac{1}{3}\big)^{1/4} q_0(s\sigma(s))^{-1}\Delta s = \infty.\end{aligned} \tag{5.8}$$

From (5.7) and (5.8), we conclude that for $T_2 > T_1 \geq q$,

$$\limsup_{t\to\infty} \int_{T_2}^{t} \{\varphi(s)\theta_2(s) - \frac{(\varphi_+^{\Delta}(s))^{\alpha+1}}{(\alpha+1)^{\alpha+1}[\varphi(s)\Theta_{n-2}(s,T_1)]^{\alpha}}\}\Delta s = \infty.$$

Hence, (3.5) and (3.19) are satisfied. By Theorem 3.5, (5.1) is oscillatory.

Example 5.2. Consider the equation

$$\begin{aligned}&\big[r(t)|y^{\Delta^{n-1}}(t)|^{\frac{1}{2}-1} y^{\Delta^{n-1}}(t)\big]^{\Delta} + q_0(t)|x(\delta_0(t))|^{\frac{1}{2}-1}x(\delta_0(t)) \\ &+ \sum_{i=1}^{2} q_i(t)|x(\delta_i(t)))|^{\beta_i-1}x(\delta_i(t)) = 0\end{aligned} \tag{5.9}$$

for $t \in [t_0,\infty)_{\mathbb{T}}$, where $\mathbb{T} = h\mathbb{N} := \{hc : h > 0, c \in \mathbb{N}\}$, $t_0 = h$, $n \geq 3$ is odd, $y(t) = x(t)+p(t)x(\tau(t))$, $m = 2$, $r(t) = (t+\sigma(t))^{-1/2}$, $p(t) \equiv p = 1/2 < 1$, $\tau(t) \leq t$, $\alpha = 1/2$, $\beta_1 = 3/4$, $\beta_2 = 1/4$, $q_0(t) = q_0 t^{-1/2}$, $q_1(t) = q_1 t^{-1/2}$, $q_2(t) = q_2 t^{-1/2}$, $q_i > 0$ $(i = 0,1,2)$.

We choose $\delta_i(t) = \delta_i \times (t + h)$, $\delta_i \geq 1$ $(i = 0,1,2)$. Noting

$$\int_{t_0}^{\infty} r^{-\frac{1}{\alpha}}(s)\Delta s = \int_{h}^{\infty} (s+\sigma(s))\Delta s = \infty$$

and taking $k = 1$, we find that (A1)–(A3) are satisfied. By direct computation, we obtain

$$\theta_1^*(t) = q_0(t) + \sum_{i=1}^{2} q_i(t) = q_0 t^{-1/2} + q_1 t^{-1/2} + q_2 t^{-1/2} := Q_1 t^{-1/2},$$

where $Q_1 = q_0 + q_1 + q_2 > 0$. It follows that

$$\int_{h}^{\infty} \theta_1^*(u)\Delta u = \int_{h}^{\infty} Q_1 u^{-1/2}\Delta u = \infty.$$

Hence, (3.1) with θ_1^* instead of θ_1 is satisfied. By Theorem 4.1, every solution of (5.9) is either oscillatory or tends to zero eventually.

Acknowledgments. This project is partially supported by the NSFC of China (11426066 and 11426068), and by the NSF of Guangdong University of Education (2014jcjs03 and 2015ybzz01). The author would like to thank Professors Yu Huang and Qiru Wang for their helpful discussions.

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