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EVOLUTIONARY $p(x)$-LAPLACIAN EQUATION FREE FROM THE LIMITATION OF THE BOUNDARY VALUE

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Abstract. In this article we consider the evolutionary $p(x)$-Laplacian equation

$$u_t = \mathrm{div}(\rho^\alpha |\nabla u|^{p(x)-2}\nabla u), \quad (x,t) \in \Omega \times (0,T),$$

where $\rho(x) = \mathrm{dist}(x, \partial\Omega)$. If the diffusion coefficient degenerates on the boundary, then and the solution may be free from any limitations of the boundary condition.

1. Introduction

Consider the usual evolutionary p-Laplacian equation

$$u_t = \mathrm{div}(|\nabla u|^{p-2}\nabla u), \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1.1}$$

where $\Omega \subset R^N$ is a bounded domain with appropriately smooth boundary. The equation arises in the fields of mechanics, physics and biology [4, 7, 16, 17, 20, 33]. In the theory of non-Newtonian fluids, the quantity p is the characteristic of the medium. The media with $p > 2$ is called dilatant fluids, those with $p = 2$ are Newtonian fluids, and those with $p < 2$ are called pseudoplastics. Note that if $p = 2$, equation (1.1) is known as the classical heat conduction equation, and the solution of equation has infinite propagation speed of disturbance. This property seems unreasonable. So, when $p \neq 2$, equation (1.1) can be better to reflect the actual physical situation of the heat conduction. In particular, when $p > 2$ the solution of equation has finite propagation speed of disturbance, see [7]. Much attention was dedicated to its well-posedness [3, 9, 11, 12, 15, 18, 22, 23, 24, 25, 26, 27, 28, 29, 30] and the references therein.

Yin-Wang [21] considered

$$u_t = \mathrm{div}(\rho^\alpha |\nabla u|^{p-2}\nabla u), \quad (x,t) \in Q_T, \tag{1.2}$$

where $\rho(x) = \mathrm{dist}(x, \partial\Omega)$ is the distance function from the boundary. An obvious character of the equation is that, the diffusion coefficient $\rho^\alpha(x)$ depends on the distance to the boundary. Since $\rho^\alpha(x)$ vanishes on the boundary $\partial\Omega$, it seems that there is no heat flux across the boundary. However, Yin-Wang showed that the fact might not coincide with what we image. In fact, the exponent α which characterizes the vanishing ratio of the diffusion coefficient, does determine the behavior of the heat transfer near the boundary.

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If $p(x)$ is only a measurable function, a new kind of fluids of prominent technological interest has recently emerged: the so-called electrorheological fluids. This model includes parabolic equations which are nonlinear with respect to the gradient of the thought solution, and with variable exponents of nonlinearity [1, 14]

$$u_t = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u), \quad (x,t) \in Q_T. \tag{1.3}$$

A natural functional space used to study the well-posedness of the solutions of equation (1.3) is $W^{1,p(x)}(\Omega)$. In what follows, we denote

$$p^+ = \operatorname{ess\,sup}_{\overline{\Omega}} p(x), p^- = \operatorname{ess\,inf}_{\overline{\Omega}} p(x).$$

In particular, we assume that

$$1 < p^- \le p(x), \quad \forall x \in \Omega,$$

and quote some new function spaces with variable exponents [5, 8]:

$$L^{p(x)}(\Omega) = \{u : u \text{ is a measurable real-valued function, } \int_\Omega |u(x)|^{p(x)}dx < \infty\},$$

which is equipped with the Luxemburg's norm

$$|u|_{L^{p(x)}}(\Omega) = \inf\{\lambda > 0 : \int_\Omega \Big|\frac{u(x)}{\lambda}\Big|^{p(x)}dx \le 1\},$$

and is a separable, uniformly convex Banach space.

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\},$$

which is endowed with the norm

$$|u|_{W^{1,p(x)}} = |u|_{L^{p(x)}(\Omega)} + |\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W^{1,p(x)}(\Omega).$$

$W_0^{1,p(x)}(\Omega)$ is the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}$. Let us recall some properties of these function spaces, see [5, 8].

Lemma 1.1. *(i) The spaces* $\big(L^{p(x)}(\Omega), |\cdot|_{L^{p(x)}(\Omega)}\big)$, $\big(W^{1,p(x)}(\Omega), |\cdot|_{W^{1,p(x)}(\Omega)}\big)$ *and* $W_0^{1,p(x)}(\Omega)$ *are reflexive Banach spaces.*

(ii) $p(x)$-Hölder's inequality. Let $q_1(x)$ and $q_2(x)$ be real functions with $\frac{1}{q_1(x)} + \frac{1}{q_2(x)} = 1$ and $q_1(x) > 1$. Then, the conjugate space of $L^{q_1(x)}(\Omega)$ is $L^{q_2(x)}(\Omega)$. And for any $u \in L^{q_1(x)}(\Omega)$ and $v \in L^{q_2(x)}(\Omega)$, we have

$$\Big|\int_\Omega uv dx\Big| \le 2|u|_{L^{q_1(x)}(\Omega)}|v|_{L^{q_2(x)}(\Omega)}.$$

(iii) We have

$$\text{If } |u|_{L^{p(x)}(\Omega)} = 1, \text{ then } \int_\Omega |u|^{p(x)}dx = 1;$$

$$\text{If } |u|_{L^{p(x)}}(\Omega) > 1, \text{ then } |u|^{p^-}_{L^{p(x)}} \le \int_\Omega |u|^{p(x)}dx \le |u|^{p^+}_{L^{p(x)}};$$

$$\text{If } |u|_{L^{p(x)}}(\Omega) < 1, \text{ then } |u|^{p^+}_{L^{p(x)}} \le \int_\Omega |u|^{p(x)}dx \le |u|^{p^-}_{L^{p(x)}}.$$

(iv) If $p_1(x) \le p_2(x)$, then $L^{p_1(x)}(\Omega) \supset L^{p_2(x)}(\Omega)$.
(v) If $p_1(x) \le p_2(x)$, then $W^{1,p_1(x)}(\Omega) \hookrightarrow W^{1,p_2(x)}(\Omega)$.

(vi) $p(x)$-Poincaré inequality: if $p(x) \in C(\Omega)$, then there is a constant $C > 0$ such that

$$|u|_{L^{p(x)}}(\Omega) \le C|\nabla u|_{L^{p(x)}(\Omega)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

This implies that $|\nabla u|_{L^{p(x)}}(\Omega)$ and $|u|_{W^{1,p(x)}(\Omega)}$ are equivalent norms of $W_0^{1,p(x)}(\Omega)$.

Zhikov [34] showed that

$$W_0^{1,p(x)}(\Omega) \neq \{v \in W_0^{1,p(x)}(\Omega) \mid v|_{\partial\Omega} = 0\} = \mathring{W}^{1,p(x)}(\Omega).$$

This fact provides us a good idea which may be used to study the well-posedness of evolutionary p-Laplacian equation [3, 9, 11, 15, 18, 12, 22, 23, 24, 25, 26, 27, 28, 29, 30]. However, if $p(x)$ satisfies the so-called logarithmic Hölder continuity condition

$$|p(x) - p(y)| \le \omega(|x-y|), \forall x, y \in Q_T, \quad |x-y| < \frac{1}{2}, \tag{1.4}$$

with

$$\limsup_{s\to 0^+} \omega(s) \ln\Big(\frac{1}{s}\Big) = C < \infty,$$

then (see [35])

$$W_0^{1,p(x)}(\Omega) = \mathring{W}^{1,p(x)}(\Omega).$$

Using this fact, Antontsev-Shmarev [2] established existence and uniqueness results for (1.3). Later, many researchers have been interested in studying (1.3), see [2, 13, 10, 31, 19].

In this article, we consider the equation

$$u_t = \operatorname{div}(\rho^\alpha |\nabla u|^{p(x)-2}\nabla u), \quad (x,t) \in Q_T, \tag{1.5}$$

with $\alpha > 0$.

If we want to consider its initial boundary value problem, usually we need the initial value condition

$$u|_{t=0} = u_0(x), \quad x \in \Omega. \tag{1.6}$$

Note that equation (1.5) is degenerate on the boundary. Can we impose the general homogeneous boundary value condition as follows?

$$u|_{\Gamma_T} = 0,\ (x,t) \in \Gamma_T = \partial\Omega \times (0,T). \tag{1.7}$$

In this study, we introduce the Fichera-Oleǐnik theory to study how to propose the boundary condition of equation (1.5) with $p^- > 1$.

Definition 1.2. A function $u(x,t)$ is said to be a solution of (1.5) with initial condition (1.6), if $u \in L^\infty(Q_T)$, $u_t \in L^2(Q_T)$, $\rho^\alpha|\nabla u|^{p(x)} \in L^1(Q_T)$ and

$$\iint_{Q_T} (-u\varphi_t + \rho^\alpha |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\varphi)\, dx\, dt = 0, \tag{1.8}$$

for any function $\varphi \in C_0^\infty(Q_T)$.

Definition 1.3. A function $u(x,t)$ is said to be a solution of equation (1.5) with the initial-boundary conditions (1.6)-(1.7), if u satisfies Definition 1.2 and the boundary condition (1.7) is satisfied in the sense of the trace.

We summarize our main result as follows.

Theorem 1.4. *Suppose $p^- > 1$, $u_0 \in L^\infty(\Omega)$, and $\rho^\alpha|\nabla u_0|^{p^+} \in L^1(\Omega)$. If $\alpha < p^- - 1$, then there exists a unique solution of equation* (1.5) *with the initial-boundary conditions* (1.6)-(1.7). *While, if $\alpha > p^+ - 1$, then there exists a unique solution of* (1.5) *with the initial value* (1.6).

Theorem 1.4 implies that, the solution of (1.5) is free from any limitations of the boundary condition provided that $\alpha > p^+ - 1$. When $p^+ - 1 \geq \alpha \geq p^- - 1$, the problem of the well-posedness of equation (1.5) remains open.

2. Fichera-Oleinik theory and its applications

If for any real vector $\xi = (\xi_1, \xi_2, \cdots, \xi_m)$ and any point $x \in \Omega$,

$$a^{rs}\xi_r\xi_s \geqslant 0, \tag{2.1}$$

the second-order equation of the form

$$L(u) = a^{rs}(x)u_{x_r x_s} + b^r(x)u_{x_r} + c(x)u = f(x), \tag{2.2}$$

is called the second-order equation with nonnegative characteristic form in Ω. Obviously, it contains elliptic equation, parabolic equation, first-order equation (the case $a^{rs}\xi_r\xi_s \equiv 0$), ultra parabolic equation, Brown motion equation, and Tricomi equation on the half-plane and so on.

Consider the first-boundary value problem of equation (2.2) in Ω, Fichera once dealt with this problem in [6]. Suppose that on $\bar{\Omega} = \Omega \cup \sum$, all the points x and all $\xi \in R^n$ satisfy the condition (2.1), and $a^{rs} \in C^2(\Omega), b^r \in C^1(\Omega), c \in C^0(\Omega)$. Let $\{n_s\}$ be the unit inner normal vector of $\partial\widetilde{\Omega}$. The Fichera function is defined as

$$b(x) \equiv (b^r - a^{rs}_{x_s})n_r. \tag{2.3}$$

We denote

$$\begin{gathered}
\Sigma^0 = \{x \in \Sigma : a^{rs}n_r n_s = 0\}, \\
\Sigma_1 = \{x \in \Sigma^0 : (b_r - a^{rs}_{x_s})n_r > 0\}, \\
\Sigma_2 = \{x \in \Sigma^0 : (b_r - a^{rs}_{x_s})n_r < 0\}, \\
\Sigma_0 = \{x \in \Sigma^0 : (b_r - a^{rs}_{x_s})n_r = 0\}, \\
\Sigma_3 = \Sigma \backslash \Sigma^0.
\end{gathered}$$

The first boundary value problem of equation (2.2) is quoted as follows, in $\bar{\Omega} = \Omega \cup \sum$, to find a function u such that

$$L(u) = f(x), x \in \Omega, \tag{2.4}$$

$$u(x) = g(x), x \in \Sigma_2 \cup \Sigma_3, \tag{2.5}$$

where f is a given function in Ω, and g is a given function on $\Sigma_2 \cup \Sigma_3$. Clearly, if (2.2) is an elliptic equation, then (2.4)-(2.5) is the usual Dirichlet problem. For the cylindrical region, (2.4)-(2.5) consists of the mixed problem, also known as parabolic equations with the initial boundary values.

Consider the evolutionary $p(x)$-Laplacian equation

$$u_t = \operatorname{div}(\rho^\alpha|\nabla u|^{p(x)-2}\nabla u), \quad (x,t) \in Q_T. \tag{2.6}$$

Since

$$\operatorname{div}(\rho^\alpha|\nabla u|^{p(x)-2}\nabla u)$$

$$
\begin{aligned}
&= \alpha\rho^{\alpha-1}|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\rho + \rho^{\alpha}|\nabla u|^{p(x)-2}\nabla p\cdot\nabla u\\
&\quad + \rho^{\alpha}|\nabla u|^{p(x)-4}[(p(x)-2)u_{x_ix_j}u_{x_i}u_{x_j} + |\nabla u|^2 u_{x_ix_i}],
\end{aligned}
$$

we rewrite (2.6) as

$$
u_t = a^{ij}(x,t)\frac{\partial^2 u}{\partial x_i\partial x_j} + \beta_i(x,t)\frac{\partial u}{\partial x_i}, \tag{2.7}
$$

where

$$
\begin{gathered}
a^{ij}(x,t) = \rho^{\alpha}|\nabla u|^{p(x)-2}[\delta_{ij} + (p(x)-2)|\nabla u|^{-2}u_{x_i}u_{x_j}],\\
\beta_i = \alpha\rho^{\alpha-1}|\nabla u|^{p(x)-2}\rho_{x_i} + \rho^{\alpha}|\nabla u|^{p(x)-2}p_{x_i}.
\end{gathered}
$$

Then

$$
\begin{aligned}
a^{ij}_{x_j} &= \alpha\rho^{\alpha-1}\rho_{x_j}|\nabla u|^{p(x)-4}[\delta_{ij}|\nabla u|^2 + (p(x)-2)u_{x_i}u_{x_j}]\\
&\quad + \rho^{\alpha}|\nabla u|^{p(x)-4}[p_{x_j}|\nabla u|^2\log|\nabla u| + (p(x)-2)u_{x_k}u_{x_kx_j} + p_{x_j}u_{x_j}u_{x_i}]\\
&\quad + (p(x)-2)\rho^{\alpha}|\nabla u|^{p(x)-6}u_{x_j}u_{x_i}[p_{x_j}|\nabla u|^2\log|\nabla u| + (p(x)-4)u_{x_k}u_{x_kx_j}]\\
&\quad + (p(x)-2)|\nabla u|^{p(x)-4}(u_{x_ix_j}u_{x_j} + u_{x_i}u_{x_jx_j}).
\end{aligned}
$$

If we compare (2.7) with (2.5), according to Fichera-Oleinik theory, the initial value condition is necessary. As for the boundary condition of equation (2.6), let us discuss it as follows.

Case 1: When $\alpha > 1$, by $\rho|_{\partial\Omega} = 0$, then $(\beta_i - \alpha^{ij}_{x_i})n_i \equiv 0, x \in \partial\Omega$,

$$
\Sigma_2 \cup \Sigma_3 = \emptyset.
$$

Case 2: When $\alpha \le 1$, by the fact of that $\rho_{x_j} = n_j$, it has

$$
\begin{aligned}
&(\beta^i - \alpha^{ij}_{x_j})n_i\\
&= \alpha\rho^{\alpha-1}|\nabla u|^{p(x)-4}[\rho_{x_i}|\nabla u|^2 - \rho_{x_j}(\delta_{ij}|\nabla u|^2 + (p(x)-2)u_{x_i}u_{x_j})]n_i\\
&= -(p(x)-2)\alpha\rho^{\alpha-1}(|\nabla u|^{p(x)-4}u_{x_i}u_{x_j})n_in_j.
\end{aligned} \tag{2.8}
$$

In this case, it is difficult to know that the Fichera function $(\beta^i - \alpha^{ij}_{x_j})n_i$ is negative or not, except the dimension of spatial variable, $N = 1$. When $\alpha < 1$ and $p(x) \le p^+ < 2$, (2.8) is transformed into

$$
(\beta - a_x)n = -(p(x)-2)\alpha\rho^{\alpha-1}|u_x|^{p(x)-2} = +\infty > 0. \tag{2.9}
$$

When $\alpha < 1$ and $p(x) \ge p^- > 2$, (2.8) is transformed into

$$
(\beta - a_x)n = -(p(x)-2)\alpha\rho^{\alpha-1}|u_x|^{p(x)-2} = -\infty < 0. \tag{2.10}
$$

If $\alpha = 1$ and $p^+ < 2$, then we have

$$
(\beta - a_x)n = -(p(x)-2)\alpha\rho^{\alpha-1}|u_x|^{p(x)-2} = -(p(x)-2)\alpha > 0. \tag{2.11}
$$

If $\alpha = 1$ and $p^- > 2$, then we get

$$
(\beta - a_x)n = -(p(x)-2)\alpha\rho^{\alpha-1}|u_x|^{p(x)-2} = -(p(x)-2)\alpha < 0. \tag{2.12}
$$

In a word, if $p^+ < 2$ and $N = 1$, we may conjecture that $\Sigma_2 \cup \Sigma_3 = \emptyset$. If $p^- > 2$ and $N = 1$, we may conjecture that $\Sigma_2 \cup \Sigma_3 = \partial\Omega$. While, $N > 1$, the partial boundary $\Sigma_2 \cup \Sigma_3 \subseteq \partial\Omega$ may be very complicated.

Certainly, equation(1.5) is a degenerate parabolic equation, it only has a weak solution generally, so the above linearization is informal. We just give some ideas to the partial boundary condition to assure the well-posedness of the weak solutions. Whether these ideas are true or not remains to be verified.

3. Existence of solutions

In this section, we consider the initial value problem of equation (1.5).

Theorem 3.1. *If $p^- > 1$, and*

$$u_0 \in L^\infty(\Omega), \quad \rho^\alpha |\nabla u_0|^{p^+} \in L^1(\Omega), \tag{3.1}$$

then there exists a weak solution of (1.5) *with initial condition* (1.6). *Moreover, if $\alpha \geq 1$ and $u_0 \in C_0^\infty(\Omega)$, then $u_t \in L^\infty(Q_T)$.*

If $u_0 \in C_0^\infty(\Omega)$, we consider the regularized problem

$$u_{\varepsilon t} - \operatorname{div}(\rho_\varepsilon^\alpha (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p(x)-2}{2}} \nabla u_\varepsilon) = 0, \quad (x,t) \in Q_T, \tag{3.2}$$

$$u_\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T), \tag{3.3}$$

$$u_\varepsilon(x,0) = u_0(x), \quad x \in \Omega. \tag{3.4}$$

where $\rho_\varepsilon = \rho + \varepsilon$, $\varepsilon > 0$. Just as we had done on the usual evolutionary p-Laplacian equation, we are able to prove that the above problem has a unique weak solution u_ε, which satisfies

$$u_\varepsilon \in L^\infty(Q_T), \quad u_{\varepsilon t} \in L^2(Q_T), \quad u_\varepsilon \in L^\infty(0,T; W_0^{1,p(x)}(\Omega)). \tag{3.5}$$

Lemma 3.2. *If $u_0 \in C_0^\infty(\Omega)$, then the solution u_ε of initial boundary-value problem* (3.2)-(3.4) *is weakly star convergent and strongly convergent to $u \in L^r_{\rm loc}(Q_T)$, and the limit function u is the solution of equation* (1.5) *with the initial condition* (1.6). *Here, if $p^- < 2$ and $1 < r < p^{-*} = \frac{Np^-}{N-p^-}$ as usual, while $p^- \geq 2$ and $r = 2$.*

Proof. By the maximum principle, there is a constant c depending on $\|u_0\|_{L^\infty(\Omega)}$ but independent on ε, such that

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \leqslant c. \tag{3.6}$$

Multiplying (3.2) by u_ε, integrating it over Q_T, we have

$$\frac{1}{2}\int_\Omega u_\varepsilon^2 dx + \iint_{Q_T} \rho_\varepsilon^\alpha (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p(x)-2}{2}} |\nabla u_\varepsilon|^2 \,dx\,dt \leqslant c. \tag{3.7}$$

If we denote $\Omega_\lambda = \{x \in \Omega : d(x) > \lambda\}$ for any given $\lambda > 0$, by that $\rho(x) > \lambda$ when $x \in \Omega_\lambda$, then

$$\int_0^T \int_{\Omega_\lambda} |\nabla u_\varepsilon|^{p^-} \,dx\,dt \leq \iint_{Q_T} \rho^\alpha (|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p(x)-2}{2}} |\nabla u_\varepsilon|^2 \,dx\,dt \leqslant c(\lambda). \tag{3.8}$$

Multiplying (3.2) by $u_{\varepsilon t}$, integrating it over Q_T, we have

$$\iint_{Q_T} (u_{\varepsilon t})^2 \,dx\,dt = \iint_{Q_T} \operatorname{div}(\rho_\varepsilon^\alpha |\nabla u_\varepsilon|^{p(x)-2} \nabla u_\varepsilon) \cdot u_{\varepsilon t} \,dx\,dt. \tag{3.9}$$

Since

$$(|\nabla u_\varepsilon|^2 + \varepsilon)^{\frac{p(x)-2}{2}} \nabla u_\varepsilon \cdot \nabla u_{\varepsilon t} = \frac{1}{2}\frac{d}{dt}\int_0^{|\nabla u_\varepsilon(x,t)|^2+\varepsilon} s^{\frac{p(x)-2}{2}} ds,$$

it follows that

$$\begin{aligned}
&\iint_{Q_T} \operatorname{div}(\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}\nabla u_\varepsilon)\cdot u_{\varepsilon t})\,dx\,dt\\
&= -\iint_{Q_T} \rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}\nabla u_\varepsilon\nabla u_{\varepsilon t}\,dx\,dt \\
&= -\frac{1}{2}\iint_{Q_T} \rho_\varepsilon^\alpha\frac{d}{dt}\int_0^{|\nabla u_\varepsilon(x,t)|^2+\varepsilon} s^{\frac{p(x)-2}{2}}ds\,dx\,dt.
\end{aligned} \tag{3.10}$$

By (3.9)-(3.10), we have

$$\iint_{Q_T}(u_{\varepsilon t})^2\,dx\,dt+\iint_{Q_T}\rho_\varepsilon^\alpha\frac{d}{dt}\int_0^{|\nabla u_\varepsilon(x,t)|^2} s^{\frac{p(x)-2}{2}}ds\,dx\,dt\leqslant c,$$

and

$$\iint_{Q_T}(u_{\varepsilon t})^2\,dx\,dt\le c+c\int_\Omega \rho_\varepsilon^\alpha|\nabla u_{\varepsilon,0}|^{p(x)}dx\le c. \tag{3.11}$$

Differentiating (3.2) with t, and denoting $w=u_{\varepsilon t}$, we obtain

$$\begin{aligned}
\frac{\partial w}{\partial t} &= (\rho+\varepsilon)^\alpha(p(x)-2)(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-4}{2}}u_{x_k}u_{x_i}w_{x_kx_i}\\
&\quad+(\rho+\varepsilon)^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}w_{x_ix_i}\\
&\quad+(p(x)-2)\nabla[(\rho+\varepsilon)^\alpha]\cdot\nabla u_\varepsilon(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p-4}{2}}u_{x_k}w_{x_k}\\
&\quad+\nabla[(\rho+\varepsilon)^\alpha](|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}\cdot\nabla w\\
&\quad+(\rho+\varepsilon)^\alpha(p(x)-2)(p(x)-4)(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-6}{2}}u_{x_j}u_{x_ix_j}u_{x_i}w_{x_k}u_{x_k}\\
&\quad+(\rho+\varepsilon)^\alpha(p(x)-2)(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-4}{2}}\big(u_{x_i}u_{x_ix_k}w_{x_k}+u_{x_k}u_{x_kx_i}w_{x_i}\\
&\quad+u_{x_k}u_{x_ix_i}w_{x_k}\big).
\end{aligned}$$

We can rewrite it as

$$\frac{\partial w}{\partial t}=a_{ij}\frac{\partial^2 w}{\partial x_i\partial x_j}+f_i(x,t,w)w_{x_i},$$

where

$$a_{ij}=(\rho+\varepsilon)^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}(\delta_{ij}+(p(x)-2)(|\nabla u_\varepsilon|^2+\varepsilon)^{-1}u_{x_i}u_{x_j}).$$

$$\begin{aligned}
f_i(x,t,w) &= (p(x)-2)\nabla[(\rho+\varepsilon)^\alpha]\cdot\nabla u_\varepsilon(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-4}{2}}u_{x_i}\\
&\quad+(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}[(\rho+\varepsilon)^\alpha]_{x_i}\\
&\quad+(\rho+\varepsilon)^\alpha(p(x)-2)(p(x)-4)(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-6}{2}}u_{x_j}u_{x_kx_j}u_{x_k}u_{x_i}\\
&\quad+(\rho+\varepsilon)^\alpha(p(x)-2)(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-4}{2}}\big(u_{x_k}u_{x_kx_i}+u_{x_i}u_{x_ix_k}\\
&\quad+u_{x_i}u_{x_kx_k}\big).
\end{aligned}$$

Clearly, w satisfies

$$\begin{gathered}
w(x,t)=0,(x,t)\in\partial\Omega\times[0,T],\\
w(x,0)=\operatorname{div}((\rho+\varepsilon)^\alpha(|\nabla u_0|^2+\varepsilon)^{\frac{p(x)-2}{2}}\nabla u_0),\quad x\in\Omega.
\end{gathered}$$

If we denote

$$a_0=(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}},$$

then
$$\min\{p(x)-1,1\}a_0|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \max\{p(x)-1,1\}a_0|\xi|^2.$$
By the maximum principle, we have
$$\sup_{\Omega\times(0,T)} |u_{\varepsilon t}| \leq \sup_{\Omega} |\operatorname{div}((\rho+\varepsilon)^\alpha(|\nabla u_0|^2+\varepsilon)^{\frac{p(x)-2}{2}}\nabla u_0| \leq c, \tag{3.12}$$
because $u_0(x) \in C_0^\infty(\Omega), \alpha \geq 1$.

By (3.6),(3.8) and (3.11), we know that there exist a subsequence (still denoted as u_ε) of u_ε, a $n-$dimensional vector function $\overrightarrow{\zeta} = (\zeta_1, \cdots, \zeta_n)$, such that
$$\begin{gathered}
u_\varepsilon \rightharpoonup *u, \quad \text{in } L^\infty(Q_T),\\
u_\varepsilon \to u, \quad \text{in } L^r_{\rm loc}(Q_T),\\
\nabla u_\varepsilon \rightharpoonup \nabla u \quad \text{in } L^{p(x)}_{\rm loc}(Q_T),\\
\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}} \rightharpoonup \overrightarrow{\zeta} \quad \text{in } L^{\frac{p(x)}{p(x)-1}}(Q_T).
\end{gathered}$$
Here, if $p^- \geq 2$, $r=2$, while $p^- < 2$, $1 < r < p^{-*} = \frac{Np^-}{N-p^-}$ as usual.

Since for any function $\varphi \in C_0^\infty(Q_T)$,
$$\iint_{Q_T} (-u_\varepsilon\varphi_t + \rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}\nabla u_\varepsilon\cdot\nabla\varphi)\,dx\,dt = 0, \tag{3.13}$$
if $\varepsilon \to 0$, then
$$\iint_{Q_T} (\frac{\partial u}{\partial t}\varphi + \overrightarrow{\zeta}\cdot\nabla\varphi)\,dx\,dt = 0. \tag{3.14}$$
As in [32], we can prove that
$$\iint_{Q_T} \rho^\alpha|\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi\,dx\,dt = \iint_{Q_T} \overrightarrow{\zeta}\cdot\nabla\varphi\,dx\,dt \tag{3.15}$$
for any function $\varphi \in C_0^\infty(Q_T)$, then u is the solution of equation (1.5) with the initial condition (1.6). Thus, we have completed the proof. □

If u_0 only satisfies (3.1), we choose $u_{\varepsilon,0} \in C_0^\infty(\Omega)$, then $\|u_{\varepsilon,0}\|_{L^\infty(\Omega)}$ and
$$\|\rho_\varepsilon^\alpha|\nabla u_{\varepsilon,0}|^{p^+}\|_{L^1(\Omega)}$$
are uniformly bounded, and $u_{\varepsilon,0}$ converge to u_0 in $W^{1,p^+}_{\rm loc}(\Omega)$. We consider the regularized problem
$$u_{\varepsilon t} - \operatorname{div}(\rho_\varepsilon^\alpha(|\nabla u_\varepsilon|^2+\varepsilon)^{\frac{p(x)-2}{2}}\nabla u_\varepsilon) = 0, (x,t) \in Q_T, \tag{3.16}$$
$$u_\varepsilon(x,t) = 0, \quad (x,t) \in \partial\Omega\times(0,T), \tag{3.17}$$
$$u_\varepsilon(x,0) = u_{\varepsilon,0}(x), \quad x \in \Omega, \tag{3.18}$$
where $\rho_\varepsilon = \rho+\varepsilon,\ \varepsilon > 0$. For any $u_{\varepsilon,0} \in C_0^\infty(\Omega)$, $\rho_\varepsilon^\alpha|\nabla u_{\varepsilon,0}|^{p^+} \in L^1(\Omega)$, the above problem has a unique weak solution, and hence for any $\varphi \in C_0^\infty(Q_T)$, u_ε satisfies
$$\iint_{Q_T} (u_{\varepsilon t}\varphi + \rho_\varepsilon^\alpha|\nabla u_\varepsilon|^{p(x)-2}\nabla u_\varepsilon\cdot\nabla\varphi)\,dx\,dt = 0. \tag{3.19}$$
As in the proof of Lemma 3.2, except that the uniformly bounded estimate of $u_{\varepsilon t}$, we can prove the following lemma.

Lemma 3.3. *If $p^- > 1$, u_0 satisfies* (3.1). *Then the solution u_ε of initial boundary value problem* (3.16)-(3.18) *is convergent to u weakly star, and strongly convergent to $u \in L^r_{\rm loc}(Q_T)$, and u is the solution of equation* (1.5) *with the initial condition* (1.6).

Clearly, Theorem 3.1 is a direct corollary of Lemmas 3.2 and 3.3.

4. Uniqueness of the solution

Lemma 4.1. *If $\alpha < p^- - 1$, let $u(x,t)$ be the solution of* (1.5) *with the initial condition* (1.6). *Then there exists a constant $\gamma > 1$, such that*

$$\iint_{Q_T} |\nabla u|^\gamma \,dx\,dt \le c. \tag{4.1}$$

Proof. Since $\frac{\alpha}{p^- -1} < 1$ and $p^- - \alpha > 1$, there exists a constant $\beta \in (\frac{\alpha}{p^- -1}, 1)$ such that $p^- - \frac{\alpha}{\beta} > 1$. Because of $\beta < 1$ and $p^- - \frac{\alpha}{\beta} > 1$, there exists $\gamma \in (1, p^- - \frac{\alpha}{\beta})$ such that $\beta\gamma < 1$. Then we find

$$\begin{aligned}
&\iint_{Q_T} |\nabla u|^\gamma \,dx\,dt \\
&= \iint_{\{(x,t)\in Q_T;\rho^\beta|\nabla u|\leqslant 1\}} |\nabla u|^\gamma \,dx\,dt + \iint_{\{(x,t)\in Q_T;\rho^\beta|\nabla u|> 1\}} |\nabla u|^\gamma \,dx\,dt \\
&\leqslant \iint_{Q_T} \rho^{-\beta\gamma} \,dx\,dt + \iint_{Q_T} \rho^\alpha |\nabla u|^{\alpha/\beta+\gamma} \,dx\,dt \\
&\leqslant \iint_{Q_T} \rho^{-\beta\gamma} \,dx\,dt + \iint_{Q_T} \rho^\alpha (1 + |\nabla u|^{p^-}) \,dx\,dt \leqslant c.
\end{aligned}$$

∎

Thus, if $\alpha < p^- - 1$, $u(x,t)$ is the solution of equation (1.5) with the initial condition (1.6), then we can define the trace of u on the boundary of Ω.

Theorem 4.2. *If $\alpha < p^- - 1$, let u and v be two weak solutions of* (1.5) *with different initial values $u(x,0)$ and $v(x,0)$ respectively, and with the same homogeneous boundary condition*

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{4.2}$$

Then

$$\int_\Omega |u(x,t) - v(x,t)| dx \le \int_\Omega |u_0 - v_0| dx, \quad \forall t \in [0,T).$$

Proof. From the definition of the weak solution, $\rho^\alpha |\nabla u|^{p(x)}, \rho^\alpha |\nabla v|^{p(x)} \in L^1(Q_T)$, and for all $\varphi \in C_0^\infty(Q_T)$, we have

$$\iint_{Q_T} \varphi \frac{\partial(u-v)}{\partial t} \,dx\,dt = -\iint_{Q_T} \rho^\alpha (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v) \cdot \nabla\varphi \,dx\,dt. \tag{4.3}$$

For any given positive integer n, let $g_n(s)$ be an odd function. When $s \ge 0$, it is defined as

$$g_n(s) = \begin{cases} 1, & s > 1/n, \\ n^2 s^2 e^{1-n^2 s^2}, & s \leqslant 1/n. \end{cases}$$

Choosing $g_n(u-v)$ as the test function in (4.3), we have

$$\begin{aligned}&\iint_{Q_T} g_n(u-v)\frac{\partial(u-v)}{\partial t}\,dx\,dt\\&+\iint_{Q_T}\rho^{\alpha}(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\cdot\nabla(u-v)g_n'(u-v)=0.\end{aligned}\tag{4.4}$$

Thus we further have

$$\int_{\Omega} g_n(u-v)\frac{\partial(u-v)}{\partial t}dx=\frac{d}{dt}\|u-v\|_1,$$
$$\iint_{Q_T}\rho^{\alpha}(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\cdot\nabla(u-v)g_n'(u-v)\,dx\,dt\geqslant 0,$$

Let $n\to\infty$ in (4.4). Then $\frac{d}{dt}\|u-v\|_1\leqslant 0$. This implies

$$\int_{\Omega}|u(x,t)-v(x,t)|dx\leqslant\int_{\Omega}|u_0-v_0|dx,\quad \forall t\in[0,T).$$

In addition, if $u_0(x)=v_0(x)$, by the arbitrariness of t,

$$u(x,t)=v(x,t)\quad \text{a.e. in } (x,t)\in Q_T;$$

then the solution of (1.5) with the homogeneous boundary value is unique. $\square$

Theorem 4.3. *Suppose that $u_0\in L^{\infty}(\Omega)$ and $\rho^{\alpha}|\nabla u_0|^{p^+}\in L^1(\Omega)$. If $\alpha>p^+-1$, then for any $t\in[0,T)$, we have*

$$\int_{\Omega}[u(x,t)-v(x,t)]^2dx\leq\int_{\Omega}[u(x,0)-v(x,0)]^2dx.\tag{4.5}$$

In other words, the solution of equation (1.5) *is free from any limitations of the boundary condition.*

Proof. Denote $\Omega_\varepsilon=\{x\in\Omega: dist(x,\partial\Omega)>\varepsilon\}$ as before, and let $\xi_\varepsilon\in C_0^{\infty}(\Omega_\varepsilon)$ such that $\xi_\varepsilon=1$ on $\Omega_{2\varepsilon}$, $0\leq\xi_\varepsilon\leq 1$ and $|\nabla\xi_\varepsilon|\leq c/\varepsilon$. From the definition of the weak solution, we have

$$\begin{aligned}&\iint_{Q_T}\varphi\frac{\partial(u-v)}{\partial t}\,dx\,dt\\&=-\iint_{Q_T}\rho^{\alpha}(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\nabla\varphi\,dx\,dt,\quad\forall\varphi\in C_0^{\infty}(Q_T).\end{aligned}\tag{4.6}$$

For any fixed $s\in[0,T)$, after an approximate procedure, we may choose $\chi_{[0,s]}(u-v)\xi_\varepsilon$ as a test function in the above equality, where $\chi_{[0,s]}$ is the characteristic function on $[0,s]$. Thus we have

$$\iint_{Q_s}\varphi\frac{\partial(u-v)}{\partial t}\,dx\,dt=-\iint_{Q_s}\rho^{\alpha}(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v)\nabla\varphi\,dx\,dt,$$

and

$$
\begin{aligned}
&\int_\Omega [u(x,s)-v(x,s)]^2 \xi_\varepsilon dx \\
&= \int_\Omega [u(x,0)-v(x,0)]^2 \xi_\varepsilon dx \\
&\quad - 2\iint_{Q_s} \xi_\varepsilon \rho^\alpha (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla(u-v)\,dx\,dt \\
&\quad - 2\iint_{Q_s} (u-v)\rho^\alpha (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla \xi_\varepsilon\,dx\,dt \\
&\le \int_\Omega [u(x,0)-v(x,0)]^2 \xi_\varepsilon dx \\
&\quad - 2\iint_{Q_s} (u-v)\rho^\alpha (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla \xi_\varepsilon\,dx\,dt.
\end{aligned} \tag{4.7}
$$

Since

$$
\begin{aligned}
&\big| -2\iint_{Q_s} (u-v)\rho^\alpha (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla \xi_\varepsilon\,dx\,dt\big| \\
&\le 2\iint_{Q_s} |u-v|\rho^\alpha (|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1})|\nabla \xi_\varepsilon|\,dx\,dt \\
&\le c\int_0^T \int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}} \big[\frac{p(x)-1}{p(x)}\rho^\alpha(|\nabla u|^{p(x)} + |\nabla v|^{p(x)}) + \frac{1}{p(x)}\rho^\alpha|\nabla\xi_\varepsilon|^{p(x)}\big]\,dx\,dt \\
&\le c\int_0^T \int_{\Omega_\varepsilon\setminus\Omega_{2\varepsilon}} \big[\frac{p(x)-1}{p(x)}\rho^\alpha(|\nabla u|^{p(x)} + |\nabla v|^{p(x)}) + \frac{1}{p(x)}\varepsilon^{\alpha-p(x)}\big]\,dx\,dt,
\end{aligned} \tag{4.8}
$$

by $\alpha > p^+ - 1$ and $\alpha - p(x) > -1$, using (4.8) yields

$$
\lim_{\varepsilon\to 0}\big| -2\iint_{Q_s} (u-v)\rho^\alpha (|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v)\nabla \xi_\varepsilon\,dx\,dt\big| = 0. \tag{4.9}
$$

Let $\varepsilon \to 0$. By (4.7)-(4.9), the stability (4.5) is obviously true. The proof is complete. □

By Theorem 3.1, Theorem 4.2 and Theorem 4.3, we arrive at Theorem 1.4.

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