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LORENTZ ESTIMATES FOR ASYMPTOTICALLY REGULAR ELLIPTIC EQUATIONS IN QUASICONVEX DOMAINS

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ABSTRACT. We derive a global Lorentz estimate of the gradient of weak solutions to nonlinear elliptic problems with asymptotically regular nonlinearity in quasiconvex domains. Here, we prove its Lorentz estimate for such an asymptotically regular elliptic problem by constructing a regular problem via Poisson's formula, and quasiconvex domain locally approximated by convex domain.

1. INTRODUCTION

Let Ω be a bounded domain in $\mathbb{R}^n$ with $n \geq 2$, and $1 < p < \infty$ be a fixed real number. The main purpose of this paper is to attain a global estimate of the gradient of weak solutions in Lorentz spaces for the following zero Dirichlet problem of nonlinear elliptic equations:

$$\begin{gathered} \operatorname{div} \mathbf{a}(x, Du) = \operatorname{div}(|\mathbf{f}|^{p-2}\mathbf{f}), \quad \text{in } \Omega, \\ u = 0, \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where the vector-valued function $\mathbf{a}(x, Du) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is asymptotically regular (for details to see Definition 1.1), and $\mathbf{f}$ is any given vector-valued function in Lorentz spaces $L^{\gamma,q}(\Omega, \mathbb{R}^n)$ with $1 < p \leq \gamma < \infty$ and $0 < q \leq \infty$. A weak solution of the Dirichlet problem (1.1) is understood in the distributional sense, if $u \in W_0^{1,p}(\Omega)$ satisfies

$$\int_\Omega \langle \mathbf{a}(x, Du), D\phi\rangle dx = \int_\Omega \langle |\mathbf{f}|^{p-2}\mathbf{f}, D\phi\rangle dx, \quad \text{for all } \phi \in W_0^{1,p}(\Omega).$$

Recently, there have been a lot of research activities about regular elliptic problems, see the papers by Byun et al. [8, 9, 10, 11] and references therein. We notice that these papers are concerned with the Calderón-Zygmund estimates or Orlicz estimates to elliptic and parabolic equations defined in the domain of Reifenberg flat sense. Lorentz spaces are a two-parameter scale of spaces which refine Lebesgue spaces in some sense. Since the pioneering work of Talenti [23] based on symmetrization, there were a large of literature on the topic of Lorentz regularity to elliptic and parabolic PDEs. In particular, Mengesha-Phuc in [19] used a kind of geometrical approach to prove the weighted Lorentz regularity of the gradient for

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quasilinear elliptic p-Laplacian equations, and Zhang-Zhou [24] extended their results to the setting of quasilinear $p(x)$-Laplacian. Meanwhile, Baroni in [3, 4] made use of another approach, which is called Large-M-inequality principle introduced by Acerbi-Mingione in [1], to prove the Lorentz estimates of gradient for evolutionary p-Laplacian systems and obstacle parabolic p-Laplacian, respectively.

The objective of this paper is mainly devoted to considering Lorentz regularity of the gradient to the Dirchlet problems (1.1) by focusing on those two optimal conditions on the operator $\mathbf{a}(x,\xi)$ and $\partial\Omega$; that is to say, one is the smoothness on coefficients and the other is the geometry of $\partial\Omega$. To the boundary geometry of domain, the concept of Reifenberg flatness is already so general that it includes very rough domains like Koch snowflake, see [9, 10, 11, 14] for the precise concept of Reifenberg flat domains. However, as indicated in [7, 17, 18] Reifenberg flatness excludes some geometrical simple domains such as polygons. To this end, similar to the paper [7] we introduce the concept of quasiconvex domain, roughly speaking, whose boundary can be approximated from inside and outside by two convex surfaces in all scales, rather than two hyperplanes for Reifenberg flat domains. Very recently, there have been many interesting regularity problems to elliptic and parabolic PDEs defined over a quasiconvex domain. For example, Jia-Li-Wang developed global regularity in Sobolev space $W^{1,p}$ and Orlicz space $W_0^1L^\psi$ with $\psi\in\nabla_2\cap\triangle_2$ for linear divergence elliptic equations in [17] and [18], respectively. Byun-Kwon-So-Wang [7] extended the global Calderón-Zygmund estimates like $\|Du\|_{L^q(\Omega)}\lesssim\|\mathbf{f}\|_{L^q(\Omega)}$ for all $q\in[p,\infty)$ in quasiconvex domains to the setting of p-Laplacian elliptic equations.

Another point in this paper is that $\mathbf{a}(x,\xi)$ is assumed an asymptotically regular. Chipot and Evans [13] first introduced the notion of asymptotically regular in the elliptic framework, and Raymond [20] considered the Lipschitz regularity of solutions to asymptotically regular problems with p-growth. Since then, there is a large of literature on the topic of asymptotically regular. Scheven and Schmidt in [21, 22] obtained a local higher integrability and a local partial Lipschitz continuity with a singular set of positive measure for the gradient Du to the system which exhibits a certain kind of elliptic behavior near infinity, respectively. Furthermore, a global Lipschitz regularity result was extended by Foss in [15]. Very recently, Byun-Oh-Wang [12] proved global Calderón-Zygmund estimates for nonhomogeneous asymptotically regular elliptic and parabolic problems in divergence form in the Reifenberg flat domain by covering the given asymptotically regular problems to suitable regular problems. Later, Byun-Cho-Oh [6] extended the same conclusions to the setting of nonlinear obstacle elliptic problems. Zhang-Zheng [25] also further extended the work of Byun-Oh-Wang [12] to the case of obstacle parabolic problems in the scale of Lorentz spaces.

Our consideration is inspired by [7, 12, 19] regarding the Lorentz scales by refining Lebesgue spaces and the minimal smooth assumptions imposed on the nonlinearity "coefficients" and the geometry of domain. More precisely, our aim is to prove a global Lorentz estimate of the gradient for nonlinear elliptic problem with asymptotically regular nonlinearity in a quasiconvex domain as mentioned above. That is a natural refined outgrowth of Byun-Oh-Wang's paper [12] and Byun-Kwon-So-Wang's paper [7] in the following two aspects, Indeed, the Lebesgue space L^γ is a special case of Lorentz space $L^{\gamma,q}$ when $q=\gamma$ and the (δ,R)-Reifenberg flat domain in [12] is also a special case of (δ,σ,R)-quasiconvex domain. To attain our aim, some ideas from the papers [7, 12] are employed in our main proof. For example, to get the global Lorentz estimate we will make use of an equivalent representation of Lorentz norm, the Hardy-Littlewood maximal functions, and the Poisson formula by constructing a regular problem from the given irregular problem. Before stating the main result, let us give some basic concepts and facts.

We first recall that the Lorentz space $L^{\gamma,q}(\Omega)$ with $1 \le \gamma < \infty, 0 < q < \infty$ is the set of measurable function $g : \Omega \to \mathbb{R}$ such that

$$\|g\|^q_{L^{\gamma,q}(\Omega)} := q \int_0^\infty \Big(\mu^\gamma |\{\xi \in \Omega : |g(\xi)| > \mu\}|\Big)^{q/\gamma} \frac{d\mu}{\mu} < +\infty.$$

While the Lorentz space $L^{\gamma,\infty}$ for $1 \le \gamma < \infty, q = \infty$ is set to be the usual Marcinkiewicz space $\mathcal{M}^\gamma(\Omega)$ with quasinorm

$$\|g\|_{L^{\gamma,\infty}} = \|g\|_{\mathcal{M}^\gamma(\Omega)} := \sup_{\mu>0} \Big(\mu^\gamma |\{\xi \in \Omega : |g(\xi)| > \mu\}|\Big)^{1/\gamma} < +\infty.$$

The local variant of such spaces is defined in the usual way. Moreover, we note that by Fubini's theorem there holds

$$\|g\|^\gamma_{L^\gamma(\Omega)} = \gamma \int_0^\infty \Big(\mu^\gamma |\{\xi \in \Omega : |g(\xi)| > \mu\}|\Big) \frac{d\mu}{\mu} = \|g\|^\gamma_{L^{\gamma,\gamma}(\Omega)},$$

so that $L^\gamma(\Omega) = L^{\gamma,\gamma}(\Omega)$; cf. [3, 4, 5].

Asymptotically regular $\mathbf{a}(x,\xi)$ says the case that it is getting closer to some vector-valued function $\mathbf{b}(x,\xi)$ as $|\xi|$ goes to infinity, where $\mathbf{b}(x,\xi)$ satisfies the following assumptions:

(H1) $\mathbf{b}(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is measurable in x and differential in ξ, and satisfies the ellipticity and growth conditions:

$$\begin{gathered} \langle \partial_\xi \mathbf{b}(x,\xi)\eta, \eta\rangle \ge \lambda |\xi|^{p-2}|\eta|^2, \\ |\mathbf{b}(x,\xi)| + |\xi||\partial_\xi \mathbf{b}(x,\xi)| \le \Lambda |\xi|^{p-1}, \end{gathered} \tag{1.2}$$

for almost every $x \in \Omega$ and all $\xi, \eta \in \mathbb{R}^n$, where the structural constants satisfy $0 < \lambda \le 1 \le \Lambda \le \infty$.

(H2) ((δ, R)-vanishing) $\mathbf{b}(x,\xi)$ is (δ,R)-vanishing if we have

$$\omega_{\mathbf{b}}(R) := \sup_{0<r\le R} \sup_{x_0\in\Omega} \fint_{B_r(x_0)\cap\Omega} \beta(\mathbf{b}, B_r(x_0))(x)dx \le \delta,$$

where

$$\beta(\mathbf{b}, B_r(x_0))(x) := \sup_{\xi\in\mathbb{R}^n} \frac{|\mathbf{b}(x,\xi) - \overline{\mathbf{b}}_{B_r(x_0)}(\xi)|}{(1+|\xi|)^{p-1}}, \quad \overline{\mathbf{b}}_{B_r(x_0)}(\xi) = \fint_{B_r(x_0)\cap\Omega} \mathbf{b}(x,\xi)dx.$$

Definition 1.1 (Asymptotically δ-Regular)**.** Let $\mathbf{b}(x,\xi)$ satisfies the assumption (H1). Then we say that $\mathbf{a}(x,\xi)$ is asymptotically δ-regular with $\mathbf{b}(x,\xi)$ if there exists a uniformly bounded nonnegative function $\theta : [0,\infty) \to [0,\infty]$ such that

$$\limsup_{\rho\to 0} \theta(\rho) \le \delta$$

and

$$|\mathbf{a}(x,\xi) - \mathbf{b}(x,\xi)| \le \theta(|\xi|)(1 + |\xi|^{p-1})$$

for almost every $x \in \Omega$ and all $\xi \in \mathbb{R}^n$.

Remark 1.2. (i) From Definition 1.1, we can easily conclude that

$$\lim_{|\xi|\to\infty}\sup_{x\in\Omega}\frac{|\mathbf{a}(x,\xi)-\mathbf{b}(x,\xi)|}{|\xi|^{p-1}}\le 2\delta, \tag{1.3}$$

namely, for any sufficiently small $\delta>0$, $\mathbf{a}(x,\xi)$ is in a regular range as $|\xi|$ is near infinity. Throughout the paper we always assume that $\mathbf{a}(x,\xi)$ is asymptotically δ-regular with $\mathbf{b}(x,\xi)$ satisfying the assumption H1, where δ is to be determined later.

(ii) The above assumption (1.2) implies that the following monotonicity condition: for all $\xi,\eta\in\mathbb{R}^n$ and for almost every $x\in\mathbb{R}^n$,

$$\langle\mathbf{b}(x,\xi)-\mathbf{b}(x,\eta),\xi-\eta\rangle\ge\begin{cases}\nu(n,p,\lambda)(|\xi|+|\eta|)^{p-2}|\xi-\eta|^2, & \text{if } 1<p<2,\\ \nu(n,p,\lambda)|\xi-\eta|^p, & \text{if } p\ge 2.\end{cases}$$

(iii) By Browder-Minty Theorem, it is well known that under the basic assumption H1, the problem (1.1) has a unique weak solution provided $\mathbf{f}\in L^p(\Omega,\mathbb{R}^n)$ and $|\Omega|<\infty$, with the estimate

$$\|Du\|_{L^p(\Omega)}\le C(\lambda,p)\|\mathbf{f}\|_{L^p(\Omega)}. \tag{1.4}$$

(iv) The assumption that $\mathbf{b}(x,\xi)$ is (δ,R)-vanishing refines the assumption that $\mathbf{b}(x,\xi)$ is VMO_x, that is to say the nonlinearity $\mathbf{b}(x,\xi)$ has small BMO semi-norm uniformly with respect to the independent variables.

Next we introduce the definition of quasiconvex domain, see [7, Definition 1.3].

Definition 1.3. A bounded domain Ω is said to be (δ,σ,R)-quasiconvex if for all $x\in\partial\Omega, 0<r\le R$, the following properties hold:

(i) there exists a ball $B_{\sigma r}(x_0)\subset\Omega_r(x)$, where x_0 is relative to x and $\sigma\in(0,\frac14)$ is a uniform constant;

(ii) there exist a hyperplane $A(x,r)$ containing x, a unit normal vector $\mathbf{n}(x,r)$ to $A(x,r)$, and a half space $H(x,r)=\{y+t\mathbf{n}(x,r): y\in L(x,r), t\ge-\delta r\}$ such that

$$\Omega_r(x)\subset H(x,r)\cap B_r(x).$$

We would like to remark two points. The constant δ here is to be chosen in the range $(0,\frac{1}{2^{n+1}})$. By scaling the problem (1.1), we can take $R=1$ or any number bigger than 1, while δ is invariant under such scaling, see [7, Lemma 2.6].

Let us summarize our main result as follows.

Theorem 1.4. *Assume $1<p\le\gamma<\infty, 0<q\le\infty$ and $0<\sigma<\frac14$. Let $u\in W_0^{1,p}(\Omega)$ be the solution to Dirichlet problem (1.1) with the vector-valued function $\mathbf{a}(x,\xi)$ and $\mathbf{f}\in L^{\gamma,q}(\Omega)$. Then there exists a small $\delta=\delta(\sigma,n,p,\gamma,q,\lambda,\Lambda)>0$ such that if $\mathbf{a}(x,\xi)$ is asymptotically δ-regular with $\mathbf{b}(x,\xi)$ satisfying the assumptions H1 and H2, and Ω is (δ,σ,R)-quasiconvex, then $Du\in L^{\gamma,q}(\Omega)$ with the estimate*

$$\|Du\|_{L^{\gamma,q}(\Omega)}\le C\|F\|_{L^{\gamma,q}(\Omega)}, \tag{1.5}$$

for some positive constant $C=C(n,\lambda,\Lambda,p,\gamma,q,\theta)$ (except in the case $q=\infty$, where it depends only on $n,\lambda,\Lambda,p,\gamma,\theta$).

The rest of this article is organized as follows. In section 2, we state some properties of quasiconvex domains, Lorentz spaces and Hardy-Littlewood maximal function. Section 3 is devoted to proving Theorem 1.4. On the basis of global

Lorentz regularity for a regular problem, we prove our main result by taking a transformation from given asymptotically regular problem to a suitable regular problem.

2. Preliminaries

We begin this section by introducing some properties of quasiconvex domains. Set

$$\Omega_r(x) = \Omega \cap B_r(x), \quad \partial_\omega \Omega(x) = \partial\Omega \cap B_r(x),$$

and by

$$D(E,F) = \max\{\sup_{x\in E} dist(x,F), \sup_{y\in F} dist(y,E)\}$$

we denote the Hausdorff distance between two sets E and F in $\mathbb{R}^n$. It is clear that the quasiconvex domains are $W^{1,p}$ extension domains (see [16]) in which the extension theorem and Sobolev embedding theorem are available. The property (ii) in Definition 1.3 implies that quasiconvex domains are locally approximated by convex domains in the following sense, see [7, Lemma 3.3].

Lemma 2.1. *If Ω is a (δ,σ,R)-quasiconvex domain, then for each $x \in \partial\Omega$ and for every $r \in (0, \frac{R}{2})$, there exist two convex domains $F_r(x)$ and $F_r^*(x)$ such that*

$$F_r^*(x) \subset \Omega_r(x) \subset F_r(x) \quad D(F_r^*(x), F_r(x)) \le \frac{34\delta r}{\sigma^3}. \tag{2.1}$$

It is worthwhile noting that

$$F_r(x) = \cap_{y\in\partial_\omega\Omega_r(x)} H(y,2r) \cap B_r(x),$$

$$F_r^*(x) = \big\{x_0 + \big(1 - \frac{16r\delta}{\sigma^3}\big)(y-y_0) : y \in F_r(x)\big\},$$

where $H(y,2r)$ and $x_0 \in \Omega_r(x)$ are given in Definition 1.3.

The next lemma states some useful embedding relations in Lorentz spaces, see [19].

Lemma 2.2. *Let Ω be a bounded measurable subset of $\mathbb{R}^n$. Then the following holds:*

1. *If $0 < q_1, q_2 \le \infty$ and $p < \eta < \gamma < \infty$, then $L^{\gamma,q_1}(\Omega) \subset L^{\eta,q_2}(\Omega)$;*
2. *If $0 < q_1 < q_2 \le \infty$ and $p < \gamma < \infty$, then $L^{\gamma,q_1}(\Omega) \subset L^{\gamma,q_2}(\Omega) \subset L^{\gamma,\infty}(\Omega) \subset L^{\gamma-\varepsilon}(\Omega)$ for any $\varepsilon > 0$ such that $\gamma - \varepsilon > p$.*

One of the main tools in our approach is the Hardy-Littlewood maximal function, which allows us to control the local behavior of a function. For a function $g \in L^1_{\rm loc}(\mathbb{R}^n)$, the Hardy-Littlewood maximal function of g is defined by

$$\mathcal{M}g(x) = \sup_{r>0} \fint_{B_r(x)} |g(y)| dy.$$

Further, for a function defined on a bounded domain $U \subset \mathbb{R}^n$, we can define the Hardy-Littlewood maximal function locally by

$$\mathcal{M}_U g := \mathcal{M}(g\chi_U),$$

where χ is the standard characteristic function on U. We recall two basic properties of the Hardy-LIttlewood maximal function as follows:

$$|\{x\in\mathbb{R}^n : \mathcal{M}g(x) \ge \mu\}| \le \frac{C(n)}{\mu}\|g\|_{L^1(\mathbb{R}^n)}, \quad \text{for } \forall t > 0,$$

$$\|\mathcal{M}g\|_{L^p(\mathbb{R}^n)} \le C(n,p)\|g\|_{L^p(\mathbb{R}^n)}, \quad \text{for } 1 < p \le \infty.$$

Recently, this boundedness in L^p has been extended to Lorentz space by Mengesha and Phuc as follows; see [19, Lemma 3.11].

Lemma 2.3. *For any $1 < \gamma < \infty, 0 < q \le \infty$, there exists a constant $C = C(n,\gamma,q)$ such that*

$$\|\mathcal{M}g\|_{L^{\gamma,q}(\mathbb{R}^n)} \le C\|g\|_{L^{\gamma,q}(\mathbb{R}^n)}$$

for all $g \in L^{\gamma,q}(\mathbb{R}^n)$.

We will apply the following lemma to prove our global regularity estimates. This modified covering lemma accommodates the special needs for the conditions of (δ,R)-vanishing and quasiconvex domains; see [7, Lemma 2.5].

Lemma 2.4. *Assume E and F are measurable sets, $E \subset F \subset \Omega$ with $\Omega(\delta,\sigma,1)$-quasiconvex, and that there exists an $\varepsilon > 0$ such that*

(i) *$|E| < \varepsilon|B_1|$, and*
(ii) *for every $x \in B_1$, and all $r \in (0,1]$,*

$$|E \cap B_r(x)| \ge \varepsilon|B_r(x)| \quad \textit{implies} \quad B_r(x)\cap\Omega \subset F.$$

Then $|E| \le (\frac{5}{\sigma})^n \varepsilon |F|$.

We also need the following elementary characterization of functions in Lorentz spaces, see [2, Lemma 4.1] or [19, Lemma 3.12].

Lemma 2.5. *Let g be a nonnegative measurable function in a bounded domain $U \subset \mathbb{R}^n$. Let $\theta > 0$ and $\lambda > 1$ be constants. Then for any $0 < \gamma, q < \infty$, we have*

$$g \in L^{\gamma,q}(U) \Leftrightarrow S := \sum_{k\ge 1} \lambda^{tk} |\{x \in U : g(x) > \theta\lambda^k\}|^{q/\gamma} < +\infty,$$

and moreover

$$C^{-1}S \le \|g\|^t_{L^{\gamma,q}(U)} \le C(|U|^{q/\gamma} + S) \tag{2.2}$$

with constant $C = C(\theta,\lambda,q) > 0$. Analogously, for $0 < \gamma < \infty$ and $q = \infty$ we have

$$C^{-1}T \le \|g\|_{L^{\gamma,\infty}(U)} \le C(|U|^{1/\gamma} + T), \tag{2.3}$$

where T is the quantity

$$T := \sup_{k\ge 1} \lambda^k |\{x \in U : |g(x)| > \theta\lambda^k\}|^{1/\gamma}.$$

3. Proof of the main result

In this section, we are devoted to the proof of our main result based on the global estimate in Lorentz spaces for problem (1.1) with $b(x,\xi)$ satisfying the assumptions (H1) and (H2), see Theorem 3.2 below. To that end, we first introduce the following lemma; cf. [7, Lemma 4.5].

Lemma 3.1. *Assume that $u \in W_0^{1,p}(\Omega)$ is the weak solution of (1.1). Then there is a constant $N_0 = N_0(n,\lambda,\Lambda,p) > 1$ so that for any fixed $\varepsilon \in (0,1)$, one can find a small constant $\delta = \delta(\varepsilon) > 0$ such that if $\mathbf{b}$ is $(\delta, \frac{48}{\sigma})$-vanishing, Ω is $(\delta,\sigma,\frac{48}{\sigma})$-quasiconvex, and $B_r(y), 0 < r \le 1, y \in \Omega$, satisfies*

$$|\{x \in \Omega : \mathcal{M}(|Du|^p(x)) > N_0^p\} \cap B_r(y)| \ge \varepsilon|B_r(y)|,$$

then we have

$$B_r(y)\cap\Omega \subset \{x \in \Omega : \mathcal{M}(|Du|^p)(x) > 1\} \cup \{x \in \Omega : \mathcal{M}(|\mathbf{f}|^p)(x) > \delta^p\}.$$

Theorem 3.2. *Assume $1 < p \leq \gamma < \infty, 0 < q \leq \infty$ and $0 < \sigma < 1/4$. Let $u \in W_0^{1,p}(\Omega)$ be the weak solution to the Dirichlet problem (1.1) with the vector-valued function $\mathbf{b}(x,\xi)$ and $\mathbf{f} \in L^{\gamma,q}(\Omega)$. Then there exists small $\delta = \delta(\sigma, n, p, \gamma, q, \lambda, \Lambda) > 0$ such that if $\mathbf{b}(x,\xi)$ satisfies the assumptions (H1) and (H2), and Ω is (δ, σ, R)-quasiconvex, then $Du \in L^{\gamma,q}(\Omega)$ with the estimate*

$$\|Du\|_{L^{\gamma,q}(\Omega)} \leq C\|f\|_{L^{\gamma,q}(\Omega)}, \tag{3.1}$$

for some positive constant $C = C(n, \lambda, \Lambda, p, \gamma, q, \theta)$ (except in the case $q = \infty$, where it depends only on $n, \lambda, \Lambda, p, \gamma, \theta$).

Proof. Let $\varepsilon > 0$ be given, and we take $\delta > 0$ and $N_0 > 1$ as in Lemma 3.1. To that end, it suffices to show that for $\eta = \frac{p+\gamma}{2}$ there holds

$$\|\mathbf{f}\|_{L^\eta(\Omega)} \leq \delta \quad \Rightarrow \quad \|Du\|_{L^{\gamma,q}(\Omega)} \leq C. \tag{3.2}$$

In fact, by considering (3.2) and the normalization with

$$\tilde{u} = \frac{\delta u}{\|\mathbf{f}\|_{L^{\gamma,q}(\Omega)} + \mu} \quad \text{and} \quad \tilde{\mathbf{f}} = \frac{\delta \mathbf{f}}{\|\mathbf{f}\|_{L^{\gamma,q}(\Omega)} + \mu}, \quad \mu > 0,$$

we derive, after letting $\mu \to 0^+$, the desired result. Since $p \leq \eta \leq \gamma$, by Lemma 2.2 we see that the assumption $\|f\|_{L^\eta} \leq \delta$ is well defined. Therefore, under this assumption we set

$$E = \{x \in \Omega : \mathcal{M}(|Du|^p(x)) > N_0^p\},$$
$$F = \{x \in \Omega : \mathcal{M}(|Du|^p)(x) > 1\} \cup \{x \in \Omega : \mathcal{M}(|\mathbf{f}|^p)(x) > \delta^p\}.$$

Then, using the weak (1-1) estimate of Hardy-Littlewood maximal function, L_p-estimate (1.4), Hölder inequality and smallness of $\mathbf{f}$ in order, we can check the first hypothesis of Lemma 2.4 as follows:

$$\begin{aligned}
|E| &\leq \frac{C}{N_0^p} \int_\Omega |Du|^p dx \\
&\leq \frac{C}{N_0^p} \int_\Omega |\mathbf{f}|^p dx \\
&\leq \frac{C}{N_0^p} \|\mathbf{f}\|^p_{L^\eta(\Omega)} |\Omega|^{1-\frac{p}{\eta}} \\
&\leq C\delta^p |\Omega|^{1-\frac{p}{\eta}} \\
&\leq \varepsilon |B_1|,
\end{aligned}$$

by choosing a small $\delta = \delta(\varepsilon) > 0$, if necessary, in order to get the last inequality. Meanwhile, the second hypothesis of Lemma 2.4 follows directly from Lemma 3.1. Therefore, by Lemma 2.4 we have

$$\begin{aligned}
&|\{x \in \Omega : \mathcal{M}(|Du|^p)(x) > N_0^p\}| \\
&\leq \varepsilon_1 |\{x \in \Omega : \mathcal{M}(|Du|^p)(x) > 1\}| + \varepsilon_1 |\{x \in \Omega : \mathcal{M}(|F|^p)(x) > \delta^p\}|,
\end{aligned} \tag{3.3}$$

for $\varepsilon_1 = (5/\sigma)^n \varepsilon$. Using a simple iteration argument to (3.3), for any $\tau > 0$ we further have

$$\begin{aligned}
&|\{x \in \Omega : \mathcal{M}(|Du|^p)(x) > N_0^{kp}\}|^\tau \\
&\leq \varepsilon_2^k |\{x \in \Omega : \mathcal{M}(|Du|^p)(x) > 1\}|^\tau + \sum_{i=1}^k \varepsilon_2^i |\{x \in \Omega : \mathcal{M}(|F|^p)(x) > \delta^p N_0^{(k-i)p}\}|^\tau,
\end{aligned}$$

where $\varepsilon_2 = \max\{1, 2^{\tau-1}\}\varepsilon_1^\tau$. Then it follows that

$$\begin{aligned}
S &:= \sum_{k=1}^{\infty} N_0^{qk} |\{x \in \Omega : \mathcal{M}(|Du|^p)(x) > N_0^{kp}\}|^{q/\gamma} \\
&\le C \sum_{k=1}^{\infty} (N_0^q \varepsilon_2)^k |\{x \in \Omega : \mathcal{M}(|Du|^p)(x) > 1\}|^{q/\gamma} \\
&\quad + C \sum_{k=1}^{\infty} N_0^{qk} \Big[\sum_{i=1}^{k} \varepsilon_2^i |\{x \in \Omega : \mathcal{M}(|\mathbf{f}|^p)(x) > \delta^p N_0^{(k-i)p}\}|^{q/\gamma} \Big] \\
&\le C \sum_{k=1}^{\infty} (N_0^q \varepsilon_2)^k |\Omega|^{q/\gamma} \\
&\quad + C \sum_{i=1}^{\infty} (N_0^q \varepsilon_2)^i \Big[\sum_{k=i}^{\infty} N_0^{q(k-i)} |\{x \in \Omega : \mathcal{M}(|\mathbf{f}|^p)(x) > \delta^p N_0^{(k-i)p}\}|^{q/\gamma} \Big] \\
&\le C \sum_{k=1}^{\infty} (N_0^q \varepsilon_2)^k |\Omega|^{q/\gamma} + C \sum_{i=1}^{\infty} (N_0^q \varepsilon_2)^i \|\mathcal{M}(|\mathbf{f}|^p)(x)\|_{L^{\frac{\gamma}{p},\frac{q}{p}}(\Omega)}^{\frac{q}{p}} \\
&\le C \sum_{k=1}^{\infty} (N_0^q \varepsilon_2)^k \Big(|\Omega|^{q/\gamma} + \||\mathbf{f}|^p\|_{L^{\frac{\gamma}{p},\frac{q}{p}}(\Omega)}^{\frac{q}{p}} \Big).
\end{aligned}$$

Now choosing ε sufficiently small so that $N_0^q \varepsilon_2 < 1$, we obtain

$$\begin{aligned}
\|Du\|_{L^{\gamma,q}(\Omega)}^q &= \||Du|^p\|_{L^{\frac{\gamma}{p},\frac{q}{p}}(\Omega)}^{\frac{q}{p}} \\
&\le C \|\mathcal{M}(|Du|^p(x)\|_{L^{\frac{\gamma}{p},\frac{q}{p}}(\Omega)}^{\frac{q}{p}} \\
&\le C \Big(|\Omega|^{q/\gamma} + \|\mathbf{f}\|_{L^{\gamma,q}(\Omega)}^q \Big) \le C.
\end{aligned}$$

This completes the proof. □

The main ingredient to prove Theorem 1.4 is to use Poisson's formula to construct a regular Dirichlet problem whose nonlinearity satisfies the assumptions (H1) and (H2). Here, we first define a vector-valued function $\mathbf{c}(x,\xi) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$|\xi|^{p-1} \mathbf{c}(x,\xi) = \mathbf{a}(x,\xi) - \mathbf{b}(x,\xi). \tag{3.4}$$

Then, from (1.3) it yields that for any sufficiently small $\delta > 0$ there exists a positive constant $M = M(\delta)$ such that

$$|\xi| \ge M \Rightarrow |\mathbf{c}(x,\xi)| \le 2\delta, \tag{3.5}$$

uniformly in $x \in \mathbb{R}^n$. For any fixed point $x \in \mathbb{R}^n$, we consider the Poisson integral

$$P[\mathbf{c}(x,\cdot)](\xi) := \int_{\partial B_M} \mathbf{c}(x,\eta) K(\xi,\eta) d\sigma(\eta) \quad \text{for } \xi \in B_M,$$

where

$$K(\xi,\eta) = \frac{M^2 - |\xi|^2}{M\omega_{n-1}|\xi - \eta|^n} \quad \text{for all } \xi \in B_M \text{ and } \eta \in \partial B_M$$

is the Poisson kernel for the ball $B_M \subset \mathbb{R}^n$ with radius M, and ω_{n-1} is the surface area of the unit sphere ∂B_1 in $\mathbb{R}^n$. Let us denote a new vector-valued function

$\tilde{\mathbf{c}}(x,\xi)$ by

$$\tilde{\mathbf{c}}(x,\xi)=\begin{cases}\mathbf{c}(x,\xi), & \text{if } |\xi|\geq M,\\ P[\mathbf{c}(x,\cdot)](\xi), & \text{if } |\xi|<M.\end{cases} \tag{3.6}$$

Then we see that $\tilde{\mathbf{c}}(x,\xi)$ is a vector-valued function defined in $\mathbb{R}^n\times\mathbb{R}^n$. By the maximum principle and (3.5), it follows that for any $\xi\in\mathbb{R}^n$ there holds

$$|\tilde{\mathbf{c}}(x,\xi)|\leq 2\delta, \tag{3.7}$$

uniformly in $x\in\mathbb{R}^n$.

Now, by combining (3.4) with (3.6) we derive

$$\begin{aligned}\mathbf{a}(x,\xi)&=\mathbf{b}(x,\xi)+|\xi|^{p-1}\mathbf{c}(x,\xi)\\ &=\mathbf{b}(x,\xi)+|\xi|^{p-1}\tilde{\mathbf{c}}(x,\xi)+|\xi|^{p-1}\chi_{\{|\xi|<M\}}(\mathbf{c}(x,\xi)-\tilde{\mathbf{c}}(x,\xi)),\end{aligned} \tag{3.8}$$

where $\chi_{\{|\xi|<M\}}$ is the characteristic function on the set $\{\xi\in\mathbb{R}^n:|\xi|<M\}$.

Here, we introduce a new nonlinearity $\tilde{\mathbf{a}}(x,\xi)$, which is regular problem transferred from the asymptotically regular one. More precisley, for a given weak solution $u\in W_0^{1,p}(\Omega)$ of the Dirichlet problem (1.1) we define $\tilde{\mathbf{a}}(x,\xi):\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}^n$ by

$$\tilde{\mathbf{a}}(x,\xi):=\mathbf{b}(x,\xi)+|\xi|^{p-1}\tilde{\mathbf{c}}(x,Du(x)). \tag{3.9}$$

The following lemma play an important role in the proof of our main Theorem 1.5.

Lemma 3.3. *Let $u\in W_0^{1,p}(\Omega)$ be a weak solution of the Dirichlet problem* (1.1)*. Assume that $\mathbf{a}(x,\xi)$ is asymptotically δ-regular with $\mathbf{b}(x,\xi)$ satisfying the assumptions (H1) and (H2). Then we have the following conclusions:*

(i) If $0<\delta<\min\left\{\frac{\lambda}{4(p-1)},1\right\}$, then $\tilde{\mathbf{a}}(x,\xi)$ satisfies the ellipticity and growth conditions:

$$\begin{gathered}\langle\partial_\xi\tilde{\mathbf{a}}(x,\xi)\eta,\eta\rangle\geq\frac{\lambda}{2}|\xi|^{p-2}|\eta|^2,\\ |\tilde{\mathbf{a}}(x,\xi)|+|\xi||\partial_\xi\tilde{\mathbf{a}}(x,\xi)|\leq\widetilde{\Lambda}|\xi|^{p-1},\end{gathered} \tag{3.10}$$

for almost every $x\in\Omega$ and all $\xi,\eta\in\mathbb{R}^n$, where $\widetilde{\Lambda}=\Lambda+p$.

(ii) $\tilde{\mathbf{a}}(x,\xi)$ satisfies the $(5\delta,R)$-vanishing condition.

Proof. (i) For any given $0<\delta<\min\left\{\frac{\lambda}{4(p-1)},1\right\}$, by (3.9) and (3.7) it follows that

$$|\tilde{\mathbf{a}}(x,\xi)|\leq|\mathbf{b}(x,\xi)|+2|\xi|^{p-1}. \tag{3.11}$$

Since $\mathbf{b}(x,\xi)$ and $|\xi|^{p-1}$ are differentiable in ξ it implies

$$\begin{aligned}\partial_\xi\tilde{\mathbf{a}}(x,\xi)&=\partial_\xi\mathbf{b}(x,\xi)+\tilde{\mathbf{c}}(x,Du(x))D_\xi(|\xi|^{p-1})^T\\ &=\partial_\xi\mathbf{b}(x,\xi)+\tilde{\mathbf{c}}(x,Du(x))[(p-1)|\xi|^{p-3}\xi]^T,\end{aligned} \tag{3.12}$$

and further using (3.7) and $\delta\leq 1$, we obtain

$$|\partial_\xi\tilde{\mathbf{a}}(x,\xi)|\leq|\partial_\xi\mathbf{b}(x,\xi)|+2(p-1)|\xi|^{p-2}. \tag{3.13}$$

Then, by (3.11), (3.13) and (1.2) it follows that

$$\begin{aligned}|\tilde{\mathbf{a}}(x,\xi)|+|\xi||\partial_\xi\tilde{\mathbf{a}}(x,\xi)|&\leq|\mathbf{b}(x,\xi)|+2|\xi|^{p-1}+|\xi||\partial_\xi\mathbf{b}(x,\xi)|+2(p-1)|\xi|^{p-1}\\ &\leq\Lambda|\xi|^{p-1}+2|\xi|^{p-1}+2(p-1)|\xi|^{p-1}\\ &=\tilde{\Lambda}|\xi|^{p-1},\end{aligned}$$

where $\tilde{\Lambda} = \Lambda + 2p$. On the other hand, by (3.12), (1.2) and (3.7) we conclude that

$$\begin{aligned}\langle \partial_\xi \tilde{\mathbf{a}}(x,\xi)\eta,\eta\rangle &= \langle \partial_\xi \mathbf{b}(x,\xi)\eta,\eta\rangle + (p-1)|\xi|^{p-3}\tilde{\mathbf{c}}(x,Du(x))\xi^T\eta\cdot\eta\\ &\geq \lambda|\xi|^{p-2}|\eta|^2 - 2\delta(p-1)|\xi|^{p-2}|\eta|^2\\ &= (\lambda - 2\delta(p-1))|\xi|^{p-2}|\eta|^2\\ &\geq \frac{\lambda}{2}|\xi|^{p-2}|\eta|^2.\end{aligned}$$

Considering $0 < \delta \leq \frac{\lambda}{4(p-1)}$ we notice that $\lambda - 2\delta(p-1) \geq \frac{\lambda}{2}$. So (i) is proved.

(ii) Let $0 < r \leq R$ and $y \in \mathbb{R}^n$. Then, for any $\xi \in \mathbb{R}^n$ and any $\varepsilon > 0$ it follows from (3.9) and (3.7) that

$$\begin{aligned}|\tilde{\mathbf{a}}(x,\xi) - \overline{\tilde{\mathbf{a}}}_{B_r(y)}(\xi)| &\leq |\mathbf{b}(x,\xi) - \overline{\mathbf{b}}_{B_r(y)}(\xi)| + 2\varepsilon|\xi|^{p-1} + 2\varepsilon|\xi|^{p-1}\\ &= |\mathbf{b}(x,\xi) - \overline{\mathbf{b}}_{B_r(y)}(\xi)| + 4\varepsilon|\xi|^{p-1}.\end{aligned}$$

So

$$\begin{aligned}\omega_{\tilde{\mathbf{a}}}(R) &:= \sup_{0<r\leq R}\sup_{\xi\in\mathbb{R}^n} \fint_{B_r(y)} \frac{\tilde{\mathbf{a}}(x,\xi) - \overline{\tilde{\mathbf{a}}}_{B_r(y)}(\xi)}{(1+|\xi|)^{p-1}}dx\\ &\leq \sup_{0<r\leq R}\sup_{\xi\in\mathbb{R}^n} \fint_{B_r(y)} \frac{\mathbf{b}(x,\xi) - \overline{\mathbf{b}}_{B_r(y)}(\xi)}{(1+|\xi|)^{p-1}}dx + 4\varepsilon.\end{aligned}$$

Since $\mathbf{b}(x,\xi)$ is (δ,R)-vanishing, we know that there exists $R_0 > 0$ such that for any $0 < R \leq R_0$ we have

$$\omega_{\tilde{\mathbf{a}}}(R) \leq \varepsilon + 4\varepsilon = 5\varepsilon$$

namely, $\tilde{\mathbf{a}}(x,\xi)$ satisfies the $(5\delta,R)$-vanishing only if we choose $\delta = \varepsilon$. So (ii) is proved. □

We are now ready to prove our main result.

Proof of Theorem 1.4. From (3.8) and (3.9), for any given $0 < \delta < 1$ there exists a positive constant $M = M(\delta) > 1$ and a vector-valued function $\tilde{\mathbf{c}}(x,Du)$ such that $\tilde{\mathbf{c}}(x,Du) \leq 2\delta$ and

$$\begin{aligned}&\mathbf{a}(x,Du)\\ &= \mathbf{b}(x,Du) + |Du|^{p-1}\tilde{\mathbf{c}}(x,Du) + |Du|^{p-1}\chi_{\{|Du|<M\}}(\mathbf{c}(x,Du) - \tilde{\mathbf{c}}(x,Du))\\ &= \tilde{\mathbf{a}}(x,Du) + |Du|^{p-1}\chi_{\{|Du|<M\}}(\mathbf{c}(x,Du) - \tilde{\mathbf{c}}(x,Du)),\end{aligned}$$

which implies

$$div\mathbf{a}(x,Du) = div\tilde{\mathbf{a}}(x,Du) + \operatorname{div}(|Du|^{p-1}\chi_{\{|Du|<M\}}(\mathbf{c}(x,Du) - \tilde{\mathbf{c}}(x,Du))).$$

Thus from (1.1) and the above equality, we see that $u \in W_0^{1,p}(\Omega)$ is a weak solution of

$$\begin{aligned}\operatorname{div}\tilde{\mathbf{a}}(x,Du) &= \operatorname{div}(|\mathbf{f}|^{p-2}\mathbf{f}) - \operatorname{div}(|Du|^{p-1}\chi_{\{|Du|<M\}}(\mathbf{c}(x,Du) - \tilde{\mathbf{c}}(x,Du)))\\ &= \operatorname{div}(|\mathbf{f}|^{p-2}\mathbf{f} + |Du|^{p-1}\chi_{\{|Du|<M\}}(\tilde{\mathbf{c}}(x,Du) - \mathbf{c}(x,Du)))\\ &= \operatorname{div}(|\mathbf{g}|^{p-2}\mathbf{g}),\end{aligned} \tag{3.14}$$

where

$$\mathbf{g} = \frac{|\mathbf{f}|^{p-2}\mathbf{f} + |Du|^{p-1}\chi_{\{|Du|<M\}}(\tilde{\mathbf{c}}(x,Du) - \mathbf{c}(x,Du))}{||\mathbf{f}|^{p-2}\mathbf{f} + |Du|^{p-1}\chi_{\{|Du|<M\}}(\tilde{\mathbf{c}}(x,Du) - \mathbf{c}(x,Du))|^{\frac{p-2}{p-1}}}$$

if

$$\big||\mathbf{f}|^{p-2}\mathbf{f}+|Du|^{p-1}\chi_{\{|Du|<M\}}(\tilde{\mathbf{c}}(x,Du)-\mathbf{c}(x,Du))\big|\neq 0,$$

while $\mathbf{g}=0$ if

$$\big||\mathbf{f}|^{p-2}\mathbf{f}+|Du|^{p-1}\chi_{\{|Du|<M\}}(\tilde{\mathbf{c}}(x,Du)-\mathbf{c}(x,Du))\big|=0.$$

Then it is clear that $|\mathbf{g}|^{p-1}$ belongs to $L^{\gamma,q}$ locally in Ω with

$$\|\mathbf{g}\|_{L^{\gamma,q}(\Omega)}=q\int_0^\infty(\mu^\gamma|\{z\in\Omega:|\mathbf{g}(z)|>\mu\}|)^{q/\gamma}\frac{d\mu}{\mu}.$$

Let

$$\mathbf{h}=|\mathbf{f}|^{p-2}\mathbf{f}+|Du|^{p-1}\chi_{\{|Du|<M\}}(\tilde{\mathbf{c}}(x,Du)-\mathbf{c}(x,Du)),\tag{3.15}$$

this yields

$$|\mathbf{g}|=|\mathbf{h}|^{\frac{1}{p-1}}\Rightarrow|\mathbf{g}|^{p-1}=|\mathbf{h}|.\tag{3.16}$$

Then we obtain

$$\mu^{p-1}<|\mathbf{g}(z)|^{p-1}=|\mathbf{h}(z)|\leq|\mathbf{f}(z)|^{p-1}+4|Du|^{p-1}\chi_{\{|Du|<M\}},$$

and

$$\begin{aligned}&|\{z\in\Omega:|\mathbf{g}(z)|>\mu\}|\\&\leq|\{z\in\Omega:|\mathbf{f}(z)|>\frac{\mu}{2^{\frac{1}{p-1}}}\}|+|\{z\in\Omega:4|Du(z)|^{p-1}\chi_{\{|Du|<M\}}>\frac{\mu}{2^{\frac{1}{p-1}}}\}|.\end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathbf{g}\|_{L^{\gamma,q}(\Omega)}&\leq q\int_0^\infty\Big(\mu^\gamma|\{z\in\Omega:|\mathbf{f}(z)|>\frac{\mu}{2^{\frac{1}{p-1}}}\}|\Big)^{q/\gamma}\frac{d\mu}{\mu}\\&\quad+q\int_0^\infty\Big(\mu^\gamma|\{z\in\Omega:2|Du(z)|^{p-1}\chi_{\{|Du|<M\}}>\frac{\mu}{2^{\frac{1}{p-1}}}\}|\Big)^{q/\gamma}\frac{d\mu}{\mu}\\&=2^{\frac{q}{p-1}}q\int_0^\infty\Big(\mu^\gamma|\{z\in\Omega:|\mathbf{f}(z)|>\mu\}|\Big)^{q/\gamma}\frac{d\mu}{\mu}\\&\quad+2^{\frac{q}{p-1}}q\int_0^\infty\Big(\mu^\gamma|\{z\in\Omega:4|Du(z)|^{p-1}\chi_{\{|Du|<M\}}>\mu\}|\Big)^{q/\gamma}\frac{d\mu}{\mu}.\end{aligned}$$

Note that

$$|\{z\in\Omega:4|Du(z)|^{p-1}\chi_{\{|Du|<M\}}>\mu\}|\leq|\{z\in\Omega:4M^{p-1}>\mu\}|,$$

it follows that

$$\begin{aligned}&\|\mathbf{g}\|_{L^{\gamma,q}(\Omega)}\\&\leq 2^{\frac{q}{p-1}}\|\mathbf{f}\|_{L^{\gamma,q}(\Omega)}+2^{\frac{q}{p-1}}q\int_0^\infty\big(\mu^\gamma|\{z\in\Omega:4M^{p-1}>\mu\}|\big)^{q/\gamma}\frac{d\mu}{\mu}\\&\leq 2^{\frac{q}{p-1}}\|\mathbf{f}\|_{L^{\gamma,q}(\Omega)}+2^{\frac{q}{p-1}}q\int_0^{4M^{p-1}}\big(\mu^\gamma|\{z\in\Omega:4M^{p-1}>\mu\}|\big)^{q/\gamma}\frac{d\mu}{\mu}\\&\quad+2^{\frac{q}{p-1}}q\int_{4M^{p-1}}^\infty\big(\mu^\gamma|\{z\in\Omega:4M^{p-1}>\mu\}|\big)^{q/\gamma}\frac{d\mu}{\mu}\\&=2^{\frac{q}{p-1}}\|\mathbf{f}\|_{L^{\gamma,q}(\Omega)}+2^{\frac{q}{p-1}}q\int_0^{4M^{p-1}}(\mu^\gamma|\Omega|)^{q/\gamma}\frac{d\mu}{\mu}+0\\&=2^{\frac{q}{p-1}}\|\mathbf{f}\|_{L^{\gamma,q}(\Omega)}+2^{\frac{q}{p-1}}q|\Omega|^{q/\gamma}\int_0^{4M^{p-1}}\mu^{q-1}d\mu\end{aligned}$$

$$
\begin{aligned}
&= 2^{\frac{q}{p-1}} \|\mathbf{f}\|_{L^{\gamma,q}(\Omega)} + 2^{\frac{q}{p-1}} |\Omega|^{q/\gamma} (4M^{p-1})^q \\
&\le C(\|\mathbf{f}\|_{L^{\gamma,q}(\Omega)} + 1), \qquad (3.17)
\end{aligned}
$$

where $C = C(n, \delta, p, \gamma, q, \theta, |\Omega|)$ is a positive constant.

Recalling Lemma 3.3 and using (3.14) and (3.17), we employ Theorem 3.2 with $\mathbf{b}(x, \xi)$ replaced by $\tilde{\mathbf{a}}(x, \xi)$ and $\mathbf{f}$ replaced by $\mathbf{g}$, respectively, which completes the proof. $\square$

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