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HIGH-ORDER TOPOLOGICAL ASYMPTOTIC EXPANSION FOR STOKES EQUATIONS

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ABSTRACT. We use the topological sensitivity analysis method to solve various optimization problems. It consists of studying the asymptotic expansion of the objective function relative to a perturbation of the domain topology. This expansion becomes insufficient in some applications when it is limited to the first order topological derivative. We present a new topological sensitivity analysis for the Stokes equations based on a high order asymptotic expansion. The derived result is valid for different class of shape functions.

1. Introduction

The topological sensitivity technique is an optimization method used for different applications [1, 2, 3, 4, 13]. The main idea consists on developing of an asymptotic expansion of the objective function in relation to the domain topological perturbation. Many operators has been studied in the case of this method such as, the Laplace operator, the Stokes system, the Helmoltz equations, ... [8, 9, 11, 12]. The majority of the existing works using topological sensitivity method are limited to the first order expansion which is sufficient in the case where the size of the domain to be detected is of infinitesimal size and not close to the boundary. However, In the case where this constraint is not ensured or if the first order term in the asymptotic expansion is equal to zero at some critical points, we need an extension of the expansion to the high order term. This concept was studied by Rocha et al [5, 6] in the case of two dimensional Laplace operator and for a second order topological asymptotic. Hassine et al. [10] generalized this work to three dimensional case and for higher order development. We present in this work an extension of this concept to the Stokes equations

$$
\begin{gathered}
-\Delta u+\nabla p=F \quad \text{in } \Omega\\
\nabla . u=0 \quad \text{in } \Omega\\
u=0 \quad \text{on } \Gamma,
\end{gathered}
$$

where $\Omega \subset \mathbb{R}^3$ with smooth boundary $\partial\Omega=\Gamma$, u is the velocity, p the pressure and F is an external force.

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We define $\omega_{z,\varepsilon}$ a small geometry perturbation of Ω that is centered at $z \in \Omega$ and has the form $\omega_{z,\varepsilon} = z + \varepsilon\omega$, where $\omega \in \mathbb{R}^3$ is a given fixed and bounded regular domain containing the origin.

Let the shape function j be defined by

$$j(\Omega \backslash \overline{\omega_{z,\varepsilon}}) = J_\varepsilon(u_\varepsilon) \tag{1.1}$$

where $J_\varepsilon \in H^1(\Omega \backslash \overline{\omega_{z,\varepsilon}})^3$ and u_ε the solution to the Stokes problem in the perturbed domain $\Omega_{z,\varepsilon} = \Omega \backslash \overline{\omega_{z,\varepsilon}}$ with homogeneous Dirichlet condition on $\partial\omega_{z,\varepsilon}$

$$\begin{gathered} -\Delta u_\varepsilon + \nabla p_\varepsilon = F \quad \text{in } \Omega_{z,\varepsilon} \\ \nabla . u_\varepsilon = 0 \quad \text{in } \Omega_{z,\varepsilon} \\ u_\varepsilon = 0 \quad \text{on } \Gamma \\ u_\varepsilon = 0 \quad \text{on } \partial\omega_{z,\varepsilon}. \end{gathered} \tag{1.2}$$

The weak formulation of (1.2) consists in finding $u_\varepsilon \in V_\varepsilon$ that satisfies

$$a_\varepsilon(u_\varepsilon, \omega) = l_\varepsilon(\omega), \quad \forall \omega \in V_\varepsilon, \tag{1.3}$$

where

$$V_0 = \{v \in (H_0^1(\Omega_{z,\varepsilon}))^3 : \nabla \cdot v = 0\}, \tag{1.4}$$

$$a_\varepsilon(v, \omega) = \int_{\Omega_{z,\varepsilon}} \nabla u : \nabla v \, dx = \int_{\Omega_{z,\varepsilon}} \operatorname{tr}(\nabla u \cdot \nabla v) \, dx, \quad \forall v, \omega \in V_0, \tag{1.5}$$

$$l_\varepsilon(\omega) = \int_{\Omega_{z,\varepsilon}} F.\omega \, dx, \quad \forall \omega \in V_0. \tag{1.6}$$

Note that Problem (1.3) has a unique solution [7].

The aim of this work is to derive a high order topological asymptotic expansion for j relative to the presence of the geometry perturbation $\omega_{z,\varepsilon}$ in the domain Ω. The idea is to develop $j(\Omega_{z,\varepsilon}) - j(\Omega)$ with respect to ε and establishing an asymptotic formula on the form

$$j(\Omega_{z,\varepsilon}) - j(\Omega) = \sum_{k=1}^{N} f_k(\varepsilon)\delta_j^k(z) + o(f_N(\varepsilon)) \tag{1.7}$$

where

- f_k, $1 \le k \le N$ are positive scalar functions verifying $f_{k+1}(\varepsilon) = o(f_k(\varepsilon))$ and
$$\lim_{\varepsilon \to} f_k(\varepsilon) = 0,$$
- δ_j^k denotes the k^{th} topological derivative of the shape function j.

This work is a generalization of the topological sensitivity method. The presented result is of higher interest and is valid for different shape functions.

We begin by presenting the asymptotic formulation in section 2. Section 3 is devoted to the main result corresponding to the high order asymptotic expansion formula. Finally, an application of the developed result is presented for two different shape function examples.

2. Asymptotic formula for the velocity variation

In this section, we discuss the influence of the geometry perturbation $\omega_{z,\varepsilon}$ on the Stokes solution $(u_\varepsilon, p_\varepsilon)$. More precisely, we derive an asymptotic formula outlining the velocity field u_ε (resp. the pressure field p_ε) variation with respect to the perturbation size ε. We begin our analysis by the next preliminary estimate

Lemma 2.1. *If the perturbation $\omega_{z,\varepsilon}$ is strictly embedded into Ω, then the perturbed Stokes solution $(u_\varepsilon, p_\varepsilon)$ satisfies*

$$\begin{gathered} u_\varepsilon(x) - u_0(x) = W_0((x-z)/\varepsilon) + O(\varepsilon) \quad \text{in } \Omega_{z,\varepsilon}, \\ p_\varepsilon(x) - p_0(x) = \frac{1}{\varepsilon} Q_0((x-z)/\varepsilon) + O(\varepsilon) \quad \text{in } \Omega_{z,\varepsilon} \end{gathered}$$

where the leading term (W_0, Q_0) is defined as the solution to the Stokes exterior problem

$$\begin{gathered} -\Delta W_0 + \nabla Q_0 = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\omega}, \\ \nabla \cdot W_0 = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\omega}, \\ W_0 \to 0 \quad \text{at } \infty \\ W_0 = -u_0(z) \quad \text{on } \partial\omega. \end{gathered} \tag{2.1}$$

The proof of the above lemma is similar to that in [1, Proposition 3.1]; so we mit it. Next, we will give a generalization of this estimate to the high-order case. The obtained asymptotic behavior is illustrated by the following result.

Theorem 2.2. *If the geometry perturbation $\omega_{z,\varepsilon} = z + \varepsilon\omega$ is strictly embedded in the fluid flow domain Ω, then the velocity and pressure fields satisfy the following asymptotic behavior*

$$u_\varepsilon(x) = \sum_{k=0}^{N} \varepsilon^k [U_k(x) + W_k((x-z)/\varepsilon))] + O(\varepsilon^{N+1}) \quad \text{in } \Omega_{z,\varepsilon}, \tag{2.2}$$

$$p_\varepsilon(x) = \sum_{k=0}^{N} \varepsilon^k [P_k(x) + \frac{1}{\varepsilon} Q_k((x-z)/\varepsilon))] + O(\varepsilon^{N+1}) \quad \text{in } \Omega_{z,\varepsilon}, \tag{2.3}$$

where $(U_k, P_k)_{0\le k\le N}$ are smooth functions, solutions to a sequence of Stokes problems in Ω, and $(W_k, Q_k)_{0\le k\le N}$ are smooth functions, solutions to a sequence of exterior problems in $\mathbb{R}^3 \setminus \overline{\omega}$.

Proof. The sequences $(U_k, P_k)_{0\le k\le N}$ and $(W_k, Q_k)_{0\le k\le N}$ are constructed using an iterative process with $(U_0, P_0) = (u_0, p_0)$ and (W_0, Q_0) is the solution to (2.1). The proof is made in three steps.

Step 1: We derive the asymptotic behavior of the functions W_k, $0 \le k \le N$ relative to ε. Due to a single layer potential [7], W_k, $0 \le k \le N$ can be written as

$$W_k(y) = \int_{\partial\omega} E(y-t)\, \eta_k(t) ds(t), \quad \forall y \in \mathbb{R}^3 \setminus \overline{\omega},$$

where E is the fundamental solution of Stokes system in $\mathbb{R}^3$ and η_k is the solution to a boundary integral equation defined on $\partial\omega$. It is easy to see that for each $x \in \mathbb{R}^3 \setminus \overline{\omega_{z,\varepsilon}}$ we have

$$W_k((x-z)/\varepsilon) = \int_{\partial\omega} E((x-z)/\varepsilon - t)\eta_k(t) ds(t) = \varepsilon \int_{\partial\omega} E((x-z) - \varepsilon t)\eta_k(t) ds(t).$$

From the fact that $\omega_{z,\varepsilon}$ is not close to the boundary $\partial\Omega$, one can remark that for all $t \in \partial\omega$ and for all x in a neighborhood of Γ the function $\Pi_{x-z,t} : \varepsilon \mapsto \Pi_{x-z,t}(\varepsilon) = \varepsilon\, E((x-z) - \varepsilon\, t)$ is smooth with respect to ε and admits the asymptotic expansion

$$\Pi_{x-z,t}(\varepsilon) = \sum_{p=1}^{N} \frac{\varepsilon^p}{p!} \Pi^{(p)}_{x-z,t}(0) + O(\varepsilon^{N+1}),$$

where $\Pi^{(p)}_{x-z,t}(0)$ is the p-th derivative of $\Pi_{x-z,t}$ at $\varepsilon = 0$. It depends on the p-th derivative of the function E at the point $x - z$. Consequently, the function $x \mapsto W_k((x-z)/\varepsilon)$ satisfies the following asymptotic behavior

$$W_k((x-z)/\varepsilon) = \sum_{p=1}^{N} \varepsilon^p\, W_k^{(p)}(x-z) + O(\varepsilon^{N+1}), \tag{2.4}$$

with $W_k^{(p)}$ is the smooth function defined in $\mathbb{R}^3 \setminus \overline{\omega}$ by

$$W_k^{(p)}(x-z) = \frac{1}{p!} \int_{\partial\omega} \Pi^{(p)}_{x-z,t}(0) \eta_k(t) ds(t), \quad \forall x \in \mathbb{R}^3 \setminus \overline{\omega}. \tag{2.5}$$

Step 2: We are now ready to present the leading terms of the expected formulas. Let us suppose that we have already derived the terms (U_i, P_i) and (W_i, Q_i) for all $0 \le i \le k-1$. The k-th order term is described by the function $x \mapsto (U_k(x), P_k(x)) + (W_k((x-z)/\varepsilon), \frac{1}{\varepsilon} Q_k((x-z)/\varepsilon))$ which is constructed as follows:

- (U_k, P_k) depends on W_j, $0 \le j \le k-1$ and solves the interior problem
$$\begin{aligned} -\Delta U_k + \nabla P_k = 0 \quad &\text{in } \Omega, \\ \nabla \cdot U_k = 0 \quad &\text{in } \Omega, \\ U_k = -\sum_{p=1}^{k} W_{k-p}^{(p)}(x-z) \quad &\text{on } \Gamma, \end{aligned} \tag{2.6}$$
with $W_j^{(p)}$ is defined by (2.5).
- (W_k, Q_k) depends on U_j, $0 \le j \le k$ and solves the exterior problem
$$\begin{aligned} -\Delta W_k + \nabla Q_k = 0 \quad &\text{in } \mathbb{R}^3 \setminus \overline{\omega}, \\ \nabla \cdot W_k = 0 \quad &\text{in } \mathbb{R}^3 \setminus \overline{\omega}, \\ W_k \to 0 \quad &\text{at } \infty \\ W_k = -U_k(z) - \sum_{p=1}^{k} \frac{1}{p!} D^p U_{k-p}(z)(y^p) \quad &\text{on } \partial\omega, \end{aligned} \tag{2.7}$$
where $D^p U_{k-p}(z)$ is the p-th derivative of the function U_{k-p} and $y^p = (y, \dots, y) \in (\mathbb{R}^3)^p$.

Step 3: We check that the used iterative process leads to the expected asymptotic formulas. Posing $R_{N,\varepsilon}(x) = \sum_{k=0}^{N} \varepsilon^k [U_k(x) + W_k((x-z)/\varepsilon))] - u_\varepsilon$ and $S_{N,\varepsilon}(x) = \sum_{k=0}^{N} \varepsilon^k [P_k(x) + \frac{1}{\varepsilon} Q_k((x-z)/\varepsilon))] - p_\varepsilon$. One can easily verify that $(R_{N,\varepsilon}, S_{N,\varepsilon})$ solves the Stokes system in $\Omega_{z,\varepsilon}$

$$\begin{aligned} -\Delta R_{N,\varepsilon} + \nabla S_{N,\varepsilon} = 0 \quad &\text{in } \Omega_{z,\varepsilon}, \\ \nabla \cdot R_{N,\varepsilon} = 0 \quad &\text{in } \Omega_{z,\varepsilon}, \end{aligned} \tag{2.8}$$

and satisfies the boundary conditions:

- on $\partial\omega_{z,\varepsilon}$: Using the systems (2.6)-(2.7), the multi-linearity of $D^pU_{k-p}(z)$, Taylor's Theorem and the fact that $\|x-z\| = O(\varepsilon)$ on $\partial\omega_{z,\varepsilon}$, one can derive
$$R_{N,\varepsilon}(x) = \sum_{k=0}^{N} \varepsilon^k \Big[U_k(x) - \sum_{p=0}^{N-k} \frac{1}{p!} D^pU_k(z)((x-z)^p)\Big] = O(\varepsilon^{N+1}).$$
- on Γ: From (2.6), (2.7) and the asymptotic expansion (2.4), one can obtain
$$\begin{aligned} R_{N,\varepsilon}(x) &= \varepsilon^N W_N((x-z)/\varepsilon) + \sum_{k=0}^{N-1} \varepsilon^k \big[W_k((x-z)/\varepsilon) \\ &\quad - \sum_{p=1}^{N-k} \varepsilon^p W_k^{(p)}(x-z)\big] = O(\varepsilon^{N+1}). \end{aligned}$$

□

3. High-order topological asymptotic expansion

We derive in this section a high-order terms in the topological asymptotic expansion for the Stokes operator. The obtained results are an extension of the the topological derivative notion for the high-order case and are valid for all shape function j defined by
$$j(\Omega_{z,\varepsilon}) = J_\varepsilon(u_\varepsilon),$$
with J_ε is a scalar function in $H^1(\Omega_{z,\varepsilon})^3$, satisfying the following hypothesis:

(H1) The function J_0 is differentiable with respect to u.
There exist real numbers $\delta^1 J(z), \dots, \delta^N J(z)$, such that
$$J(u_\varepsilon) - J_0(u_0) = DJ_0(u_0)(u_\varepsilon - u_0) + \sum_{k=1}^{N} \varepsilon^k \delta^k J(z) + o(\varepsilon^N), \quad \forall \varepsilon > 0.$$

In the term $DJ_0(u_0)(u_\varepsilon - u_0)$, the velocity field u_ε is extended by zero inside the domain $\omega_{z,\varepsilon}$. Its extension will be denoted by u_ε throughout the rest of the paper.

Under hypothesis (H1), the variation of the shape function j reads
$$j(\Omega_{z,\varepsilon}) - j(\Omega) = \int_{\Omega_{z,\varepsilon}} \nabla(u_0 - u_\varepsilon) : \nabla v_0 dx + \sum_{k=1}^{N} \varepsilon^k \,\delta^k J(z) + o(\varepsilon^N),$$
where u_0 and v_0 are respectively solutions to the Stokes and its associated adjoint problems. Using Green formula and Theorem 2.2, the integral term can be decomposed as
$$\begin{aligned} &\int_{\Omega_{z,\varepsilon}} \nabla(u_0 - u_\varepsilon) : \nabla v_0 dx \\ &= \int_{\omega_{z,\varepsilon}} \nabla u_0 : \nabla v_0 dx - \sum_{k=0}^{N} \varepsilon^k \int_{\partial\omega_{z,\varepsilon}} \nabla_x W_k((x-z)/\varepsilon)) n \cdot v_0 ds \\ &\quad - \sum_{k=1}^{N} \varepsilon^k \int_{\partial\omega_{z,\varepsilon}} \nabla U_k(x) n(x) \cdot v_0(x) ds + O(\varepsilon^{N+1}). \end{aligned} \tag{3.1}$$

To derive the high-order topological asymptotic expansion for j, we establish an estimate for all terms on the right side of equality (3.1).

3.1. **Preliminary estimates.**

Lemma 3.1. *The first integral term in* (3.1) *satisfies*

$$\int_{\omega_{z,\varepsilon}} \nabla u_0 : \nabla v_0 dx = \sum_{k=3}^{N} \varepsilon^k \, \mathcal{G}^{1,k-3}_{u_0,v_0}(z) + O(\varepsilon^{N+1}),$$

where the functions $z \mapsto \mathcal{G}^{1,k}_{u_0,v_0}(z)$, $0 \le k \le N$ *are defined in* Ω *by*

$$\mathcal{G}^{1,k}_{u_0,v_0}(z) = \sum_{p=0}^{k} \frac{1}{p!(k-p)!} \int_{\omega} \nabla^{(p+1)} u_0(z)(y^p) : \nabla^{(k-p+1)} v_0(z)(y^{k-p}) dy, \tag{3.2}$$

with $y^k = (y, \dots, y) \in (\mathbb{R}^3)^k$ *and* $\nabla^{(k)} w(z)$ *denotes the* k*-th derivative of the function* w *at the point* z.

Lemma 3.2. *The second integral term in* (3.1) *satisfies*

$$\sum_{k=0}^{N} \varepsilon^k \int_{\partial\omega_{z,\varepsilon}} \nabla_x W_k((x-z)/\varepsilon)) n \cdot v_0 ds = -\sum_{k=1}^{N} \varepsilon^k \mathcal{G}^{2,k-1}_{W,v_0}(z) + O(\varepsilon^{N+1}),$$

where the functions $z \mapsto \mathcal{G}^{2,k}_{W,v_0}(z)$, $0 \le k \le N$ *are defined in* Ω *by*

$$\mathcal{G}^{2,k}_{W,v_0}(z) = -\sum_{p=0}^{k} \frac{1}{p!} \int_{\partial\omega} \nabla_y W_{k-p}(y) n(y) \cdot [\nabla^{(p)} v_0(z)(y^p)] ds(y). \tag{3.3}$$

Lemma 3.3. *The third integral term in* (3.1) *satisfies*

$$\sum_{k=1}^{N} \varepsilon^k \int_{\partial\omega_{z,\varepsilon}} \nabla U_k(x) n(x) \cdot v_0(x) ds = -\sum_{k=3}^{N} \varepsilon^k \mathcal{G}^{3,k-3}_{U,v_0}(z) + O(\varepsilon^{N+1}).$$

where the functions $z \mapsto \mathcal{G}^{3,k}_{U,v_0}(z)$, $0 \le k \le N$ *are defined in* Ω *by*

$$\begin{aligned} &\mathcal{G}^{3,k}_{U,v_0}(z) \\ &= -\sum_{p=0}^{k} \sum_{q=0}^{p} \frac{1}{q!(p-q)!} \int_{\partial\omega} [\nabla^{(q+1)} U_{k-p+1}(z)(y^q)] n(y) \cdot [\nabla^{(p-q)} v_0(z)(y^{p-q})] ds(y). \end{aligned}$$

3.2. **Asymptotic expansion.** Based on the previous estimates, we derive in theorem 3.4 a high-order topological asymptotic expansion valid for all shape function that meets hypothesis (H1). Propositions 3.5 and 3.6 are devoted to two particular examples of shape functions.

Theorem 3.4. *Let* $\omega_{z,\varepsilon} = z + \varepsilon\omega$ *be a geometry perturbation strictly embedded in* Ω. *If* J_ε *satisfies* (H1), *then the associated shape function* j *satisfies*

$$j(\Omega_{z,\varepsilon}) - j(\Omega) = \sum_{k=1}^{N} \varepsilon^k \delta^k j(z) + o(\varepsilon^N),$$

where $\delta^k j$ *is the* $k-$*th topological derivative order, defined in* Ω *by*

$$\delta^k j(z) = \begin{cases} \mathcal{G}^{2,k-1}_{W,v_0}(z) + \delta^k J(z) & \text{if } k = 1,2 \\ \mathcal{G}^{1,k-3}_{u_0,v_0}(z) + \mathcal{G}^{2,k-1}_{W,v_0}(z) + \mathcal{G}^{3,k-3}_{U,v_0}(z) + \delta^k J(z) & \text{if } 3 \le k \le N. \end{cases}$$

To discuss hypothesis (H1). We consider two examples of shape functions satisfying (H1) and we derive their variations $\delta^1 J$, $\delta^2 J$, $\dots$, and $\delta^N J$.

Proposition 3.5. *Let $g \in L^2(\Omega)^3$ be a given function. The function $u \mapsto J_\varepsilon(u) = \int_{\Omega_{z,\varepsilon}} g \cdot u dx$, for $u \in H^1(\Omega_{z,\varepsilon})$ satisfies (H1) with*

$$DJ_0(w) = \int_\Omega g \cdot w \, dx, \quad \forall w \in H^1(\Omega),$$

and $\delta^k J(z) = 0$ in Ω $k = 1, \dots, N$.

Proposition 3.6. *Let U_d be a given desired state, smooth in $\omega_{z,\varepsilon}$. The function $u \mapsto J_\varepsilon(u) = \int_{\Omega_{z,\varepsilon}} |\nabla u - \nabla U_d|^2 \, dx$, $u \in H^1(\Omega_{z,\varepsilon})$ satisfies (H1) with*

$$DJ_0(w) = 2 \int_\Omega \nabla(u_0 - U_d) : \nabla w dx, \quad \forall w \in H^1(\Omega),$$

and

$$\delta^k J(z) = \begin{cases} \mathcal{G}^{2,k-1}_{W,u_0}(z) & \text{if } k = 1, 2 \\ \mathcal{G}^{2,k-1}_{W,u_0}(z) + \mathcal{G}^{1,k-3}_{u_0,u_0}(z) + \mathcal{G}^{1,k-3}_{U_d,U_d}(z) + \mathcal{G}^{3,k-3}_{U,u_0}(z) & \text{if } 3 \le k \le N. \end{cases}$$

Conclusion. The present work generalizes the topological derivative notion for the high-order case. The obtained results are based on the asymptotic formulas describing the variations of the velocity and pressure fields relative to the presence of a geometry perturbation $\omega_{z,\varepsilon} = z + \varepsilon\omega$ in the fluid flow domain Ω. The presented mathematical analysis is general. It can be extended to different partial differential equations.

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