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# A UNIQUENESS THEOREM ON THE INVERSE PROBLEM FOR THE DIRAC OPERATOR

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Abstract. In this article, we consider an inverse problem for the Dirac operator. We show that a particular set of eigenvalues is sufficient to determine the unknown potential functions.

## 1. Introduction

The inverse spectral problem for a differential operator consists of recovering the operator from its spectral data. In 1929, Ambarzumyan [2] was the first to discuss the following statement

> If $q \in C[0,\pi]$ and $\{n^2 : n = 0,1,2,\dots\}$ is the spectral set of the boundary value problem
> $$-y'' + q(x)y = \lambda y, \quad x \in [0,\pi], \tag{1.1}$$
> with Neumann boundary conditions
> $$y'(0,\lambda) = y'(\pi,\lambda) = 0, \tag{1.2}$$
> then $q(x) \equiv 0$ in $[0,\pi]$.

McLaughlin and Rundell [17] discussed the inverse problem for the Sturm-Liouville equation (1.1) with the boundary conditions $y(0,\lambda) = 0$ and $y'(\pi,\lambda) + H_k y(\pi,\lambda) = 0$ and showed that the spectral data, for a fixed $n$ ($n = 0,1,2,\dots$), $\{\lambda_n(q,H_k)\}_{k=1}^{+\infty}$ is equivalent to two spectra of boundary value problems with the equation (1.1) and one common boundary condition at $x = 0$ and two different boundary conditions at $x = \pi$. By using McLaughlin and Rundell's method [17], Koyunbakan [13] considered a singular Sturm-Liouville problem. Using the spectral data in [17] and Hochstadt and Lieberman's method, Wang [28] discussed the inverse problem for indefinite Sturm-Liouville operators on the finite interval $[a,b]$. However, we are motivated by inverse spectral problems for Dirac operators with the above spectral data which are particular set of eigenvalues. As far as we know, inverse spectral problems for Dirac operators have not been considered with the spectral data before.

We consider the system of Dirac operators $L := L(p,q,H_k)$
$$Ly = By' + Q(x)y = \lambda y, \quad 0 \le x \le \pi, \tag{1.3}$$

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with the boundary conditions

$$y_1(0, \lambda) = 0, \tag{1.4}$$

$$y_2(\pi, \lambda) + H_k y_1(\pi, \lambda) = 0, \tag{1.5}$$

where

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} p(x) & q(x) \\ q(x) & -p(x) \end{pmatrix}, \quad y(x) = \begin{pmatrix} y_1(x) \\ y_2(x) \end{pmatrix}$$

and $(p(x), q(x))$ are potential functions which are real valued functions in space $L^2[0, \pi]$.

We consider another Dirac operator $\widetilde{L} := \widetilde{L}(\widetilde{p}, \widetilde{q}, H_k)$ which is defined as:

$$\widetilde{L}y = B\widetilde{y}' + \widetilde{Q}(x)\widetilde{y} = \lambda\widetilde{y}, \quad 0 \le x \le \pi, \tag{1.6}$$

with the boundary conditions

$$\widetilde{y}_1(0, \lambda) = 0, \tag{1.7}$$

$$\widetilde{y}_2(\pi, \lambda) + H_k \widetilde{y}_1(\pi, \lambda) = 0, \tag{1.8}$$

where

$$\widetilde{Q}(x) = \begin{pmatrix} \widetilde{p}(x) & \widetilde{q}(x) \\ \widetilde{q}(x) & -\widetilde{p}(x) \end{pmatrix},$$

$H_k \in \mathbb{R}$, $0 < H_1 < H_2 < \dots < H_k < H_{k+1} < \dots$, the potentials $(\widetilde{p}(x), \widetilde{q}(x))$ are real valued functions, $(\widetilde{p}(x), \widetilde{q}(x)) \in L^2[0, \pi]$ and $\lambda$ is a spectral parameter.

The Dirac equation is a modern presentation of the relativistic quantum mechanics of electrons intended new mathematical outcomes accessible to a wider audience. It treats in some dept the relativistic invariance of a quantum theory, self-adjointness and spectral theory, qualitative features of relativistic bound and scattering states and the external field problem in quantum electrodynamics, without neglecting the interpretational difficulties and limitations of the theory.

Inverse problems for Dirac system were studied by Moses [18], Prats and Toll [24], Verde [27], Gasymov and Levitan [7] and Panakhov [22, 23]. It is well known [8] that two spectra uniquely determine the matrix-valued potential function. In particular, in reference [11], eigenfunction expansions for one dimensional Dirac operators describing the motion of a particle in quantum mechanics are discussed.

Direct or inverse spectral problem for Dirac and Sturm-Liouville operators were extensively studied in [1, 3, 4, 5, 9, 12, 15, 18, 20, 21, 25, 26, 28]. However, the results on direct or inverse spectral problems of the Dirac operator are less than classical Sturm-Liouville operator, this leads to additional difficulties in connection with inverse spectral problem by the spectral data in [17].

In this article, by using the spectral data in [17] and Hochstadt and Lieberman's [10] method a uniqueness theorem for Dirac operator on the interval $[0, \pi]$ will be established, i.e., for a fixed index $n$ ($n = 0, \pm 1, \pm 2, \dots$), we show that if the spectral set $\{\lambda_n(p, q, H_k)\}_{k=1}^{+\infty}$ for distinct $H_k$ can be measured, then the spectral set $\{\lambda_n(p, q, H_k)\}_{k=1}^{+\infty}$ is sufficient to determine the potential functions $(p(x), q(x))$.

**Lemma 1.1** ([14]). *Let the function* $\varphi(x, \lambda) = \begin{pmatrix} \varphi_1(x, \lambda) \\ \varphi_2(x, \lambda) \end{pmatrix}$ *be the solution of* (1.3) *satisfying the initial condition*

$$\varphi(0, \lambda) = \begin{pmatrix} \varphi_1(0, \lambda) \\ \varphi_2(0, \lambda) \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} = (0, -1)^T.$$

*Then $\varphi(x,\lambda)$ satisfies the integral equations*

$$\varphi(x,\lambda)=\begin{pmatrix}\sin\lambda x\\ -\cos\lambda x\end{pmatrix}+\int_0^x K(x,t)\begin{pmatrix}\sin\lambda t\\ -\cos\lambda t\end{pmatrix}dt, \tag{1.9}$$

*where kernel $K(x,t)$ is symmetric matrix-valued functions whose entries are continuously differentiable in both of its variables.*

**Lemma 1.2.** *The eigenvalues $\lambda_n$ $(n\neq 0)$ of the boundary-value problem (1.3)-(1.5) for the coefficient $H=H_k$ in (1.5) are the roots of (1.5) and satisfy the asymptotic formulae:*

$$\lambda_n=\lambda_n^0+\epsilon_n, \tag{1.10}$$

*where $\{\epsilon_n\}\in l_2$ ($l_2$ consist of sequences $\{x_n\}$ such that $\sum_{n=1}^{\infty}|x_n|^2<\infty$) and $\lambda_n^0$ are the zeros of $\Delta_0(\lambda):=-\cos\lambda\pi+H\sin\lambda\pi$, i.e.,*

$$\lambda_n^0=n+\frac{1}{\pi}\arctan\frac{1}{H}.$$

*Proof.* Let $\varphi(x,\lambda)=\begin{pmatrix}\varphi_1(x,\lambda)\\ \varphi_2(x,\lambda)\end{pmatrix}$ be the solution of (1.3) satisfying the initial condition

$$\varphi(0,\lambda)=\begin{pmatrix}\varphi_1(0,\lambda)\\ \varphi_2(0,\lambda)\end{pmatrix}=\begin{pmatrix}0\\ -1\end{pmatrix},$$

and $\varphi_1(0,\lambda)=0,\varphi_2(\pi,\lambda)+H\varphi_1(\pi,\lambda)=0$. The characteristic function $\Delta(\lambda)$ of the problem $L$ is defined by the following relation:

$$\Delta(\lambda)=\varphi_2(\pi,\lambda)+H\varphi_1(\pi,\lambda),$$

and the zeros of $\Delta(\lambda)$ coincide with the eigenvalues of the problem $L$.

Using Lemma 1.1, we obtain

$$\begin{aligned}\Delta(\lambda)=-\cos\lambda\pi+H\sin\lambda\pi&+\int_0^\pi (K_{21}(\pi,t)+HK_{11}(\pi,t))\sin\lambda t dt\\ &-\int_0^\pi (K_{22}(\pi,t)+HK_{12}(\pi,t))\cos\lambda t dt.\end{aligned}$$

Since the eigenvalues are zeros of $\Delta(\lambda)$, we can write the equation

$$\begin{aligned}&-\cos\lambda\pi+H\sin\lambda\pi+\int_0^\pi (K_{21}(\pi,t)+HK_{11}(\pi,t))\sin\lambda t\,dt\\ &-\int_0^\pi (K_{22}(\pi,t)+HK_{12}(\pi,t))\cos\lambda t dt=0.\end{aligned}$$

Denote

$$\begin{gathered}G_n=\{\lambda\in\mathbb{C}:|\lambda|=|\lambda_n^0|+\beta,\ n=0,\pm1,\pm2,\dots\},\\ G_\delta=\{\lambda:|\lambda-\lambda_n^0|\geq\delta,\ n=0,\pm1,\pm2,\dots\},\end{gathered}$$

where $\delta$ is sufficiently small number ($\delta\ll\beta$).

Since $|\Delta_0(\lambda)|>C_\delta\exp(|\tau|\pi)$ for $\lambda\in\overline{G}_\delta$, from [16],

$$\begin{aligned}&\lim_{|\lambda|\to\infty}e^{-|\tau|\pi}(\Delta(\lambda)-\Delta_0(\lambda))\\ &=\lim_{|\lambda|\to\infty}\Big(e^{-|\tau|\pi}\int_0^\pi (K_{21}(\pi,t)+HK_{11}(\pi,t))\sin\lambda t\,dt\end{aligned}$$

$$- e^{-|\tau|\pi}\int_0^\pi (K_{22}(\pi,t)+HK_{12}(\pi,t))\cos\lambda t\,dt\Big)=0$$

and $|\Delta(\lambda)-\Delta_0(\lambda)|<C_\delta\exp(|\tau|\pi)$ for sufficiently large $n$ and $\lambda\in G_n$, we have

$$|\Delta_0(\lambda)|>|\Delta(\lambda)-\Delta_0(\lambda)|,$$

where $\tau=\operatorname{Im}\lambda$.

Using the Rouché's theorem, we conclude that, for sufficiently large $n$, the functions $\Delta_0(\lambda)$ and $\Delta_0(\lambda)+\{\Delta(\lambda)-\Delta_0(\lambda)\}=\Delta(\lambda)$ have the same number of zeros inside the contour $G_n$, namely $2n+1$ zeros $\lambda_{-n},\dots,\lambda_0,\dots,\lambda_n$. Thus the eigenvalues $\lambda_n$ are of the form $\lambda_n=\lambda_n^0+\epsilon_n$, where

$$\lim_{n\to\infty}\epsilon_n=0.$$

Substituting $\lambda_n^0+\epsilon_n$ for $\lambda_n$ in the last equality and using the fact that $\Delta_0(\lambda_n^0+\epsilon_n)=\Delta_0'(\lambda_n^0)[1+o(1)]\epsilon_n$. We conclude that $\epsilon_n\in l_2$. The proof is complete. $\square$

## 2. Main results and proofs

**Lemma 2.1.** *Let $\sigma(L_{k_j}):=\{\lambda_n(p,q,H_{k_j})\}$ $(j=1,2)$ be the spectrum of the boundary value problem* (1.3)-(1.5) *for the coefficient $H_{k_j}$. If $H_{k_1}\neq H_{k_2}$, then*

$$\sigma(L_{k_1})\cap\sigma(L_{k_2})=\emptyset, \tag{2.1}$$

*where $k_j\in\mathbb{N}$, $\emptyset$ denotes an empty set.*

**Lemma 2.2.** *Let $\lambda_n(p,q,H_k)$ be the $n$-th eigenvalue of the boundary-value problem* (1.3)-(1.5). *Then the spectral set $\{\lambda_n(p,q,H_k)\}_{k=1}^{+\infty}$ is a bounded infinite set.*

The above lemema plays an important role in the proof of next theorem.

**Theorem 2.3.** *Let $\lambda_n(p,q,H_k)$ be the $n$-th eigenvalue of the boundary-value problem* (1.3)-(1.5) *and $\lambda_n(\widetilde{p},\widetilde{q},H_k)$ be the $n$-th eigenvalue of the boundary-value problem* (1.6)-(1.8)*, for a fixed index $n(n\in\mathbb{Z})$. If*

$$\lambda_n(p,q,H_k)=\lambda_n(\widetilde{p},\widetilde{q},H_k)\quad \textit{for all } k\in\mathbb{N},$$

*then*

$$(p(x),q(x))=(\widetilde{p}(x),\widetilde{q}(x))\quad \textit{a.e. on } [0,\pi].$$

*Proof of Lemma 2.1.* Suppose that the conclusion is false. Denote $\lambda_{n_j}(H_{k_j})=\lambda_{n_j}(p,q,H_{k_j})$, $j=1,2$. Then there exists $\lambda_{n_1}(H_{k_1})=\lambda_{n_2}(H_{k_2})\in\mathbb{R}$, where $\lambda_{n_j}(H_{k_j})\in\sigma(L_{k_j})$ $n_j\in\mathbb{Z}$. Let $\varphi_j(x,\lambda_{n_j}(H_{k_j}))$ be the solution of (1.3)-(1.5) corresponding to the eigenvalue $\lambda_{n_j}(H_{k_j})$ and that it satisfies the initial conditions $\varphi_{j,1}(0,\lambda_{n_j}(H_{k_j}))=0$ where $\varphi_j=(\varphi_{j,1},\varphi_{j,2})^T$. We get

$$B\varphi_1'(x,\lambda_{n_1}(H_{k_1}))+Q(x)\varphi_1(x,\lambda_{n_1}(H_{k_1}))=\lambda_{n_1}(H_{k_1})\varphi_1(x,\lambda_{n_1}(H_{k_1})), \tag{2.2}$$

and

$$B\varphi_2'(x,\lambda_{n_2}(H_{k_2}))+Q(x)\varphi_2(x,\lambda_{n_2}(H_{k_2}))=\lambda_{n_2}(H_{k_2})\varphi_2(x,\lambda_{n_2}(H_{k_2})). \tag{2.3}$$

If we multiply (2.2) by $\varphi_2(x,\lambda_{n_2}(H_{k_2}))$, and (2.3) by $\varphi_1(x,\lambda_{n_1}(H_{k_1}))$ respectively (in the sense of scalar product i.e.

$$\langle(\varphi_{1,1},\varphi_{1,2})^T,(\varphi_{2,1},\varphi_{2,2})^T\rangle=\varphi_{1,1}\varphi_{2,1}+\varphi_{1,2}\varphi_{2,2}$$

and subtract from each other and integrate from 0 to $\pi$, we obtain

$$\varphi_{2,2}(x,\lambda_{n_2}(H_{k_2}))\varphi_{1,1}(x,\lambda_{n_1}(H_{k_1}))-\varphi_{2,1}(x,\lambda_{n_2}(H_{k_2}))\varphi_{1,2}(x,\lambda_{n_1}(H_{k_1}))\,|_{x=0}^{\pi}=0. \tag{2.4}$$

Using the initial conditions, we obtain

$$\varphi_{2,2}(\pi,\lambda_{n_2}(H_{k_2}))\varphi_{1,1}(\pi,\lambda_{n_1}(H_{k_1}))-\varphi_{2,1}(\pi,\lambda_{n_2}(H_{k_2}))\varphi_{1,2}(\pi,\lambda_{n_1}(H_{k_1}))=0. \tag{2.5}$$

On the other hand, note the equality

$$\begin{aligned} &\varphi_{2,2}(\pi,\lambda_{n_2}(H_{k_2}))\varphi_{1,1}(\pi,\lambda_{n_1}(H_{k_1}))-\varphi_{2,1}(\pi,\lambda_{n_2}(H_{k_2}))\varphi_{1,2}(\pi,\lambda_{n_1}(H_{k_1}))\\ &=\varphi_{1,1}(\pi,\lambda_{n_1}(H_{k_1}))[\varphi_{2,2}(\pi,\lambda_{n_2}(H_{k_2}))+H_{k_2}\varphi_{2,1}(\pi,\lambda_{n_2}(H_{k_2}))]\\ &\quad-\varphi_{2,1}(\pi,\lambda_{n_2}(H_{k_2}))[\varphi_{1,2}(\pi,\lambda_{n_1}(H_{k_1}))+H_{k_1}\varphi_{1,1}(\pi,\lambda_{n_1}(H_{k_1}))]\\ &\quad+(H_{k_1}-H_{k_2})\varphi_{1,1}(\pi,\lambda_{n_1}(H_{k_1}))\varphi_{2,1}(\pi,\lambda_{n_2}(H_{k_2}))\\ &=(H_{k_1}-H_{k_2})\varphi_{1,1}(\pi,\lambda_{n_1}(H_{k_1}))\varphi_{2,1}(\pi,\lambda_{n_2}(H_{k_2})). \end{aligned} \tag{2.6}$$

Since $H_{k_1}-H_{k_2}\neq 0$, if $\varphi_{1,1}(\pi,\lambda_{n_1}(H_{k_1}))\varphi_{2,1}(\pi,\lambda_{n_2}(H_{k_2}))=0$, then

$$\varphi_{1,1}(\pi,\lambda_{n_1}(H_{k_1}))=0 \text{ or } \varphi_{2,1}(\pi,\lambda_{n_2}(H_{k_2}))=0. \tag{2.7}$$

This and (1.5) yield

$$\varphi_{1,1}(\pi,\lambda_{n_1}(H_{k_1}))=\varphi_{1,2}(\pi,\lambda_{n_1}(H_{k_1}))=0, \tag{2.8}$$

or

$$\varphi_{2,1}(\pi,\lambda_{n_2}(H_{k_2}))=\varphi_{2,2}(\pi,\lambda_{n_2}(H_{k_2}))=0. \tag{2.9}$$

Then (2.8) and (2.9) yield

$$\varphi_1(x,\lambda_{n_1}(H_{k_1}))\equiv 0 \quad\text{or}\quad \varphi_2(x,\lambda_{n_2}(H_{k_2}))\equiv 0 \quad\text{on } [0,\pi], \tag{2.10}$$

where $\varphi_1=(\varphi_{1,1},\varphi_{1,2})^T$ and $\varphi_2=(\varphi_{2,1},\varphi_{2,2})^T$. This is impossible. Thus, we obtain

$$\varphi_{2,2}(\pi,\lambda_{n_2}(H_{k_2}))\varphi_{1,1}(\pi,\lambda_{n_1}(H_{k_1}))-\varphi_{2,1}(\pi,\lambda_{n_2}(H_{k_2}))\varphi_{1,2}(\pi,\lambda_{n_1}(H_{k_1}))\neq 0. \tag{2.11}$$

It is obvious that the contradiction between (2.5) and (2.11) implies that (2.1) holds. Hence the proof is complete. □

*Proof of Lemma 2.2.* We prove the lemma by two steps. For the problem $L_1 := L_1(q)$, $\mu_n$ is the $n$-th eiegenvalue with boundary conditions $\varphi_1(0)=\varphi_1(\pi)=0$, for the problem $L_2 := L_2(q,h)$, $\lambda_n$ is the $n$-th eigenvalue problem (1.3)-(1.5) with $H=H_k$.

**Step 1.** We show that (see [6])

$$\mu_n<\lambda_n\leq\mu_{n+1}. \tag{2.12}$$

From the Green identity, we have

$$\begin{aligned} &[\varphi_2(x,\lambda)\varphi_1(x,\mu)-\varphi_1(x,\lambda)\varphi_2(x,\mu)]\big|_{x=0}^{x=\pi}\\ &=(\mu-\lambda)\int_0^\pi[\varphi_1(x,\mu)\varphi_1(x,\lambda)+\varphi_2(x,\mu)\varphi_2(x,\lambda)]dx. \end{aligned}$$

Hence, we have

$$\begin{aligned} &(\mu-\lambda)\int_0^\pi[\varphi_1(x,\mu)\varphi_1(x,\lambda)+\varphi_2(x,\mu)\varphi_2(x,\lambda)]dx\\ &=[\varphi_2(\pi,\lambda)\varphi_1(\pi,\mu)-\varphi_1(\pi,\lambda)\varphi_2(\pi,\mu)]=d(\mu)\Delta(\lambda)-d(\lambda)\Delta(\mu), \end{aligned}$$

where $d(\mu)=\varphi_1(\pi,\mu)$, $\Delta(\lambda)=\varphi_2(\pi,\lambda)+H\varphi_1(\pi,\lambda)$.

When $\lambda \to \mu$, we obtain

$$\int_0^\pi [\varphi_1^2(x,\mu) + \varphi_2^2(x,\mu)]dx = d^{\cdot}(\mu)\Delta(\mu) - d(\mu)\Delta^{\cdot}(\mu),$$

with $\Delta^{\cdot}(\mu) = \frac{d}{d\mu}\Delta(\mu)$ and $d^{\cdot}(\mu) = \frac{d}{d\mu}d(\mu)$. In particular, this yields

$$\alpha_n = -\Delta^{\cdot}(\mu_n)d(\mu_n),$$
$$\frac{1}{d^2(\mu)}\int_0^\pi [\varphi_1^2(x,\mu) + \varphi_2^2(x,\mu)]dx = -\frac{d}{d\mu}(\frac{\Delta(\mu)}{d(\mu)}),$$

for $-\infty < \mu < \infty$ and $d(\mu) \neq 0$, where $\alpha_n$ are norming constants.

Thus the function $\frac{\Delta(\mu)}{d(\mu)}$ is monotonically decreasing on $R - \{\lambda_n : n \in Z\}$ with

$$\lim_{\mu \to \lambda_n} \frac{\Delta(\mu)}{d(\mu)} = \pm\infty.$$

Consequently from the asymptotic behavior of $\lambda_n$ and $\mu_n$, we prove (2.12).

**Step 2.** We show that the following formula holds,

$$\lambda_n(H_0) < \dots < \lambda_n(H_{k+1}) < \lambda_n(H_k) < \dots. \tag{2.13}$$

Let $\varphi(x, \lambda_n(H))$ be the solution of the boundary value problem (1.3)-(1.5) corresponding to the eigenvalue $\lambda_n(H)$ and that it satisfies the initial conditions $\varphi_1(0, \lambda_n(H)) = 0$, $\varphi_2(0, \lambda_n(H)) = -1$ and $\varphi_1(0, \lambda_n(H + \Delta H)) = 0$, $\varphi_2(0, \lambda_n(H + \Delta H)) = -1$. We have

$$B\varphi'(x, \lambda_n(H)) + Q(x)\varphi(x, \lambda_n(H)) = \lambda_n(H)\varphi(x, \lambda_n(H)), \tag{2.14}$$

$$\begin{aligned} &B\varphi'(x, \lambda_n(H + \Delta H)) + Q(x)\varphi(x, \lambda_n(H + \Delta H)) \\ &= \lambda_n(H + \Delta H)\varphi(x, \lambda_n(H + \Delta H)), \end{aligned} \tag{2.15}$$

where $\Delta H$ is the increment of $H$. Multiplying (2.14) by $\varphi(x, \lambda_n(H + \Delta H))$, and multiplying (2.15) by $\varphi(x, \lambda_n(H))$ and subtracting from each other and integrating from 0 to $\pi$, we obtain, from the initial conditions at zero,

$$\begin{aligned} &\Delta\lambda_n(H) \int_0^\pi \Big[\varphi_1(x, \lambda_n(H))\varphi_1(x, \lambda_n(H + \Delta H)) \\ &+ \varphi_2(x, \lambda_n(H))\varphi_2(x, \lambda_n(H + \Delta H))\Big]dx \\ &= \Delta H \varphi_1(\pi, \lambda_n(H))\varphi_1(\pi, \lambda_n(H + \Delta H)), \end{aligned} \tag{2.16}$$

where $\Delta\lambda_n(H) = \lambda_n(H + \Delta H) - \lambda_n(H)$.

It is well known that $\varphi(x, \lambda_n(H))$ and $\lambda_n(H)$ are real and continuous with respect to $H$. Letting $\Delta H \to 0$, we have

$$\frac{\partial\lambda_n(H)}{\partial H} = \frac{\varphi_1^2(\pi, \lambda_n(H))}{\int_0^\pi [\varphi_1^2(x, \lambda_n(H)) + \varphi_2^2(x, \lambda_n(H))]dx} > 0. \tag{2.17}$$

This implies that (2.13) holds. Therefore, from Step 1 and Step 2 we have that the spectral set $\{\lambda_n(p, q, H_k)\}_{k=1}^\infty$ is a bounded infinite set. The proof is complete $\square$

Finally, using Lemma 2.2, the properties of entire functions and the result of [29], we have shown that Theorem 2.3 holds.

*Proof of Theorem 2.3.* By Lemma 1.1 the solutions to Equation (1.3) satisfying $\varphi(0,\lambda) = (0,-1)^T$, and solutions of (1.6) satisfying $\widetilde{\varphi}(0,\lambda) = (0,-1)^T$ can be respectively expressed in the integral forms:

$$\varphi(x,\lambda) = \begin{pmatrix} \sin\lambda x \\ -\cos\lambda x \end{pmatrix} + \int_0^x K(x,t) \begin{pmatrix} \sin\lambda t \\ -\cos\lambda t \end{pmatrix} dt, \tag{2.18}$$

$$\widetilde{\varphi}(x,\lambda) = \begin{pmatrix} \sin\lambda x \\ -\cos\lambda x \end{pmatrix} + \int_0^x \widetilde{K}(x,t) \begin{pmatrix} \sin\lambda t \\ -\cos\lambda t \end{pmatrix} dt, \tag{2.19}$$

where kernels $K(x,t)$ and $\widetilde{K}(x,t)$ are symmetric matrix-valued functions whose entries are continuously differentiable in both of its variables.

If we multiply (1.3) by $\widetilde{\varphi}(x,\lambda)$ and (1.6) by $\varphi(x,\lambda)$ respectively (in the sense of scalar product in $\mathbb{R}^2$) and subtract from each other, then we obtain

$$\frac{d}{dx}\{\varphi_1(x,\lambda)\widetilde{\varphi}_2(x,\lambda) - \widetilde{\varphi}_1(x,\lambda)\varphi_2(x,\lambda)\} = \langle [Q(x) - \widetilde{Q}(x)]\varphi(x,\lambda), \widetilde{\varphi}(x,\lambda)\rangle. \tag{2.20}$$

Integrating the last equality from 0 to $\pi$ with respect to the variable $x$, we give

$$\{\varphi_1(x,\lambda)\widetilde{\varphi}_2(x,\lambda) - \widetilde{\varphi}_1(x,\lambda)\varphi_2(x,\lambda)\}\big|_{x=0}^{\pi} = \int_0^\pi \langle [Q(x) - \widetilde{Q}(x)]\varphi(x,\lambda), \widetilde{\varphi}(x,\lambda)\rangle dx. \tag{2.21}$$

Because $\varphi(x,\lambda)$ and $\widetilde{\varphi}(x,\lambda)$ satisfy the same initial conditions, it follows that

$$\varphi_1(0,\lambda)\widetilde{\varphi}_2(0,\lambda) - \widetilde{\varphi}_1(0,\lambda)\varphi_2(0,\lambda) = 0. \tag{2.22}$$

Define

$$P(x) = Q(x) - \widetilde{Q}(x), \quad p_1(x) = p(x) - \widetilde{p}(x), \quad q_1(x) = q(x) - \widetilde{q}(x), \tag{2.23}$$

and

$$H(\lambda) := \int_0^\pi \langle P(x)\varphi(x,\lambda), \widetilde{\varphi}(x,\lambda)\rangle dx. \tag{2.24}$$

Considering the properties of $\varphi(x,\lambda)$ and $\widetilde{\varphi}(x,\lambda)$, we conclude that $H(\lambda)$ is an entire function in $\lambda$. Because the first term of (2.21) for $\lambda = \lambda_n(p,q,H_k)$ and $x = \pi$ is zero, then

$$H(\lambda_n(p,q,H_k)) = 0.$$

From Lemma 2.2, we see that the spectral set $\{\lambda_n(p,q,H_k)\}_{k=1}^{+\infty}$ is a bounded infinite set. Hence, there exists $\lambda_{n0}(p,q) \in \mathbb{R}$, such that $\lambda_{n0}(p,q)$ is a finite accumulation point of the spectrum set $\{\lambda_n(p,q,H_k)\}_{k=1}^{+\infty}$. It is well known that the set of zeros of every entire function which is not identically zero hasn't any finite accumulation point. Therefore

$$H(\lambda) = 0, \quad \forall\lambda \in \mathbb{C}.$$

We can show from (2.24) that

$$\begin{aligned} H(\lambda) = \int_0^\pi p_1(x)\Big\{ &-\cos 2\lambda x + \int_0^x R_1(x,t)\exp(2i\lambda t)dt \\ &- \int_0^x R_2(x,t)\exp(-2i\lambda t)dt\Big\}dx + \int_0^\pi q_1(x)\Big\{ -\sin 2\lambda x \\ &+ \int_0^x R_3(x,t)\exp(2i\lambda t)dt + \int_0^x R_4(x,t)\exp(-2i\lambda t)dt\Big\}dx = 0 \end{aligned} \tag{2.25}$$

where $R_l(x,t)$, $l=\overline{1,4}$ are piecewise-continuously differentiable on $0 \le t \le x \le \pi$. Moreover, by using Euler's formula, (2.25) can be written as

$$\begin{aligned} &\int_0^\pi f_1(x)\Big\{\exp(2i\lambda x)+\int_0^x S_{11}(x,t)\exp(2i\lambda t)dt \\ &+\int_0^x S_{12}(x,t)\exp(-2i\lambda t)dt\Big\}dx+\int_0^\pi f_2(x)\{\exp(-2i\lambda x) \\ &+\int_0^x S_{21}(x,t)\exp(2i\lambda t)dt+\int_0^x S_{22}(x,t)\exp(-2i\lambda t)dt\}dx=0, \end{aligned} \tag{2.26}$$

where

$$f_1(x)=-\frac{1}{2i}(q_1(x)+ip_1(x)),\quad f_2(x)=\frac{1}{2i}(q_1(x)-ip_1(x)),\quad i=\sqrt{-1}, \tag{2.27}$$

and the matrix $S(x,t)=(S_{ij}(x,t))$, $i,j=1,2$ with entries being piecewise-continuously differentiable on $0\le t\le x\le\pi$. By changing the order of integration, (2.26) can be written as

$$\begin{aligned} &\int_0^\pi \exp(2i\lambda t)\Big[f_1(t)+\int_t^\pi (f_1(x)S_{11}(x,t)+f_2(x)S_{21}(x,t))dx]dt \\ &+\int_0^\pi \exp(-2i\lambda t)\Big[f_2(t)+\int_t^\pi (f_1(x)S_{12}(x,t)+f_2(x)S_{22}(x,t))dx\Big]dt=0, \end{aligned}$$

or

$$\int_0^\pi \Big\langle e_0(\lambda t),\ f(t)+\int_t^\pi S(x,t)f(x)dx\Big\rangle dt=0. \tag{2.28}$$

Here $e_0(\lambda t)=(\exp(2i\lambda t),\exp(-2i\lambda t))^{\ T}$ and $f(x)=(f_1(x),\ f_2(x))^{\ T}$. Thus from the completeness of the functions $e_0(\lambda t)$, it follows that

$$f(t)+\int_t^\pi S(x,t)f(x)dx=0,\quad \text{for } 0<t<\pi.$$

But this equation is a homogeneous Volterra integral equation and has only the zero solution. Thus we have $f(x)=(f_1(x),\ f_2(x))^T=0$ on the interval $[0,\pi]$. From (2.27)), it holds

$$q_1(x)+ip_1(x)=0=q_1(x)-ip_1(x),$$

i.e. $q_1(x)=0$ and $p_1(x)=0$. From (2.23) we obtain

$$(p(x),q(x))=(\widetilde{p}(x),\widetilde{q}(x)),$$

a.e. on $[0,\pi]$. This result completes the proof. □

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## References


[1] M. S. Agranovich; *Spectral problems for the Dirac system with spectral parameter in local boundary conditions*, Functional Analysis and its Applications, 35 (2001), 161-175.

[2] V. A. Ambarzumyan; *Über eine frage der eigenwerttheorie*, Zeitschrift für Physik, 53 (1929), 690-695.

[3] E. Bairamov, Y. Aygar, T. Koprubasi; *The spectrum of eigenparameter-dependent discrete Sturm–Liouville equations*, Journal of computational and applied mathematics, 235 (2011), 4519-4523.

[4] A. Boumenir; *The sampling method for sturm-liouville problems with the eigenvalue parameter in the boundary condition*, Numerical functional analysis and optimization, 21 (2000), 67-75.

[5] A. Boumenir; *A finite inverse problem by the determinant method*, Operators and matrices, 3 (2009), 547-556.
[6] G. Freiling, V.A. Yurko; *Inverse Sturm–Liouville Problems and Their Applications*, NOVA Science Publishers, New York, 2001.
[7] M. G. Gasymov, B. M. Levitan; *Inverse problem for Dirac's system*, Dokl. Akad. Nauk. SSSR, 167 (1966), 967-970.
[8] M. G. Gasymov, T. T. Dzhabiev; *Determination of a system of Dirac differential equations using two spectra*, in: Proceedings of School-Seminar on the Spectral Theory of Operators and Representations of Group Theory [in Russian] (Elm. Baku, 1975), 46-71.
[9] M. G. Gasymov; *Inverse problem of the scattering theory for Dirac system of order* $2n$, Tr. Mosk. Mat. Obshch., 19 (1968), 41-112.
[10] H. Hochstadt, B. Lieberman; *An inverse Sturm-Liouville problem with mixed given data,* SIAM J. Appl. Math., 34 (1978), 676-680.
[11] I. Joa, A. Minkin; *Eigenfunction estimate for a Dirac operator*, Acta Math. Hungar., 76 (1997), 337-349.
[12] N. B. Kerimov; *A boundary value problem for the Dirac system with a spectral parameter in the boundary conditions*, Differ. Equ., 38 (2002), 164-174.
[13] H. Koyunbakan; *Inverse spectral problem for some singular differential operators*, Tamsui Oxford journal of Mathematical Sciences, 25 (2009), 277-283.
[14] B. M. Levitan, I. S. Sargsjan; *Sturm-Liouville and Dirac operators, Kluwer Academic Publishers*, Dodrecht, Boston, London, 1991.
[15] M. M. Malamud; *Uniqueness questions in inverse problems for systems of differential equations on a finite interval*, Trans. Moscow Math. Soc., 60 (1999), 204-262.
[16] V. A. Marchenko; *Sturm-Liouville operators and their applications* [in Russian], Naukova Dumka, Kiev (1977).
[17] J. R. McLaughlin, W. Rundell; *A uniqueness theorem for an inverse Sturm-Liouville problem*, Journal of Mathematical Physics, 28 (1987), 1471-1472.
[18] H. E. Moses; *Calculation of the scattering potentials from reflection coefficient*, Bull. Amer. Phys. Soc., 4 (1956), 240.
[19] O. S. Mukhtarov, K. Aydemir; *Eigenfunction expansion for Sturm-Liouville problems with transmission conditions at one interior point*, Acta Mathematica Scientia, 35 (2015), 639-649.
[20] I. M. Nabiev; *On reconstruction of Dirac operator on the segment*, Proceeding of IMM of Nas of Azerbaijan, 18 (2003), 97-102.
[21] A. S. Ozkan, R. Kh. Amirov; *An interior inverse problem for the impulsive Dirac operator*, Tamkang Journal of Mathematics, 42 (2011), 259-263.
[22] E. S. Panakhov; *The defining of Dirac system in two incompletely set collection of eigenvalues*, Dokl. An AzSSR, 5 (1985), 8-12.
[23] E. S. Panakhov; *Inverse problem for Dirac system in two partially settled spectrum*, VINITY, 3304 (1981), 1-29.
[24] F. Prats, J. S.Toll; *Construction of the Dirac equation central potential from phase shifts and bound states*, Phys. Rev., 113 (1959), 363-370.
[25] M. Sat, E. S. Panakhov, K. Tas; *Wellposedness of the Inverse Problem for Dirac Operator*, Chinese Journal of Mathematics, 2013, (2013).
[26] C. T. Shieh, Y. P. Wang; *Inverse Problems for Sturm–Liouville Equations with Boundary Conditions Linearly Dependent on the Spectral Parameter from Partial Information*, Results in Mathematics, 65 (2014), 105-119.
[27] M. Verde; *The inversion problem in wave mechanics and dispertion relations*, Nuclear Phys., 9 (1958), 255-266.
[28] Y. P. Wang; *A uniqueness theorem for indefinite Sturm-Liouville operators*, Appl. Math. J. Chinese Univ., 27 (2012), 345-352.
[29] C. F. Yang; *Hochstadt-Lieberman theorem for Dirac operator with eigenparameter dependent boundary conditions*, Nonlinear Anal., 74 (2011), 2475-2484.

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