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MULTIPLICITY AND CONCENTRATION BEHAVIOR OF SOLUTIONS FOR A QUASILINEAR PROBLEM INVOLVING N-FUNCTIONS VIA PENALIZATION METHOD

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Abstract. In this work we study the existence, multiplicity and concentration of positive solutions for the quasilinear problem

$$-\Delta_{\Phi}u + V(\epsilon x)\phi(|u|)u = f(u) \quad \text{in } \mathbb{R}^N,$$

where $\Phi(t) = \int_0^{|t|}\phi(s)sds$ is an N-function, Δ_{Φ} is the Φ-Laplacian operator, ϵ is a positive parameter, and $N \geq 2$.

1. Introduction

Many recent studies have focused on the nonlinear Schrödinger equation

$$i\epsilon\frac{\partial\Psi}{\partial t} = -\epsilon^2\Delta\Psi + (V(z)+E)\Psi - f(\Psi) \quad \text{for } z \in \mathbb{R}^N, \tag{1.1}$$

where $N \geq 1$, $\epsilon > 0$ is a parameter and V, f are continuous function verifying some conditions. This class of equation is one of the main subjects of the quantum physics, because it appears in problems which involve nonlinear optics, plasma physics and condensed matter physics.

Knowledge of the solutions for the elliptic equation

$$\begin{gathered} -\epsilon^2\Delta u + V(z)u = f(u) \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{gathered} \tag{1.2}$$

or equivalently

$$\begin{gathered} -\Delta u + V(\epsilon z)u = f(u) \quad \text{in } \mathbb{R}^N, \\ u \in H^1(\mathbb{R}^N), \end{gathered} \tag{1.3}$$

has a great importance in the study of standing-wave solutions of (1.1). In recent years, the existence and concentration of positive solutions for general semilinear elliptic equations (1.2) have been extensively studied, see for example, Floer and Weinstein [18], Oh [29, 30], Rabinowitz [32], Wang [35], Ambrosetti and Malchiodi [9], Ambrosetti, Badiale and Cingolani [8], Floer and Weinstein [19], del Pino and Felmer [14] and their references.

In the above mentioned papers, existence, multiplicity and concentration of positive solutions have been obtained in connection with the geometry of the function

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V. In some of them, it was proved that the maximum points of the solutions are close to the set

$$\mathcal{V} = \{x \in \mathbb{R}^N : V(x) = \min_{z \in \mathbb{R}^N} V(z)\},$$

when ϵ is small enough. Moreover, in a lot of problems, the multiplicity of solutions is related to topology richness of $\mathcal{V}$.

By a mountain pass argument, Rabinowitz [32] proved the existence of positive solutions of (1.2), for $\epsilon > 0$ small, whenever

$$\liminf_{|z| \to \infty} V(z) > \inf_{z \in \mathbb{R}^N} V(z) = V_0 > 0. \tag{1.4}$$

Later Wang [35] showed that these solutions concentrate at global minimum points of V as ϵ tends to 0.

Del Pino and Felmer [14] found solutions which concentrate around local minimum of V by introducing of a penalization method. More precisely, they assume that

$$V(x) \geq \inf_{z \in \mathbb{R}^N} V(z) = V_0 > 0 \quad \text{for all } x \in \mathbb{R}^N \tag{1.5}$$

and there is an open and bounded set $\Omega \subset \mathbb{R}^N$ such that

$$\inf_{z \in \Omega} V(z) < \min_{z \in \partial\Omega} V(z). \tag{1.6}$$

Existence, multiplicity and concentration of positive solutions have been also considered for quasilinear problems of the type

$$-\Delta_p u + V(\epsilon x)|u|^{p-2}u = f(u) \quad \text{in } \mathbb{R}^N,$$

and

$$-\Delta_p u - \Delta_q u + V(\epsilon x)(|u|^{p-2}u + |u|^{q-2}u) = f(u) \quad \text{in } \mathbb{R}^N.$$

Related to this subject, we cite the papers by Alves and Figueiredo [3, 4], Benouhiba and Belyacine [10], Cammaroto and Vilasi [12], Chaves, Ercole and Miyagaki [13], Figueiredo [17], Li and Liang [25] and their references.

Recently, Alves and Silva [7] showed the existence, multiplicity and concentration of positive solutions for the following class of quasilinear problems

$$\begin{gathered} -\Delta_\Phi u + V(\epsilon x)\phi(|u|)u = f(u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,\Phi}(\mathbb{R}^N), \end{gathered} \tag{1.7}$$

where $N \geq 2$, ϵ is positive parameter, the operator $\Delta_\Phi u = \operatorname{div}(\phi(|\nabla u|)\nabla u)$, where $\Phi(t) = \int_0^{|t|} \phi(s)s ds$, called Φ-Laplacian, is a natural extension of the p-Laplace operator and $V : \mathbb{R}^N \to \mathbb{R}$ is a continuous function which satisfies (1.4).

This type of operator arises in applications, such as: Nonlinear Elasticity: $\Phi(t) = (1 + t^2)^\alpha - 1$, $\alpha \in (1, \frac{N}{N-2})$; Plasticity: $\Phi(t) = t^p \ln(1 + t)$, $1 < \frac{-1+\sqrt{1+4N}}{2} < p < N - 1$, $N \geq 3$; Non-Newtonian Fluid: $\Phi(t) = \frac{1}{p}|t|^p$ for $p > 1$; Plasma Physics: $\Phi(t) = \frac{1}{p}|t|^p + \frac{1}{q}|t|^q$ where $1 < p < q < N$ with $q \in (p, p^*)$.

The reader can find more details about this subject in [15, 17, 21] and their references. Actually, we have observed that there are interesting papers, which study the existence of solution for (1.7) when $\epsilon = 1$, we would like to cite the papers [5, 11, 19, 20, 21, 26, 27, 33] and references therein. The authors know of only one publication [7], where the existence, multiplicity and concentration of solutions has been considered for a Φ-Laplacian equation.

Motivated by [3, 7, 14], in the present paper we study the existence, multiplicity and concentration of solutions for (1.7), by supposing that V satisfies the conditions (1.5)-(1.6). This way, we improve the main result proved in [7], because we are considering a more general condition on potential V. Moreover, we complement the study made in [3, 14], in the sense that we obtain the same type of results for a large class of operators. In the proof of our results, we will work with N-function theory and Orlicz-Sobolev spaces. Since we are working with a general class of function Φ, some estimates explored in [3, 14] cannot be repeated. For example, in [3], it was used interaction Moser techniques, which does not work well in our case. To we overcome this difficulty, we adapt some arguments found in [6, 22, 23, 24, 34]. Here, we also modify the nonlinearity as in [14], however our modification is more technical, see Section 2 for details.

In this article we use the following assumptions: The function $\phi : [0,+\infty) \to [0,+\infty)$ is C^1 and satisfies:

(H1) (1) $\phi(t)$, $(\phi(t)t)' > 0$, $t > 0$.
(2) There exist $l, m \in (1, N)$ such that

$$l \le m < l^* = \frac{Nl}{N-l},$$
$$l \le \frac{\phi(t)t^2}{\Phi(t)} \le m, \quad \forall t \ne 0,$$

where $\Phi(t) = \int_0^{|t|} \phi(s)s\,ds$.
(3) The function $\phi(t)/t^{m-2}$ is non-increasing in $(0,+\infty)$.
(4) The function ϕ is monotone.
(5) There exists a constant $c > 0$ such that

$$|\phi'(t)t| \le c\phi(t), \quad \forall\ t \in [0,+\infty).$$

We remark that the functions ϕ associated with each N-function mentioned in this introduction fulfill the conditions in (H1).

Hereafter, we will say that Φ belongs to the class $\mathcal{C}_m$ if

$$\Phi(t) \ge |t|^m, \quad \forall t \in \mathbb{R}.$$

Also we define

$$\gamma = \begin{cases} m, & \text{if } \Phi \in \mathcal{C}_m, \\ l, & \text{if } \Phi \notin \mathcal{C}_m. \end{cases}$$

We assume that the function $f : \mathbb{R} \to \mathbb{R}$ is C^1 and satisfies:

(H2) (1) There are functions $r, b : [0,+\infty) \to [0,+\infty)$ such that

$$\limsup_{|t|\to 0} \frac{f'(t)}{(r(|t|)|t|)'} = 0, \quad \limsup_{|t|\to +\infty} \frac{|f'(t)|}{(b(|t|)|t|)'} < +\infty.$$

(2) There exists $\theta > m$ such that $0 < \theta F(t) \le f(t)t$ for all $t > 0$, where $F(t) = \int_0^s f(s)\,ds$.
(3) The function $f(t)/t^{m-1}$ is increasing for $t > 0$.

We assume that the function r belongs to C^1 and satisfy:

(H3) (1) r is increasing.
(2) There exists a constant $\overline{c} > 0$ such that $|r'(t)t| \le \overline{c}r(t)$ for all $t \ge 0$.

(3) There exist positive constants r_1 and r_2 such that

$$r_1 \leq \frac{r(t)t^2}{R(t)} \leq r_2, \quad \forall t > 0,$$

where $R(t) = \int_0^{|t|} r(s)s\,ds$.

(4) The function R satisfies

$$\limsup_{t \to 0} \frac{R(t)}{\Phi(t)} < +\infty, \quad \limsup_{|t| \to +\infty} \frac{R(t)}{\Phi_*(t)} = 0.$$

We assume that the function b belongs to C^1 and satisfy:

(H4) (1) b is increasing.

(2) There exists a constant $\widetilde{c} > 0$ such that $|b'(t)t| \leq \widetilde{c}b(t)$ for all $t \geq 0$.

(3) There exist positive constants $b_1, b_2 \in (1, \gamma^*)$ such that

$$b_1 \leq \frac{b(t)t^2}{B(t)} \leq b_2, \quad \forall t > 0,$$

where $\gamma^* = N\gamma/(N-\gamma)$ and $B(t) = \int_0^{|t|} b(s)s\,ds$.

(4) The function B satisfies

$$\limsup_{t \to 0} \frac{B(t)}{\Phi(t)} < +\infty, \quad \limsup_{|t| \to +\infty} \frac{B(t)}{\Phi_*(t)} = 0,$$

where Φ_* is the Sobolev conjugate function, which is defined by the inverse function of

$$G_\Phi(t) = \int_0^t \frac{\Phi^{-1}(s)}{s^{1+\frac{1}{N}}} ds.$$

Using the above hypotheses, we are able to state our main result.

Theorem 1.1. *Suppose that* (H1)–(H4), (1.5) *and* (1.6) *hold. Then, for any* $\delta > 0$ *small enough, there exists* $\epsilon_\delta > 0$ *such that* (1.7) *has at least* $\operatorname{cat}_{M_\delta}(M)$ *positive solutions, for any* $0 < \epsilon < \epsilon_\delta$*, where*

$$M = \{x \in \Omega : V(x) = V_0\}, \quad M_\delta = \big\{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \leq \delta\big\}.$$

Moreover, if u_ϵ *denotes one of these solutions and* $x_\epsilon \in \mathbb{R}^N$ *is a global maximum point of* u_ϵ*, we have that*

$$\lim_{\epsilon \to 0} V(\epsilon x_\epsilon) = V_0.$$

We would like point out that, if Y is a closed subset of a topological space X, the Lusternik-Schnirelman category $\operatorname{cat}_X(Y)$ is the least number of closed and contractible sets in X which cover Y.

The plan of this article is as follows: In Section 2, we will prove the existence and multiplicity of solutions for an auxiliary problem, more precisely, by using of the Lusternik-Schnirelman category theory, we show that the auxiliary problem has at last $\operatorname{cat}_{M_\delta}(M)$ positive solutions for ϵ small enough. In Section 3, we make some estimates to prove that the solutions found are solutions of the original problem for ϵ small enough. Finally, we write an Appendix A, where we show the existence of a special function used in Section 2.

2. An auxiliary problem

In this is section, motivated by some arguments explored in Alves and Figueiredo [3], and mainly in del Pino and Felmer [14], we will show the existence and multiplicity of positive solutions for an auxiliary problem. To this end, we need to fix some notations, however if the reader does not know the main properties involving the Orlicz-Sobolev spaces, we suggest [7, Section 2] for a brief review and [1, 2, 16, 28, 31] for a more complete study.

Since we intend to find positive solutions, we will assume that

$$f(t) = 0 \quad \text{for all } t < 0. \tag{2.1}$$

Let $a, k > 0$ satisfy

$$k > \frac{(\theta - l)}{(\theta - m)} \frac{m}{l} \quad \text{and} \quad \frac{f(a)}{\phi(a)a} = \frac{V_0}{k},$$

where θ is the number given in (H2)(3). Using the above numbers, we define the function

$$\widehat{f}(s) = \begin{cases} f(s) & \text{if } s \le a \\ \frac{V_0}{k}\phi(s)s & \text{if } s > a. \end{cases}$$

Fixing $t_0 < a < t_1$ with $t_0, t_1 \approx a$, it is possible to find a function $\eta \in C^1([t_0, t_1])$ which satisfies:

- (H5) (1) $\eta(s) \le \widehat{f}(s)$ for all $s \in [t_0, t_1]$,
  - (2) $\eta(t_0) = \widehat{f}(t_0)$ and $\eta(t_1) = \widehat{f}(t_1)$,
  - (3) $\eta'(t_0) = (\widehat{f})'(t_0)$ and $\eta'(t_1) = (\widehat{f})'(t_1)$,
  - (4) The function $s \to \frac{\eta(s)}{\phi(s)s}$ is nondecreasing for all $s \in [t_0, t_1]$.

In Appendix A, we shown the existence of η. Using the functions η and $\widehat{f}$, we consider two new functions

$$\widetilde{f}(s) = \begin{cases} \widehat{f}(s) & \text{if } s \notin [t_0, t_1], \\ \eta(s) & \text{if } s \in [t_0, t_1], \end{cases}$$

$$g(x, s) = \chi_\Omega(x) f(s) + (1 - \chi_\Omega(x))\widetilde{f}(s),$$

where χ_Ω is the characteristic function related to the set Ω. From definition of g, we see that g is a Carathéodory function with

$$g(x, s) = 0, \quad \forall (x, s) \in \mathbb{R}^N \times (-\infty, 0], \tag{2.2}$$

$$g(x, s) \le f(s), \quad \forall (x, s) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.3}$$

Moreover, for each $x \in \mathbb{R}^N$, the function $s \to g(x, s)$ is of class C^1 and it satisfies the following conditions:

- (H6) (1) $\limsup_{|s| \to 0} \frac{g(x,s)}{\phi(|s|)|s|} = 0$, uniformly in $x \in \mathbb{R}^N$.
  - (2) $\limsup_{|s| \to +\infty} \frac{g(x,s)}{b(|s|)|s|} < +\infty$, uniformly in $x \in \mathbb{R}^N$.
  - (3) $0 \le \theta G(x, s) = \theta \int_0^s g(x, t)dt \le g(x, s)s$ for all $(x, s) \in \Omega \times (0, +\infty)$.
  - (4) $0 < lG(x, s) \le g(x, s)s \le \frac{V_0}{k}\phi(s)s^2$, for all $(x, s) \in \Omega^c \times (0, +\infty)$.
  - (5) The function $s \to \frac{g(x,s)}{\phi(s)s}$ is nondecreasing for each $x \in \mathbb{R}^N$ and for all $s > 0$.

Using the function g, we can consider the auxiliary problem

$$\begin{gathered} -\Delta_\Phi u + V(\epsilon x)\phi(|u|)u = g(\epsilon x, u) \quad \text{in } \mathbb{R}^N, \\ u \in W^{1,\Phi}(\mathbb{R}^N). \end{gathered} \tag{2.4}$$

Here, we would like to point out that if Ω_ϵ denotes the set Ω/ϵ, that is,

$$\Omega_\epsilon = \{x \in \mathbb{R}^N : \epsilon x \in \Omega\}$$

and u is a positive solution of (2.4) with $u(x) \le t_0$ for all $x \in \mathbb{R}^N \backslash \Omega_\epsilon$, then u is also a positive solution of (1.7).

2.1. **Preliminary results.** In what follows, we denote by $J_\epsilon : X_\epsilon \to \mathbb{R}$ the energy functional related to (2.4) given by

$$J_\epsilon(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|)dx + \int_{\mathbb{R}^N} V(\epsilon x)\Phi(|u|)dx - \int_{\mathbb{R}^N} G(\epsilon x, u)dx,$$

where X_ϵ denotes the subspace of $W^{1,\Phi}(\mathbb{R}^N)$ given by

$$X_\epsilon = \Big\{u \in W^{1,\Phi}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x)\Phi(|u|)dx < +\infty\Big\},$$

endowed with the norm

$$\|u\|_\epsilon = \|\nabla u\|_\Phi + \|u\|_{\Phi,V_\epsilon}$$

where

$$\begin{gathered} \|\nabla u\|_\Phi := \inf\big\{\lambda > 0; \int_{\mathbb{R}^N} \Phi\Big(\frac{|\nabla u|}{\lambda}\Big)dx \le 1\big\}, \\ \|u\|_{\Phi,V_\epsilon} := \inf\big\{\lambda > 0; \int_{\mathbb{R}^N} V(\epsilon x)\Phi\Big(\frac{|u|}{\lambda}\Big)dx \le 1\big\}. \end{gathered}$$

From (1.5), the embeddings $X_\epsilon \hookrightarrow L^\Phi(\mathbb{R}^N)$ and $X_\epsilon \hookrightarrow L^B(\mathbb{R}^N)$ are continuous. Using the above embeddings, a direct computation yields $J_\epsilon \in C^1(X_\epsilon, \mathbb{R})$ with

$$J_\epsilon'(u)v = \int_{\mathbb{R}^N} \phi(|\nabla u|)\nabla u \nabla v \, dx + \int_{\mathbb{R}^N} V(\epsilon x)\phi(|u|)uv \, dx - \int_{\mathbb{R}^N} g(\epsilon x, u)v \, dx,$$

for all $u, v \in X_\epsilon$. Thereby, $u \in X_\epsilon$ is a weak solution of (2.4) if, and only if, u is a critical point of J_ϵ. Furthermore, by (2.2), the critical points of J_ϵ are nonnegative.

Lemma 2.1. *Let (u_n) be a sequence $(PS)_c$. Then, (u_n) is a bounded sequence in X_ϵ.*

Proof. Since (u_n) is a $(PS)_c$ sequence for J_ϵ, there is $C_1 > 0$ such that

$$J_\epsilon(u_n) - \frac{1}{\theta}J_\epsilon'(u_n)u_n \le C_1(1 + \|u_n\|_\epsilon), \quad \forall n \in \mathbb{N}.$$

On the other hand, by (H1)(2), (H6)(3) and (H6)(4),

$$J_\epsilon(u_n) - \frac{1}{\theta}J_\epsilon'(u_n)u_n \ge C\Big(\int_{\mathbb{R}^N} \Phi(|\nabla u_n|)dx + \int_{\mathbb{R}^N} V(\epsilon x)\Phi(u_n)dx\Big),$$

where $C = \big[(1 - \frac{m}{\theta}) - \big(1 - \frac{l}{\theta}\big)\frac{m}{kl}\big] > 0$. Hence, by [19, Lemma 2.1],

$$J_\epsilon(u_n) - \frac{1}{\theta}J_\epsilon'(u_n)u_n \ge C_1\big(\xi_0(\|\nabla u_n\|_\Phi) + \xi_0(\|u_n\|_{\Phi,V_\epsilon})\big), \quad \forall n \in \mathbb{N}.$$

Now, the proof follows as in [5, Lemma 4.2]. $\square$

Lemma 2.2. *Let (u_n) be a $(PS)_d$ sequence for J_ϵ. Then for each $\tau > 0$, there exists $\rho_0 = \rho_0(\tau) > 0$ such that*

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N \setminus B_{\rho_0}(0)} \big[\Phi(|\nabla u_n|) + V(\epsilon x)\Phi(|u_n|)\big] dx < \tau.$$

Proof. For each $\rho > 0$, let $\xi_\rho \in C^\infty(\mathbb{R}^N)$ satisfy

$$\xi_\rho(x) = \begin{cases} 0, & x \in B_{\frac{\rho}{2}}(0) \\ 1, & x \notin B_\rho(0) \end{cases}$$

with $0 \le \xi_\rho(x) \le 1$ and $|\nabla \xi_\rho| \le \frac{C}{\rho}$, where C is a constant independent of ρ. Note that

$$\begin{aligned} J'_\epsilon(u_n)(\xi_\rho u_n) &= \int_{\mathbb{R}^N} \phi(|\nabla u_n|)\nabla u_n \nabla(\xi_\rho u_n) dx + \int_{\mathbb{R}^N} V(\epsilon x)\phi(|u_n|)u_n^2 \xi_\rho dx \\ &\quad - \int_{\mathbb{R}^N} g(\epsilon x, u_n)u_n \xi_\rho dx. \end{aligned}$$

Choosing $\rho > 0$ such that $\Omega_\epsilon \subset B_{\frac{\rho}{2}}(0)$, the condition (H1)(2) ensures that

$$\begin{aligned} &l \int_{\mathbb{R}^N} \xi_\rho \Big[\Phi(|\nabla u_n|) + V(\epsilon x)\Phi(|u_n|)\Big] dx \\ &\le J'_\epsilon(u_n)(\xi_\rho u_n) - \int_{\mathbb{R}^N} u_n \phi(|\nabla u_n|)\nabla u_n \nabla \xi_\rho dx + \int_{\mathbb{R}^N \setminus \Omega_\epsilon} g(\epsilon x, u_n)u_n \xi_\rho dx. \end{aligned}$$

Gathering (H6)(4) and (H1)(2),

$$\begin{aligned} &l \int_{\mathbb{R}^N} \xi_\rho \Big[\Phi(|\nabla u_n|) + V(\epsilon x)\Phi(|u_n|)\Big] dx \\ &\le J'_\epsilon(u_n)(\xi_\rho u_n) - \int_{\mathbb{R}^N} u_n \phi(|\nabla u_n|)\nabla u_n \nabla \xi_\rho \, dx + \frac{m}{k} \int_{\mathbb{R}^N} V(\epsilon x)\Phi(|u_n|)\xi_\rho \, dx. \end{aligned}$$

Since $(\xi_\rho u_n)$ is bounded in X_ϵ and $k > \frac{m}{l}$, by Hölder inequality there is a constant $C_1 > 0$ such that

$$\int_{\mathbb{R}^N} \xi_\rho \big[\Phi(|\nabla u_n|) + V(\epsilon x)\Phi(|u_n|)\big] dx \le o_n(1) + \frac{C_1}{\rho}.$$

Now, fixed $\tau > 0$, there exists $\rho_0 > 0$ such that $\frac{C_1}{\rho_0} < \tau$. Then,

$$\int_{\mathbb{R}^N \setminus B_{\rho_0}(0)} \big[\Phi(|\nabla u_n|) + V(\epsilon x)\Phi(|u_n|)\big] dx \le o_n(1) + \tau.$$

Passing to the limit in the last inequality, it follows that

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N \setminus B_{\rho_0}(0)} \big[\Phi(|\nabla u_n|) + V(\epsilon x)\Phi(|u_n|)\big] dx \le \tau.$$

□

The lemma below establishes an important property of the (PS) sequences of J_ϵ. Since the proof follows as in [5, Lemma 4.3], we will omit it.

Lemma 2.3. *Let (u_n) be a $(PS)_d$ sequence for J_ϵ with $u_n \rightharpoonup u$ in X_ϵ. Then,*

$$\nabla u_n(x) \to \nabla u(x) \quad a.\ e.\ in\ \mathbb{R}^N. \tag{2.5}$$

As a consequence of the above limit, we deduce that u is a critical point for J_ϵ.

Proposition 2.4. *The functional J_ϵ satisfies the (PS) condition.*

Proof. Let (u_n) be a $(PS)_c$ sequence for J_ϵ. From Lemma 2.1, there exists $u \in X_\epsilon$ such that

$$u_n \rightharpoonup u \quad \text{in } X_\epsilon. \tag{2.6}$$

Moreover, by Lemma 2.2, given $\tau > 0$ there is $\rho_0 > 0$ satisfying

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N \setminus B_{\rho_0}(0)} \big[\Phi(|\nabla u_n|) + V(\epsilon x)\Phi(|u_n|)\big]dx < \tau.$$

Increasing $\rho_0 > 0$ if necessary, the above limit together with Δ_2-condition gives

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} \Phi(|\nabla u_n - \nabla u|)dx \le \limsup_{n\to+\infty} \int_{B_{\rho_0}(0)} \Phi(|\nabla u_n - \nabla u|)dx + 2\tau \tag{2.7}$$

and

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} V(\epsilon x)\Phi(|u_n - u|)dx \le \limsup_{n\to+\infty} \int_{B_{\rho_0}(0)} V(\epsilon x)\Phi(|u_n - u|)dx + 2\tau.$$

By (2.6), up to a subsequence, $u_n \to u$ in $L^\Phi(B_{\rho_0}(0))$. This information combined with the last limit guarantees that

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} V(\epsilon x)\Phi(|u_n - u|)dx \le 2\tau.$$

As τ is arbitrary, we conclude that

$$\limsup_{n\to+\infty} \int_{\mathbb{R}^N} V(\epsilon x)\Phi(|u_n - u|)dx = 0. \tag{2.8}$$

Now, we will show that

$$\limsup_{n\to+\infty} \int_{B_{\rho_0}(0)} \Phi(|\nabla u_n - \nabla u|)dx = 0.$$

By Lemma 2.3,

$$\Phi(|\nabla u_n(x) - \nabla u(x)|) \to 0 \quad \text{a. e. in } B_{\rho_0}(0).$$

Moreover, from Δ_2-condition and (H1)(2), there exist constants $c_1, c_2 > 0$ such that

$$\Phi(|\nabla u_n - \nabla u|) \le c_1\phi(|\nabla u_n|)|\nabla u_n|^2 + c_2\phi(|\nabla u|)|\nabla u|^2.$$

Using again Lemma 2.3

$$c_1\phi(|\nabla u_n|)|\nabla u_n|^2 + c_2\phi(|\nabla u|)|\nabla u|^2 \to (c_1 + c_2)\phi(|\nabla u|)|\nabla u|^2 \quad \text{a. e. in } B_{\rho_0}(0).$$

On the other hand, as in [5, Lemma 4.3],

$$\int_{B_{\rho_0}(0)} (\phi(|\nabla u_n|)\nabla u_n - \phi(|\nabla u|)\nabla u)(\nabla u_n - \nabla u)dx = o_n(1),$$

and so,

$$\int_{B_{\rho_0}(0)} \phi(|\nabla u_n|)|\nabla u_n|^2dx \to \int_{B_{\rho_0}(0)} \phi(|\nabla u|)|\nabla u|^2dx.$$

Therefore,

$$\int_{B_{\rho_0}(0)} \Big[c_1\phi(|\nabla u_n|)|\nabla u_n|^2 + c_2\phi(|\nabla u|)|\nabla u|^2\Big]dx \to (c_1+c_2)\int_{B_{\rho_0}(0)} \phi(|\nabla u|)|\nabla u|^2dx.$$

Applying the Lebesgue's Theorem, we see that

$$\lim_{n\to+\infty}\int_{B_{\rho_0}(0)}\Phi(|\nabla u_n-\nabla u|)dx=0.$$

Combining the last limit with (2.7), we obtain

$$\lim_{n\to+\infty}\int_{\mathbb{R}^N}\Phi(|\nabla u_n-\nabla u|)dx=0. \tag{2.9}$$

From (2.8) and (2.9), we have the limit $u_n\to u$ in X_ϵ, which shows the (PS) condition. □

Next, we show some results involving J_ϵ and its Nehari manifold. We recall that the Nehari manifold associated to J_ϵ is given by

$$\mathcal{N}_\epsilon=\big\{u\in X_\epsilon\backslash\{0\}:J'_\epsilon(u)u=0\big\}.$$

Our first lemma ensures that the Nehari manifold has a positive distance from the origin in X_ϵ. Once it follows by standard arguments, we omit its proof.

Lemma 2.5. *For all $u\in\mathcal{N}_\epsilon$, there exists $\sigma>0$, which is independent of ϵ, such that $\|u\|_\epsilon>\sigma$.*

The next lemma establishes an important characterization of the mountain pass level, which is useful in a lot of problems. In what follows, we denote by $c_{\epsilon,1}$ and $c_{\epsilon,2}$ the following numbers

$$c_{\epsilon,1}=\inf_{u\in\mathcal{N}_\epsilon}J_\epsilon(u)\quad\text{and}\quad c_{\epsilon,2}=\inf_{u\in X_\epsilon\backslash\{0\}}\max_{t\ge0}J_\epsilon(tu).$$

Fixing the subset

$$\mathcal{A}_\epsilon=\big\{u\in X_\epsilon\,:\,u^+\neq0\quad\text{and}\quad|supp(u)\cap\Omega_\epsilon|>0\big\}$$

and the number

$$\overline{c}_{\epsilon,2}=\inf_{u\in\mathcal{A}_\epsilon}\max_{t\ge0}J_\epsilon(tu),$$

it is easy to see that $c_{\epsilon,2}=\overline{c}_{\epsilon,2}$.

Lemma 2.6. *Assume that* (H1)–(H4), (1.5) *and* (1.6) *hold. Then, for each $u\in\mathcal{A}_\epsilon$, there exists a unique $t_u>0$ such that $t_uu\in\mathcal{N}_\epsilon$ and $J_\epsilon(t_uu)=\max_{t\ge0}J_\epsilon(tu)$. Moreover,*

$$c_\epsilon=c_{\epsilon,1}=c_{\epsilon,2},$$

where c_ϵ denotes the mountain pass level associated with J_ϵ.

Proof. For each $u\in\mathcal{A}_\epsilon$, we define $h_\epsilon(t)=J_\epsilon(tu)$; that is,

$$h_\epsilon(t)=\int_{\mathbb{R}^N}\Phi(|\nabla(tu)|)dx+\int_{\mathbb{R}^N}V(\epsilon x)\Phi(|tu|)dx-\int_{\mathbb{R}^N}G(\epsilon x,tu)\,dx.$$

Existence. By a direct computation, $h_\epsilon(t)>0$ for t enough small and $h_\epsilon(t)<0$ for t sufficiently large. Thus, there is $t_u>0$ such that

$$h_\epsilon(t_u)=\max_{t\ge0}h_\epsilon(t)=\max_{t\ge0}J_\epsilon(tu).$$

From this, $h'_\epsilon(t_u)=0$, and so, $t_uu\in\mathcal{N}_\epsilon$.
Uniqueness. Suppose that there exist $t_1,t_2>0$ such that $t_1u,t_2u\in\mathcal{N}_\epsilon$ and $t_1<t_2$. Then

$$\int_{\mathbb{R}^N}\phi(|\nabla(t_1u)|)|\nabla(t_1u)|^2dx+\int_{\mathbb{R}^N}V(\epsilon x)\phi(|t_1u|)|t_1u|^2dx=\int_{[u>0]}g(\epsilon x,t_1u)t_1u\,dx,$$

$$\int_{\mathbb{R}^N} \phi(|\nabla(t_2u)|)|\nabla(t_2u)|^2dx + \int_{\mathbb{R}^N} V(\epsilon x)\phi(|t_2u|)|t_2u|^2dx = \int_{[u>0]} g(\epsilon x, t_2u)t_2u dx.$$

Setting $\upsilon(t) = \phi(t)/t^{m-2}$ for all $t > 0$, we have

$$\begin{aligned}&\int_{\mathbb{R}^N} \big(\upsilon(|t_1|\nabla u||) - \upsilon(|t_2|\nabla u||)\big)|\nabla u|^m dx + \int_{\mathbb{R}^N} V(\epsilon x)\big(\upsilon(|t_1|u||) - \upsilon(|t_2|u||)\big)|u|^m dx\\ &= \int_{[u>0]} \Big[\frac{g(\epsilon x, t_1u)}{(t_1u)^{m-1}} - \frac{g(\epsilon x, t_2u)}{(t_2u)^{m-1}}\Big]u^m dx.\end{aligned}$$

Thus, from (1.5) and (H1)(3),

$$\begin{aligned}&\int_{\mathbb{R}^N} \big(\upsilon(|t_1|\nabla u||) - \upsilon(|t_2|\nabla u||)\big)|\nabla u|^m dx\\ &+ \int_{(\mathbb{R}^N\setminus\Omega_\epsilon)\cap[u>0]} \Big[\Big(\frac{V_0}{k}\upsilon(|t_1|u||) - \frac{\widetilde{f}(t_1u)}{(t_1u)^{m-1}}\Big) - \Big(\frac{V_0}{k}\upsilon(|t_2|u||) - \frac{\widetilde{f}(t_2u)}{(t_2u)^{m-1}}\Big)\Big]u^m dx\\ &\le \int_{\Omega_\epsilon\cap[u>0]} \Big[\frac{f(t_1u)}{(t_1u)^{m-1}} - \frac{f(t_2u)}{(t_2u)^{m-1}}\Big]u^m\, dx.\end{aligned}$$

Now, consider the function

$$h(t) = \frac{V_0}{k}\upsilon(t) - \frac{\widetilde{f}(t)}{t^{m-1}}$$

and note that $h(t) = \upsilon(t)h_1(t)$, where

$$h_1(t) = \frac{V_0}{k} - \frac{\widetilde{f}(t)}{\phi(t)t}.$$

As υ, h_1 are non increasing and nonnegative, h is nonincreasing. Hence, $h(t_1u) \ge h(t_2u)$, and so,

$$\begin{aligned}0 &\le \int_{\mathbb{R}^N} \big(\upsilon(|t_1|\nabla u||) - \upsilon(|t_2|\nabla u||)\big)|\nabla u|^m dx\\ &\quad + \int_{(\mathbb{R}^N\setminus\Omega_\epsilon)\cap[u>0]} \big(h(t_1u) - h(t_2u)\big)u^m dx\\ &\le \int_{\Omega_\epsilon\cap[u>0]} \Big[\frac{f(t_1u)}{(t_1u)^{m-1}} - \frac{f(t_2u)}{(t_2u)^{m-1}}\Big]u^m dx < 0,\end{aligned}$$

which is an absurd. Therefore, $t_1 = t_2$. Now the proof follows as in [36, Theorem 4.2]. □

2.2. (PS) **condition for** J_ϵ **on** $\mathcal{N}_\epsilon$**.** In this subsection, our main goal is to study the (PS) condition for J_ϵ on $\mathcal{N}_\epsilon$, which will be used later on. In what follows, without loss of generality, we will assume that

$$V(0) = \min_{z\in\mathbb{R}^N} V(z) = V_0.$$

Using the above number, we fix the quasilinear problem

$$\begin{gathered}-\Delta_\Phi u + V_0\phi(|u|)u = f(u) \quad \text{in } \mathbb{R}^N,\\ u \in W^{1,\Phi}(\mathbb{R}^N).\end{gathered} \tag{2.10}$$

We recall that the weak solutions of (2.10) are critical points of the functional

$$E_0(u) = \int_{\mathbb{R}^N} \Phi(|\nabla u|)\, dx + V_0\int_{\mathbb{R}^N} \Phi(|u|)\, dx - \int_{\mathbb{R}^N} F(u)\, dx,$$

which is well defined in $Y = W^{1,\Phi}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_Y = \|\nabla u\|_\Phi + V_0\|u\|_\Phi.$$

For the rest of this article, we denote by d_0 the mountain pass level of E_0, and by $\mathcal{M}_0$ the Nehari manifold

$$\mathcal{M}_0 = \big\{u \in Y\backslash\{0\} : E_0'(u)u = 0\big\}.$$

The next lemma will be used to prove that J_ϵ satisfies the (PS) condition on $\mathcal{N}_\epsilon$.

Lemma 2.7. *Consider $U = \{u \in \mathcal{N}_\epsilon : J_\epsilon(u) < d_0 + 1\}$. Then, there are $\sigma_1, \sigma_2 > 0$ independents of ϵ, for ϵ small enough, such that*

(a) *$\int_{\mathbb{R}^N} \Phi(u)dx \le \sigma_1$ for all $u \in U$,*
(b) *$\int_{\Omega_\epsilon} (f'(u)u^2 - (m-1)f(u)u)dx \ge \sigma_2$ for all $u \in U$.*

Proof. (a) For any $u \in U$,

$$J_\epsilon(u) - \frac{1}{\theta}J_\epsilon'(u)u = J_\epsilon(u) < d_0 + 1.$$

On the other hand, as in the proof of Lemma 2.1, there is $C > 0$ such that

$$J_\epsilon(u) - \frac{1}{\theta}J_\epsilon'(u)u \ge CV_0 \int_{\mathbb{R}^N} \Phi(u)dx.$$

From this,

$$CV_0 \int_{\mathbb{R}^N} \Phi(u)dx \le d_0 + 1 \quad \forall u \in U,$$

and (a) is proved.

(b) The claim follows by proving that if $\epsilon_n \to 0$ and $u_n \in \mathcal{N}_{\epsilon_n}$, then

$$\liminf_{n\to+\infty} \int_{\Omega_{\epsilon_n}} (f'(u_n)u_n^2 - (m-1)f(u_n)u_n)dx > 0. \tag{2.11}$$

By a Lions' type arguments, it is possible to show that there are $z_n \in \mathbb{R}^N$ and $r, \beta > 0$ satisfying

$$\int_{\Omega_{\epsilon_n} \cap B_r(z_n)} \Phi(u_n)dx \ge \beta \quad \forall n \in \mathbb{N}.$$

Setting $\tilde{u}_n(x) = u_n(x + z_n)$, the above inequality together with (f_3) permits to conclude that

$$\liminf_{n\to+\infty} \int_{\Omega_{\epsilon_n} - z_n} (f'(\tilde{u}_n)\tilde{u}_n^2 - (m-1)f(\tilde{u}_n)\tilde{u}_n)dx > 0.$$

Once

$$\int_{\Omega_{\epsilon_n}} (f'(u_n)u_n^2 - (m-1)f(u_n)u_n)dx = \int_{\Omega_{\epsilon_n} - z_n} (f'(\tilde{u}_n)\tilde{u}_n^2 - (m-1)f(\tilde{u}_n)\tilde{u}_n)dx,$$

we obtain (2.11). This completes the proof of (b). □

Proposition 2.8. *The functional J_ϵ restricted to $\mathcal{N}_\epsilon$ satisfies the $(PS)_c$ condition for $c \in (0, d_0 + 1)$.*

Proof. Let (u_n) be a $(PS)_c$ sequence on $\mathcal{N}_\epsilon$; that is, $J_\epsilon(u_n) \to c$ and $\|J'_\epsilon(u_n)\|_* = o_n(1)$. Then, there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$J'_\epsilon(u_n) = \lambda_n L'_\epsilon(u_n) + o_n(1),$$

where $L_\epsilon(v) = J'_\epsilon(v)v$ for all $v \in X_\epsilon$. Thus,

$$\lambda_n L'_\epsilon(u_n)u_n = o_n(1). \tag{2.12}$$

We claim that $\lambda_n = o_n(1)$. In fact, note that

$$\begin{aligned} L'_\epsilon(u_n)u_n &= \int_{\mathbb{R}^N} \big[\phi'(|\nabla u_n|)|\nabla u_n| + 2\phi(|\nabla u_n|)\big]|\nabla u_n|^2 dx \\ &\quad + \int_{\mathbb{R}^N} V(\epsilon x)\big[\phi'(|u_n|)|u_n| + 2\phi(|u_n|)\big]|u_n|^2 dx \\ &\quad - \int_{\mathbb{R}^N} \big[g'(\epsilon x, u_n)u_n^2 + g(\epsilon x, u_n)u_n\big] dx. \end{aligned}$$

By (H1)(3),

$$\begin{aligned} L'_\epsilon(u_n)u_n &\le m\Big[\int_{\mathbb{R}^N} \phi(|\nabla u_n|)|\nabla u_n|^2 dx + \int_{\mathbb{R}^N} V(\epsilon x)\phi(|u_n|)|u_n|^2 dx\Big] \\ &\quad - \int_{\mathbb{R}^N} \big[g'(\epsilon x, u_n)u_n^2 + g(\epsilon x, u_n)u_n\big] dx, \end{aligned}$$

which implies

$$\begin{aligned} L'_\epsilon(u_n)u_n &\le \int_{\mathbb{R}^N} \big[(m-1)g(\epsilon x, u_n)u_n - g'(\epsilon x, u_n)u_n^2\big] dx \\ &= \int_{\Omega_\epsilon \cup [u_n < t_0]} \big[(m-1)f(u_n)u_n - f'(u_n)u_n^2\big] dx \\ &\quad + \int_{(\mathbb{R}^N\setminus\Omega_\epsilon)\cap[t_0 \le u_n \le t_1]} \big[(m-1)\eta(u_n)u_n - \eta'(u_n)u_n^2\big] dx \\ &\quad + \int_{(\mathbb{R}^N\setminus\Omega_\epsilon)\cap[u_n > t_1]} \big[(m-1)\frac{V_0}{k}\phi(u_n)u_n^2 - \frac{V_0}{k}(\phi(u_n)(u_n))'u_n^2\big] dx. \end{aligned}$$

Since $\eta'(t), (\phi(t)t)' \ge 0$ for $t > 0$, it follows that,

$$\begin{aligned} L'_\epsilon(u_n)u_n &\le \int_{\Omega_\epsilon \cup [u_n < t_0]} \big[(m-1)f(u_n)u_n - f'(u_n)u_n^2\big] dx \\ &\quad + \int_{(\mathbb{R}^N\setminus\Omega_\epsilon)\cap[t_0 \le u_n \le t_1]} (m-1)\eta(u_n)u_n dx \\ &\quad + \int_{(\mathbb{R}^N\setminus\Omega_\epsilon)\cap[u_n > t_1]} (m-1)\frac{V_0}{k}\phi(u_n)u_n^2\, dx. \end{aligned} \tag{2.13}$$

Now, the inequality

$$\eta(t) \le \frac{V_0}{k}\phi(t)t \quad \forall t \in [t_0, t_1],$$

leads to

$$\begin{aligned} L'_\epsilon(u_n)u_n &\le \int_{\Omega_\epsilon \cup [u_n < t_0]} \Big[(m-1)f(u_n)u_n - f'(u_n)u_n^2\Big] dx \\ &\quad + \int_{(\mathbb{R}^N\setminus\Omega_\epsilon)\cap[t_0 \le u_n \le t_1]} (m-1)\frac{V_0}{k}\phi(u_n)u_n^2 dx \end{aligned}$$

$$+\int_{(\mathbb{R}^N\setminus\Omega_\epsilon)\cap[u_n>t_1]}(m-1)\frac{V_0}{k}\phi(u_n)u_n^2dx,$$

and so,

$$L'_\epsilon(u_n)u_n\leq\int_{\Omega_\epsilon}\Big[(m-1)f(u_n)u_n-f'(u_n)u_n^2\Big]dx+\frac{V_0}{k}\int_{\mathbb{R}^N}\phi(u_n)u_n^2dx.$$

Then, by Lemma 2.7,

$$-L'_\epsilon(u_n)u_n\geq\sigma_2-\frac{mV_0\sigma_1}{k}.$$

Therefore, by increasing of k if necessary, we deduce that

$$-L'_\epsilon(u_n)u_n\geq C\quad\forall n\in\mathbb{N}.$$

for some $C>0$. The last inequality combine with (2.12) to give $\lambda_n=o_n(1)$. Hence $J'_\epsilon(u_n)=o_n(1)$, which permits us to conclude that (u_n) is a $(PS)_c$ sequence for J_ϵ in X_ϵ. Now, the result follows from Proposition 2.4. $\square$

As a byproduct of the arguments used in the proof of the last proposition, we have the following result.

Corollary 2.9. *The critical points of functional J_ϵ on $\mathcal{N}_\epsilon$ are critical points of J_ϵ in X_ϵ.*

2.3. **Multiplicity of solutions to** (2.4)**.** After the previous subsection, we are able to show the existence of multiple positive solutions for (2.4), by using of Lusternik-Schnirelman theory. Furthermore, we also study the behavior of the maximum points these solutions in relation to M. For each $\delta>0$ small enough, we consider $\vartheta\in C_0^\infty([0,+\infty),[0,1])$ with

$$\vartheta(s)=\begin{cases}1, & \text{if } 0\leq s\leq\frac{\delta}{2}\\ 0, & \text{if } s\geq\delta.\end{cases}$$

Using the function above, for each $y\in M$ we set

$$\Psi_{\epsilon,y}(x)=\vartheta(|\epsilon x-y|)w(\frac{\epsilon x-y}{\epsilon}),$$

where $w\in W^{1,\Phi}(\mathbb{R}^N)$ denotes a positive ground state solution of problem (P_0), which exists according to [7, Theorem 3.4]. By Lemma 2.6, there exists $t_\epsilon>0$ such that $t_\epsilon\Psi_{\epsilon,y}\in\mathcal{N}_\epsilon$ and

$$J_\epsilon(t_\epsilon\Psi_{\epsilon,y})=\max_{t\geq0}J_\epsilon(t\Psi_{\epsilon,y}).$$

From this, we can define $\widetilde{\Psi}_\epsilon:M\to\mathcal{N}_\epsilon$ by $\widetilde{\Psi}_\epsilon(y)=t_\epsilon\Psi_{\epsilon,y}$.

Lemma 2.10. *The function $\widetilde{\Psi}_\epsilon$ satisfies*

$$\lim_{\epsilon\to0}J_\epsilon(\widetilde{\Psi}_\epsilon(y))=d_0,\quad \textit{uniformly in } y\in M.$$

Proof. It is sufficient to show that for each $(y_n)\subset M$ and $(\epsilon_n)\subset\mathbb{R}^+$ with $\epsilon_n\to0$, there is a subsequence such that

$$J_\epsilon(\widetilde{\Psi}_{\epsilon_n}(y_n))\to d_0.$$

Recall firstly that $J'_{\epsilon_n}(\widetilde{\Psi}_{\epsilon_n}(y_n))\widetilde{\Psi}_{\epsilon_n}(y_n)=0$; that is,

$$\int_{\mathbb{R}^N}\widehat{\phi}(|\nabla(\widetilde{\Psi}_{\epsilon_n}(y_n)|)dx+\int_{\mathbb{R}^N}V(\epsilon_nx)\widehat{\phi}(|\widetilde{\Psi}_{\epsilon_n}(y_n)|)dx$$

$$= \int_{\mathbb{R}^N} g(\epsilon_n x, \widetilde{\Psi}_{\epsilon_n}(y_n))\widetilde{\Psi}_{\epsilon_n}(y_n)dx,$$

where $\widehat{\phi}(s) = \phi(s)s^2$ for all $s \geq 0$. Using (H1)(2) and [19, Lemma 2.1],

$$\begin{aligned}
&\int_{\mathbb{R}^N} \widehat{\phi}(|\nabla(\widetilde{\Psi}_{\epsilon_n}(y_n)|)dx + \int_{\mathbb{R}^N} V(\epsilon_n x)\widehat{\phi}(|\widetilde{\Psi}_{\epsilon_n}(y_n)|)dx \\
&\leq m\xi_1(t_{\epsilon_n})\Big[\int_{\mathbb{R}^N} \Phi(|\nabla(\Psi_{\epsilon_n,y_n})|)dx + \int_{\mathbb{R}^N} V(\epsilon_n x)\Phi(|\Psi_{\epsilon_n,y_n}|)dx\Big],
\end{aligned} \tag{2.14}$$

where $\xi_1(t) = \max\{t^l, t^m\}$. On the other hand, the change of variable $z = \frac{\epsilon_n x - y_n}{\epsilon_n}$ leads to

$$\begin{aligned}
&\int_{\mathbb{R}^N} g(\epsilon_n x, \widetilde{\Psi}_{\epsilon_n}(y_n))\widetilde{\Psi}_{\epsilon_n}(y_n)dx \\
&= \int_{\mathbb{R}^N} g(\epsilon_n z + y_n, t_{\epsilon_n}\vartheta(|\epsilon_n z|)w(z))t_{\epsilon_n}\vartheta(|\epsilon_n z|)w(z)dx.
\end{aligned}$$

Note that, if $z \in B_{\frac{\delta}{\epsilon_n}}(0)$, then $\epsilon_n z + y_n \in B_\delta(y_n) \subset M_\delta \subset \Omega$. Since $f = g$ in Ω, $\vartheta \equiv 1$ on $B_{\delta/2}(0)$ and $B_{\delta/2}(0) \subset B_{\frac{\delta}{2\epsilon_n}}(0)$, it follows that

$$\begin{aligned}
&\int_{\mathbb{R}^N} g(\epsilon_n x, \widetilde{\Psi}_{\epsilon_n}(y_n))\widetilde{\Psi}_{\epsilon_n}(y_n)dx \\
&\geq \int_{\mathbb{R}^N} f(t_{\epsilon_n}\vartheta(|\epsilon_n z|)w(z))t_{\epsilon_n}\vartheta(|\epsilon_n z|)w(z)dx \\
&\geq \int_{B_{\delta/2}(0)} f(t_{\epsilon_n}w(z))t_{\epsilon_n}w(z)dx.
\end{aligned} \tag{2.15}$$

Combining (2.14) with (2.15), we find

$$\begin{aligned}
&\int_{B_{\delta/2}(0)} \frac{f(t_{\epsilon_n}w(z))}{(t_{\epsilon_n}w(z))^{m-1}}|t_{\epsilon_n}w(z)|^m dx \\
&\leq m\xi_1(t_{\epsilon_n})\Big[\int_{\mathbb{R}^N} \Phi(|\nabla(\Psi_{\epsilon_n,y_n})|)dx + \int_{\mathbb{R}^N} V(\epsilon_n x)\Phi(|\Psi_{\epsilon_n,y_n}|)dx\Big].
\end{aligned}$$

By Proposition 2.14, we know that w is a continuous function. Then, there is $z_0 \in \mathbb{R}^N$ such that

$$w(z_0) = \min_{z \in B_{\delta/2}(0)} w(z),$$

and so, from (H2)(3),

$$\begin{aligned}
&\frac{f(t_{\epsilon_n}w(z_0))}{(t_{\epsilon_n}w(z_0))^{m-1}}\int_{B_{\delta/2}(0)} |t_{\epsilon_n}w(z)|^m dx \\
&\leq m\xi_1(t_{\epsilon_n})\Big[\int_{\mathbb{R}^N} \Phi(|\nabla(\Psi_{\epsilon_n,y_n})|)dx + \int_{\mathbb{R}^N} V(\epsilon_n x)\Phi(|\Psi_{\epsilon_n,y_n}|)dx\Big].
\end{aligned}$$

By (H2)(2), there are $c_1, c_2 > 0$ such that

$$\begin{aligned}
&\big[c_1(t_{\epsilon_n}w(z_0))^{\theta-m} - c_2(t_{\epsilon_n}w(z_0))^{-m}\big]t_{\epsilon_n}^m\int_{B_{\delta/2}(0)} |w(z)|^m dx \\
&\leq m\xi_1(t_{\epsilon_n})\Big[\int_{\mathbb{R}^N} \Phi(|\nabla(\Psi_{\epsilon_n,y_n})|)dx + \int_{\mathbb{R}^N} V(\epsilon_n x)\Phi(|\Psi_{\epsilon_n,y_n}|)dx\Big].
\end{aligned}$$

Now, we show that $(t_{\epsilon_n}$ is bounded. To do this, we will suppose that for some subsequence $t_{\epsilon_n} \to +\infty$ and $t_{\epsilon_n} \geq 1$ for all $n \in \mathbb{N}$. Thereby, $\xi_1(t_{\epsilon_n}) = t_{\epsilon_n}^m$ and

$$\begin{aligned}&\big[c_1(t_{\epsilon_n} w(z_0))^{\theta-m} - c_2(t_{\epsilon_n} w(z_0))^{-m}\big] \int_{B_{\delta/2}(0)} |w(z)|^m \, dx \\ &\leq m\Big[\int_{\mathbb{R}^N} \Phi(|\nabla(\Psi_{\epsilon_n,y_n})|)dx + \int_{\mathbb{R}^N} V(\epsilon_n x)\Phi(|\Psi_{\epsilon_n,y_n}|)dx\Big].\end{aligned}$$

The change of variable $z = \frac{\epsilon_n x - y_n}{\epsilon_n}$ together with the Lebesgue's Theorem ensures that

$$\begin{gathered}\int_{\mathbb{R}^N} \Phi(|\nabla(\Psi_{\epsilon_n,y_n})|)dx \to \int_{\mathbb{R}^N} \Phi(|\nabla w|) \, dx, \\ \int_{\mathbb{R}^N} V(\epsilon_n z + y_n)\Phi(|\Psi_{\epsilon_n,y_n}|)dx \to \int_{\mathbb{R}^N} V_0 \Phi(|w|)dx.\end{gathered}$$

Since $\theta > m$, we have

$$[c_1(t_{\epsilon_n} w(z_0))^{\theta-m} - c_2(t_{\epsilon_n} w(z_0))^{-m}\big] \to +\infty,$$

which contradicts the bove inequality. Therefore (t_{ϵ_n}) is bounded, and for some subsequence, there exists $t_0 \geq 0$ such that $t_{\epsilon_n} \to t_0$.

Now, as $\widetilde{\Psi}_{\epsilon_n}(y_n) \in \mathcal{N}_{\epsilon_n}$, we know that $\|\widetilde{\Psi}_{\epsilon_n}(y_n)\|_{\epsilon_n} \geq \sigma$ for all $n \in \mathbb{N}$. Using again the Lebesgue's Theorem, it is possible to prove that

$$E_0'(t_0 w)(t_0 w) = 0 \quad \text{and} \quad \|t_0 w\|_Y \geq \sigma.$$

Thus, $t_0 > 0$ and $t_0 w \in \mathcal{M}_0$. Furthermore, as w is a ground state solution, we must have $t_0 = 1$. Then, the limit $t_n \to 1$ together with the Lebesgue's Theorem implies that

$$\lim_{n\to+\infty} J_\epsilon(\widetilde{\Psi}_{\epsilon_n}(y_n)) = E_0(w) = d_0,$$

and the proof is complete. □

In the sequel, for any $\delta > 0$, we let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. Let the function $\chi : \mathbb{R}^N \to \mathbb{R}^N$ be given by

$$\chi(x) = \begin{cases} x, & \text{if } x \in B_\rho(0), \\ \frac{\rho x}{|x|}, & \text{if } x \in B_\rho^c(0) \end{cases}$$

and $\beta : \mathcal{N}_\epsilon \to \mathbb{R}^N$ the barycenter map given by

$$\beta(u) = \frac{\int_{\mathbb{R}^N} \chi(\epsilon x)\Phi(|u|)dx}{\int_{\mathbb{R}^N} \Phi(|u|)dx}.$$

Lemma 2.11. *The function $\widetilde{\Psi}_\epsilon$ satisfies*

$$\lim_{\epsilon\to 0} \beta(\widetilde{\Psi}_\epsilon(y)) = y, \quad \textit{uniformly in } M.$$

Proof. The lemma follows by using of the definition of $\widetilde{\Psi}_\epsilon(y)$ together with the Lebesgue's Theorem. □

Hereafter, we consider the function $h : \mathbb{R}^+ \to \mathbb{R}^+$ given by

$$h(\epsilon) = \sup_{y\in M} |J_\epsilon(\widetilde{\Psi}_\epsilon(y)) - d_0|,$$

which satisfies $\lim_{\epsilon\to 0} h(\epsilon) = 0$. Moreover, we set

$$\widetilde{\mathcal{N}}_\epsilon := \{u \in \mathcal{N}_\epsilon : J_\epsilon(u) \le d_0 + h(\epsilon)\}.$$

From Lemma 2.10, it follows that $\widetilde{\mathcal{N}}_\epsilon \neq \emptyset$ because $\widetilde{\Psi}_\epsilon(y) \in \widetilde{\mathcal{N}}_\epsilon$. Using the above notation, we have the following result.

Lemma 2.12. *Let $\delta > 0$ and $M_\delta = \{x \in \mathbb{R}^N : \operatorname{dist}(x, M) \le \delta\}$. Then, the limit below holds*

$$\lim_{\epsilon\to 0} \sup_{u\in\widetilde{\mathcal{N}}_\epsilon} \inf_{y\in M_\delta} \big|\beta(u) - y\big| = 0.$$

The proof of the above lemma follows the proof in [3, Lemma 3.7]. The next theorem is a result of multiplicity for the auxiliary problem.

Theorem 2.13. *For any $\delta > 0$ there exists $\epsilon_\delta > 0$ such that* (2.4) *has at least $\operatorname{cat}_{M_\delta}(M)$ positive solutions, for any $0 < \epsilon < \epsilon_\delta$.*

Proof. We fix a small $\epsilon > 0$. Then, by Lemmas 2.10 and 2.12, $\beta \circ \widetilde{\Psi}_\epsilon$ is homotopic to inclusion map $\mathrm{id} : M \to M_\delta$, and so,

$$\operatorname{cat}_{\widetilde{\mathcal{N}}_\epsilon}(\widetilde{\mathcal{N}}_\epsilon) \ge \operatorname{cat}_{M_\delta}(M).$$

Since that functional J_ϵ satisfies the $(PS)_c$ condition for $c \in (d_0, d_0 + h(\epsilon))$, by the Lusternik-Schnirelman theory of critical points ([36]), we can conclude that J_ϵ has at least $\operatorname{cat}_{M_\delta}(M)$ critical points on $\mathcal{N}_\epsilon$. Consequently by Corollary 2.9, J_ϵ has at least $\operatorname{cat}_{M_\delta}(M)$ critical points in X_ϵ. $\square$

Using the same approach explored in [7, Section 3], it is possible to show the following result.

Proposition 2.14. *If $u_\epsilon \in W^{1,\Phi}(\mathbb{R}^N)$ is a nontrivial solution of* (2.4)*, then u_ϵ is positive, $u_\epsilon \in L^\infty(\mathbb{R}^N) \cap C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ and $\lim_{|x|\to+\infty} u_\epsilon(x) = 0$.*

3. Multiplicity of solutions for the original problem

After the study made in Section 2, the main goal this section is to prove that the solutions obtained are solutions for the original problem when ϵ is small enough. To do this, we will show three technical results.

Proposition 3.1. *Let $\epsilon_n \to 0$ and $(u_n) \subset \mathcal{N}_{\epsilon_n}$ be such that $J_{\epsilon_n}(u_n) \to d_0$. Then, there exists a sequence $(\widetilde{y}_n) \subset \mathbb{R}^N$, such that $v_n(x) = u_n(x + \widetilde{y}_n)$ has a convergent subsequence in $W^{1,\Phi}(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n \to y \in M$, where $y_n = \epsilon_n \widetilde{y}_n$.*

The proof of the above propositions follows from the proof of [7, Proposition 5.3].

Lemma 3.2. *Let $(x_j) \subset \overline{\Omega}_{\epsilon_j}$ and (ϵ_j) be sequences with $\epsilon_j \to 0$ as $j \to +\infty$. If $v_j(x) = u_{\epsilon_j}(x + x_j)$ where u_{ϵ_j} is a solution of $(\widetilde{P}_{\epsilon_j})$ given by Theorem 2.13, then (v_j) converges uniformly on compact subsets of $\mathbb{R}^N$.*

Proof. First of all, note that v_j is a solution to the problem

$$\begin{gathered} -\Delta_\Phi v_j + V_j(x)\phi(|v_j|)v_j = g(\epsilon_j x + \overline{x}_j, v_j) \quad \text{in } \mathbb{R}^N, \\ v_j \in W^{1,\Phi}(\mathbb{R}^N), \\ v_j > 0 \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{3.1}$$

where $V_j(x) = V(\epsilon_j x + \overline{x}_j)$ and $\overline{x}_j = \epsilon_j x_j$.

Next, let $x_0 \in \mathbb{R}^N$, $R_0 > 1$, $0 < t < s < 1 < R_0$ and $\xi \in C_0^\infty(\mathbb{R}^N)$ verify

$$0 \le \xi \le 1, \quad \operatorname{supp} \xi \subset B_s(x_0), \quad \xi \equiv 1 \text{ on } B_t(x_0), \quad |\nabla \xi| \le \frac{2}{s-t}.$$

For $\zeta \ge 1$, set $\eta_j = \xi^m (v_j - \zeta)_+$ and

$$Q_j = \int_{A_{j,\zeta,s}} \Phi(|\nabla v_j|)\xi^m \, dx,$$

where $A_{j,\zeta,\rho} = \{x \in B_\rho(x_0) : v_j(x) > \zeta\}$. Using η_j as a test function and combining (1.5) with (H1)(2), we obtain

$$\begin{aligned} lQ_j \le m &\int_{A_{j,\zeta,s}} \phi(|\nabla v_j|)|\nabla v_j|\, |\nabla \xi| \xi^{m-1}(v_j - \zeta)_+ dx \\ &- V_0 \int_{A_{j,\zeta,s}} \phi(|v_j|) v_j \xi^m (v_j - \zeta)_+ dx + \int_{A_{j,\zeta,s}} g(\epsilon_j x + \overline{x}_j, v_j)\xi^m (v_j - \zeta)_+ dx. \end{aligned}$$

Now, by repeating of the same arguments found in [7, Lemma 3.5], we obtain

$$Q_j \le c_1 \Big(\int_{A_{j,\zeta,s}} \big| \frac{v_j - \zeta}{s-t} \big|^{\gamma^*} dx + (\zeta^{\gamma^*} + 1)|A_{j,\zeta,s}| \Big).$$

Using the condition $(\mathcal{C}_m)$ and the definition of ξ, we obtain

$$\int_{A_{j,\zeta,t}} |\nabla v_j|^m dx \le c_2 \Big(\int_{A_{j,\zeta,s}} \big| \frac{v_j - \zeta}{s-t} \big|^{\gamma^*} dx + (\zeta^{\gamma^*} + 1)|A_{j,\zeta,s}| \Big),$$

where the constant c_2 does not depend of ζ and $\zeta \ge \zeta_0 \ge 1$, for some constant ζ_0.

Now, fix $R_1 > 0$ and define

$$\begin{gathered} \sigma_n = \frac{R_1}{2} + \frac{R_1}{2^{n+1}}, \quad \overline{\sigma}_n = \frac{\sigma_n + \sigma_{n+1}}{2}, \\ \zeta_n = \frac{\zeta_0}{2}\Big(1 - \frac{1}{2^{n+1}}\Big), \quad Q_{n,j} = \int_{A_{j,\zeta_n,\sigma_n}} \big((v_j - \zeta_n)_+\big)^{\gamma^*} dx. \end{gathered}$$

Arguing as in proof of [7, Lemma 3.6], we see that for each $j \in \mathbb{N}$,

$$Q_{n,j} \le C A^\eta Q_{n,j}^{1+\eta} \quad \forall n \in \mathbb{N},$$

where $C, \eta > 0$ are independent of n and $A > 1$. Now, we claim that

$$Q_{0,j} \le C^{\frac{1}{\eta}} A^{-1/\eta^2}, \quad \text{for } j \approx +\infty.$$

Indeed, by Proposition 3.1, we have $v_j \to v$ in $W^{1,\Phi}(\mathbb{R}^N)$. Therefore,

$$\limsup_{\zeta_0 \to +\infty} \Big(\limsup_{j \to +\infty} Q_{0,j} \Big) = \limsup_{\zeta_0 \to +\infty} \Big(\limsup_{j \to +\infty} \int_{A_{j,\zeta_0,\sigma_0}} (v_j - \frac{\zeta_0}{4})_+^{\gamma^*} dx \Big) = 0.$$

Then, there exist $j_0 \in \mathbb{N}$ and $\zeta_0^* > 0$ such that

$$Q_{0,j} \le C^{\frac{1}{\eta}} A^{-1/\eta^2} \quad \text{for } j \ge j_0 \text{ and } \zeta_0 \ge \zeta_0^*.$$

By [24, Lemma 4.7], $\lim_{n \to +\infty} Q_{n,j} = 0$ for $j \ge j_0$. On the other hand,

$$\lim_{n \to +\infty} Q_{n,j} = \lim_{n \to +\infty} \int_{A_{j,K_n,\sigma_n}} ((v_j - \zeta_n)_+)^{\gamma^*} dx = \int_{A_{j,\frac{\zeta}{2},\frac{R_1}{2}}} ((v_j - \frac{\zeta_0}{2})_+)^{\gamma^*} dx.$$

Then

$$\int_{A_{j,\frac{\zeta_0}{2},\frac{R_1}{2}}} ((v_j - \frac{\zeta_0}{2})_+)^{\gamma^*} dx = 0, \quad \forall j \geq j_0,$$

and so,

$$v_j(x) \leq \frac{\zeta_0}{2} \quad \text{a.e in } B_{\frac{R_1}{2}}(x_0), \quad \forall j \geq j_0.$$

Since $x_0 \in \mathbb{R}^N$ is arbitrary, we deduce that $v_j(x) \leq \zeta_0/2$ a.e in $\mathbb{R}^N$ for all $j \geq j_0$; that is,

$$\|v_j\|_\infty \leq \frac{\zeta_0}{2}, \quad \forall j \geq j_0.$$

Setting $C = \max\{\frac{\zeta_0}{2}, \|v_1\|_\infty, \dots, \|v_{j_0-1}\|_\infty\}$, we derive that $\|v_j\|_\infty \leq C$ for all $j \in \mathbb{N}$.

Combining the above estimate with regularity theory, we deduce that $(v_j) \subset C^{1,\alpha}_{\rm loc}(\mathbb{R}^N)$. Moreover, there is $v \in C^{1,\alpha}_{\rm loc}(\mathbb{R}^N)$ such that

$$v_j \to v \quad \text{in} \quad C^{1,\alpha}(B_{\rho_0}(0)) \quad \forall \rho_0 > 0.$$

□

Lemma 3.3. *Let (ϵ_n) be a sequence with $\epsilon_n \to 0$ and let $(x_n) \subset \overline{\Omega}_{\epsilon_n}$ be a sequence such that $u_{\epsilon_n}(x_n) \geq \tau_0 > 0$, for all $n \in \mathbb{N}$ and some $\tau_0 > 0$, where u_{ϵ_n} is the solution of (2.4) given by Theorem 2.13. Then $\lim_{n\to+\infty} V(\overline{x}_n) = V_0$, where $\overline{x}_n = \epsilon_n x_n$.*

Proof. As Ω is bounded and $\overline{x}_n \in \overline{\Omega}$, there exists $x_0 \in \overline{\Omega}$ such that, up to a subsequence, $\overline{x}_n \to x_0$. Then, the continuity of V leads to

$$\lim_{n\to+\infty} V(\overline{x}_n) = V(x_0) \geq V_0. \tag{3.2}$$

In the sequel, we will argue by contradiction, by supposing that

$$V(x_0) > V_0. \tag{3.3}$$

From Theorem 2.13, $(u_{\epsilon_n}) \subset \widetilde{\mathcal{N}}_{\epsilon_n}$. Thus,

$$c_{\epsilon_n} \leq J_{\epsilon_n}(u_{\epsilon_n}) < d_0 + h(\epsilon_n)$$

which implies $\limsup_n c_{\epsilon_n} \leq d_0$. On the other hand, once

$$E_0(tu) \leq J_{\epsilon_n}(tu), \quad \forall t \geq 0 \text{ and } \forall u \in W^{1,\Phi}(\mathbb{R}^N),$$

we derive the inequality

$$d_0 \leq \max_{t\geq 0} E_0(tu_{\epsilon_n}) \leq \max_{t\geq 0} J_{\epsilon_n}(tu_{\epsilon_n}) \quad \forall n \in \mathbb{N},$$

which leads to $d_0 \leq \liminf_n c_{\epsilon_n}$. Therefore,

$$J_{\epsilon_n}(u_{\epsilon_n}) \to d_0 \quad \text{and} \quad J'_{\epsilon_n}(u_{\epsilon_n})u_{\epsilon_n} = 0.$$

From this, (u_{ϵ_n}) is a bounded sequence in $W^{1,\Phi}(\mathbb{R}^N)$, and so, $v_n(z) = u_{\epsilon_n}(z + x_n)$ is also a bounded sequence in $W^{1,\Phi}(\mathbb{R}^N)$. Hence, there exists $v \in W^{1,\Phi}(\mathbb{R}^N)$ such that

$$v_n \rightharpoonup v \quad \text{in} \quad W^{1,\Phi}(\mathbb{R}^N). \tag{3.4}$$

Now, the above limit, the Lemma 3.2 and the inequality $u_{\epsilon_n}(x_n) \geq \tau_0 > 0$ combine to give $v \not\equiv 0$.

For all $n \in \mathbb{N}$, let $t_n > 0$ such that $t_n v_n \in \mathcal{M}_0$. Repeating the same arguments used in the proof of Lemma 2.10, we obtain $t_n \to t_0$. Setting $\widetilde{v}_n = t_n v_n$, we see that

$$\begin{aligned}
E_0(\widetilde{v}_n) &= \int_{\mathbb{R}^N} \Phi(|\nabla(t_n v_n)|)dx + V_0 \int_{\mathbb{R}^N} \Phi(|t_n v_n|)dx - \int_{\mathbb{R}^N} F(t_n v_n)dx \\
&\le \int_{\mathbb{R}^N} \Phi(|\nabla(t_n v_n)|)dx + \int_{\mathbb{R}^N} V(\epsilon_n z + \overline{x}_n)\Phi(|t_n v_n|)dx \\
&\quad - \int_{\mathbb{R}^N} G(\epsilon_n z + \overline{x}_n, t_n v_n)dx \\
&= \int_{\mathbb{R}^N} \Phi(|\nabla(t_n u_{\epsilon_n}|)dx + \int_{\mathbb{R}^N} V(\epsilon_n z)\Phi(|t_n u_{\epsilon_n}|)dx \\
&\quad - \int_{\mathbb{R}^N} G(\epsilon_n z, t_n u_{\epsilon_n})dx \\
&= J_{\epsilon_n}(t_n u_{\epsilon_n}) \\
&\le \max_{t \ge 0} J_{\epsilon_n}(t u_{\epsilon_n}) = J_{\epsilon_n}(u_{\epsilon_n}).
\end{aligned}$$

Thereby,

$$d_0 \le E_0(\widetilde{v}_n) \le d_0 + o_n(1),$$

or equivalently, $E_0(\widetilde{v}_n) \to d_{V_0}$. Applying [7, Proposition 5.3], we obtain

$$\widetilde{v}_n \to \widetilde{v} \quad \text{in} \quad W^{1,\Phi}(\mathbb{R}^N), \tag{3.5}$$

with $\widetilde{v} = t_0 v \not\equiv 0$. Moreover, $E_0(\widetilde{v}) = d_0$ and

$$d_0 < \int_{\mathbb{R}^N} \Phi(|\nabla \widetilde{v}|)dx + \int_{\mathbb{R}^N} V(x_0)\Phi(|\widetilde{v}|)dx - \int_{\mathbb{R}^N} F(\widetilde{v})dx.$$

By (3.5) and Fatou's Lemma,

$$\begin{aligned}
d_0 &< \liminf_{n \to +\infty} \Big[\int_{\mathbb{R}^N} \big(\Phi(|\nabla \widetilde{v}_n|) + V(\epsilon_n z + \overline{x}_n)\Phi(|\widetilde{v}_n|) - F(\widetilde{v}_n)\big)dx \Big] \\
&\le \liminf_{n \to +\infty} \Big[\int_{\mathbb{R}^N} \big(\Phi(|\nabla(t_n v_n)|) + V(\epsilon_n z + \overline{x}_n)\Phi(|t_n v_n|) \\
&\quad - G(\epsilon_n z + \overline{x}, t_n v_n)\big)dx \Big] \\
&= \liminf_{n \to +\infty} J_{\epsilon_n}(t_n u_{\epsilon_n}) \\
&\le \liminf_{n \to +\infty} J_{\epsilon_n}(u_{\epsilon_n}) = d_0
\end{aligned}$$

which is an absurd. Hence, from (3.2), $\lim_{n \to +\infty} V(\overline{x}_n) = V_0$. □

Our next lemma will permit us to conclude that the solutions of the auxiliary problem are solutions for the original problem for ϵ small enough.

Lemma 3.4. *If* $\kappa_\epsilon = \sup\big\{ \max_{\partial \Omega_\epsilon} u_\epsilon : u_\epsilon \in \widetilde{\mathcal{N}}_\epsilon$ *is a solution of* $(\widetilde{P}_\epsilon)\big\}$, *then*

$$\lim_{\epsilon \to 0} \kappa_\epsilon = 0. \tag{3.6}$$

Proof. Arguing by contradiction, we will assume that

$$\liminf_{\epsilon \to 0} \kappa_\epsilon > \tau_0 > 0,$$

for some $\tau_0 > 0$. From this, there is $(\epsilon_n) \subset (0, +\infty)$ and $x_n \in \partial\Omega_{\epsilon_n}$ such that

$$u_{\epsilon_n}(x_n) = \max_{x \in \partial\Omega_{\epsilon_n}} u_{\epsilon_n}(x) > \tau_0 \quad \forall n \in \mathbb{N}.$$

Applying the Lemma 3.3, $\lim_{n\to+\infty} V(\overline{x}_n) = V_0$, where $\overline{x}_n = \epsilon_n x_n$. Since $(\overline{x}_n) \subset \partial\Omega$, there exist $x_0 \in \partial\Omega$ such that, up to a subsequence, $\overline{x}_n \to x_0$. Then, by continuity of V, we have $V(x_0) = V_0$, which contradicts (1.6). Thereby, $\lim_{\epsilon\to 0} \kappa_\epsilon = 0$. ☐

Proof of Theorem 1.1. (i) Multiplicity of positive solutions. From Theorem 2.13, for any $\delta > 0$ there exists $\epsilon_\delta > 0$ such that (2.4) has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions, for any $0 < \epsilon < \epsilon_\delta$. Let u_ϵ be a these solutions. By Lemma 3.4 there exists $\overline{\epsilon} > 0$ such that

$$\kappa_\epsilon < t_0, \quad \forall \epsilon \in (0, \overline{\epsilon}).$$

Thus, $(u_\epsilon - t_0)_+ \in W_0^{1,\Phi}(\mathbb{R}^N \backslash \Omega_\epsilon)$ and

$$\omega_\epsilon(x) = \begin{cases} 0, & \text{if } x \in \Omega_\epsilon \\ (u_\epsilon - t_0)_+, & \text{if } x \in \mathbb{R}^N \backslash \Omega_\epsilon \end{cases}$$

belongs to $W^{1,\Phi}(\mathbb{R}^N)$. Using ω_ϵ as text function, we have

$$\begin{aligned} &\int_{\mathbb{R}^N\backslash\Omega_\epsilon} \phi(|\nabla u_\epsilon|)\nabla u_\epsilon \nabla (u_\epsilon - t_0)_+ dx + \int_{\mathbb{R}^N\backslash\Omega_\epsilon} V(\epsilon x)\phi(|u_\epsilon|)u_\epsilon(u_\epsilon - t_0)_+ dx \\ &= \int_{\mathbb{R}^N\backslash\Omega_\epsilon} g(\epsilon x, u_\epsilon)(u_\epsilon - t_0)_+ dx, \end{aligned}$$

which implies

$$\begin{aligned} &\big(1 - \frac{1}{k}\big)\Big[\int_{\mathbb{R}^N\backslash\Omega_\epsilon} \phi(|\nabla(u_\epsilon - t_0)_+|)|\nabla(u_\epsilon - t_0)_+|^2 dx \\ &+ V_0 \int_{\mathbb{R}^N\backslash\Omega_\epsilon} \phi(|u_\epsilon|)u_\epsilon(u_\epsilon - t_0)_+ dx\Big] \le 0. \end{aligned}$$

By Proposition 2.14, we know that $u_\epsilon > 0$, then the above inequality gives

$$\int_{\mathbb{R}^N\backslash\Omega_\epsilon} \phi(|u_\epsilon|)u_\epsilon(u_\epsilon - t_0)_+ dx = 0,$$

from where it follows that $u_\epsilon \le t_0$ in $\mathbb{R}^N\backslash\Omega_\epsilon$. Hence, u_ϵ is a solution of (1.7) and we can conclude that (1.7) has at least $\mathrm{cat}_{M_\delta}(M)$ positive solutions for $\epsilon \in (0, \epsilon_\delta)$.

(ii) Behavior of maximum points. Finally, if u_{ϵ_n} is a solution of problem (1.7) with ϵ_n instead of ϵ. Then, $v_n(x) = u_{\epsilon_n}(x + \widetilde{y}_n)$ is a solution of the problem

$$\begin{gathered} -\Delta_\Phi v_n + V_n(x)\phi(|v_n|)v_n = f(v_n) \quad \text{in } \mathbb{R}^N, \\ v_n \in W^{1,\Phi}(\mathbb{R}^N), \\ v_n > 0 \quad \text{in } \mathbb{R}^N, \end{gathered} \tag{3.7}$$

where $V_n(x) = V(\epsilon_n x + \epsilon_n \widetilde{y}_n)$ and $(\widetilde{y}_n)$ is the sequence obtained in Proposition 3.1. Moreover, up to a subsequence, $v_n \to v$ in $W^{1,\Phi}(\mathbb{R}^N)$ and $y_n \to y$ in M, where $y_n = \epsilon_n \widetilde{y}_n$. Applying [7, Proposition 6.1] and [7, Lemma 6.4], there are $R_0 > 0$ and $q_n \in B_{R_0}(0)$ such that $v_n(q_n) = \max_{z\in\mathbb{R}^N} v_n(z)$. Hence, $x_n = q_n + \widetilde{y}_n$ is a global maximum point of u_{ϵ_n} and $\epsilon_n x_n \to y$. Since V is a continuous function, it follows that

$$\lim_{n\to+\infty} V(\epsilon_n x_n) = V(y) = V_0,$$

and the concentration behavior is proved. $\square$

4. Appendix A: Existence of the function η.

In this appendix, we show the existence of η which was used in Section 2. In what follows, we fix ϱ small enough, such that the number $\lambda = a - \varrho$ satisfies

$$f'(\lambda) > \frac{V_0}{k}(m-1)\phi(\lambda). \tag{4.1}$$

Considering $h(s) = f(s)/\phi(s)$, we have that $h'(\lambda) > V_0/k$ and $h(\lambda) < V_0\lambda/k$. Now, note that the function

$$\widehat{h}(s) = \begin{cases} h(s), & \text{if } s \le a, \\ \frac{V_0}{k}s, & \text{if } s > a \end{cases}$$

satisfies $\widehat{h}(s) = \widehat{f}(s)/\phi(s)$ and

- $\frac{h(a)}{a} = \frac{f(a)}{\phi(a)a} = \frac{V_0}{k}$,
- $\frac{h(s)}{s} = \frac{f(s)}{\phi(s)s} = \frac{s^{m-2}}{\phi(s)}\frac{f(s)}{s^{m-1}} \uparrow$ for $s > 0$,
- $h'(\lambda) > \frac{V_0}{k}$,
- $B := \frac{V_0}{k}\lambda - h(\lambda) > 0$.

The next lemma is a key step to get the function η.

Lemma 4.1. *There exist $t_0, t_1 \in (0, +\infty)$ such that $t_0 < a < t_1$ and $\widetilde{\eta} \in C^1([t_0, t_1])$, satisfying*

1. *$\widetilde{\eta}(t) \le \widehat{h}(t)$, for all $t \in [t_0, t_1]$,*
2. *$\widetilde{\eta}(t_0) = \widehat{h}(t_0)$ and $\widetilde{\eta}(t_1) = \widehat{h}(t_1)$,*
3. *$(\widetilde{\eta})'(t_0) = (\widehat{h})'(t_0)$ and $(\widetilde{\eta})'(t_1) = (\widehat{h})'(t_1)$,*
4. *The function $t \to \frac{\widetilde{\eta}(t)}{t}$ is nondecreasing for all $s \in [t_0, t_1]$.*

Proof. In what follows, for each $\delta > 0$ small enough, we fix the numbers

$$\lambda = a - \delta, \quad B = h'(\lambda) > \frac{V_0}{k}, \quad D = \frac{V_0}{k}\lambda - h(\lambda)$$

where $\frac{V_0}{k} = \frac{h(a)}{a}$. Setting the function

$$y(t) = At^2 + Bt,$$

we have $y(0) = 0$ and $y'(0) = B$.

Next, our goal is proving that there are $A < 0$ and $T > \delta$ such that

$$y(T) = \frac{V_0}{k}T + D \quad \text{and} \quad y'(T) = \frac{V_0}{k}.$$

The above equalities are equivalent to the system

$$AT^2 + BT = \frac{V_0}{k}T + D$$
$$2AT + B = \frac{V_0}{k}$$

whose solution is

$$T = \frac{2D}{B - \frac{V_0}{k}} = \frac{2(\frac{V_0}{k}\lambda - h(\lambda))}{B - \frac{V_0}{k}} > \delta, \quad \text{if } \delta \approx 0^+,$$
$$A = -\frac{1}{4}\frac{(B - \frac{V_0}{k})^2}{D}.$$

Now, we set $\widetilde{\eta} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{\eta}(t) = y(t - \lambda) + h(\lambda).$$

Note that

$$\widetilde{\eta}(\lambda) = h(\lambda), \quad \widetilde{\eta}'(\lambda) = h'(\lambda), \quad \widetilde{\eta}(T + \lambda) = \frac{V_0}{k}(T + \lambda), \quad \widetilde{\eta}'(T + \lambda) = \frac{V_0}{k}.$$

A simple computation gives

$$\widetilde{\eta}(t) \le \widehat{h}(t), \quad \forall t \in \mathbb{R},$$
$$\widetilde{\eta}'(t)t - \widetilde{\eta}(t) = At^2 - A\lambda^2 + B\lambda - h(\lambda).$$

Thus,

$$\widetilde{\eta}'(t)t - \widetilde{\eta}(t) > 0 \iff At^2 - A\lambda^2 + B\lambda - h(\lambda) > 0.$$

Moreover,

$$At^2 - A\lambda^2 + B\lambda - h(\lambda) > 0 \iff -t_* < t < t_*$$

where

$$t_* = \sqrt{\lambda^2 - \frac{(B\lambda - f(\lambda))}{A}} = T + \lambda.$$

Therefore,

$$\widetilde{\eta}'(t)t - \widetilde{\eta}(t) > 0 \quad \forall t \in [\lambda, T + \lambda),$$

from where it follows that $\widetilde{\eta}(t)/t$ is increasing in $[a - \delta, a + \tau]$, where $\tau = T - \delta > 0$. $\square$

Using the above lemma, the function $\eta(t) = \phi(t)\widetilde{\eta}(t)$ satisfies the following conditions:

- $\eta(t) \le \phi(t)\widehat{h}(t) = \widehat{f}(t)$, for all $t \in [t_0, t_1]$;
- $\eta(t_0) = \phi(t_0)\widehat{h}(t_0) = \widehat{f}(t_0)$ and $\eta(t_1) = \phi(t_1)\widehat{h}(t_1) = \widehat{f}(t_1)$;
- $$\begin{aligned} \eta'(t_0) &= \phi'(t_0)\widetilde{\eta}(t_0) + \phi(t_0)(\widetilde{\eta})'(t_0) \\ &= \phi'(t_0)\widehat{h}(t_0) + \phi(t_0)(\widehat{h})'(t_0) \\ &= \phi'(t_0)h(t_0) + \phi(t_0)h'(t_0) \\ &= \big(\phi(t)h(t)\big)'(t_0) = f'(t_0); \end{aligned}$$
- $\eta'(t_1) = (\widehat{f})'(t_1)$;
- $\frac{\eta(s)}{\phi(s)s} = \frac{\phi(s)\widetilde{\eta}(s)}{\phi(s)s} = \frac{\widetilde{\eta}(s)}{s}$;

From this, the function η satisfies conditions (H5) mentioned in Section 2.

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