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INVERSE PROBLEMS OF PERIODIC SPATIAL DISTRIBUTIONS FOR A TIME FRACTIONAL DIFFUSION EQUATION

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Abstract. For a time fractional diffusion equation and a diffusion-wave equation with Caputo partial derivative we prove the inverse problem is well posed. This problem consists in the restoration of the initial data of a classical solution in time and with values in a space of periodic spatial distributions. A time integral over-determination condition is used.

1. Introduction

Inverse problems to equations of fractional order with respect to time with different unknown quantities (coefficients, right-hand sides, initial data), under different over-determination conditions, are actively studied in connection with their applications (see, for instance, [1, 2, 4, 10, 11, 14, 17, 22, 25]). Sufficient conditions of classical solvability of a time fractional Cauchy problem and the first boundary-value problem to a time fractional diffusion equation were obtained, for example, in [8, 13, 15, 24]. Comparison on correctness of the inverse problems for an equation of fractional diffusion and its corresponding ordinary diffusion equation is made in [6]. In particular, there was determined that the problem with reverse time to an equation with a fractional derivative of order $\alpha \in (0,1)$ is correct in contrast to the corresponding problem for ordinary diffusion equation. Some studies of the inverse problems (see, for instance, [1, 5, 11]) use the integral type over-determination condition.

In this article, for a time fractional diffusion equation and diffusion-wave equation we study the inverse problem for restoration the initial data of a solution, classical in time and with values in a space of periodic spatial distributions. We use a time integral over-determination condition. Note that the solvability of some nonclassical direct problems for partial differential equations with integral initial conditions, in particular, in the space of periodic spatial variable functions have been established, for example, in [9, 20, 21].

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2. Definitions and auxiliary results

Assume that $\mathbb{N}$ is a set of natural numbers, $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, $\mathcal{D}(\mathbb{R})$ is the space of indefinitely differentiable functions with compact supports, $\mathcal{S}(\mathbb{R})$ is the space of rapidly decreasing indefinitely differentiable functions [23, p. 90], while $\mathcal{D}'(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$ are the spaces of linear continuous functionals (distributions) respectively over $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$, and the symbol (f, φ) stands for the value of the distribution f on the test function φ. Note that $\mathcal{S}'(\mathbb{R})$ is the space of slowly increasing distributions. Let $\mathcal{D}'_+(\mathbb{R}) = \{f \in \mathcal{D}'(\mathbb{R}) : f = 0 \text{ for } t < 0\}$.

We denote by $f * g$ the convolution of the distributions f and g, and use the function $f_\lambda \in \mathcal{D}'_+(\mathbb{R})$:

$$f_\lambda(t) = \begin{cases} \frac{\vartheta(t) t^{\lambda-1}}{\Gamma(\lambda)}, & \lambda > 0, \\ f'_{1+\lambda}(t), & \lambda \le 0, \end{cases}$$

where $\Gamma(\lambda)$ is the Gamma-function, $\vartheta(t)$ is the Heaviside function. Note that

$$f_\lambda * f_\mu = f_{\lambda+\mu}.$$

Recall that the Riemann-Liouville derivative $v_t^{(\alpha)}(x,t)$ of order $\alpha > 0$ is defined [23, p. 87] as

$$v_t^{(\alpha)}(x,t) = f_{-\alpha}(t) * v(x,t),$$

the regularized fractional derivative (Caputo derivative, or Caputo-Djrbashian derivative) is defined [3, 8, 19] by

$$
\begin{aligned}
{}^cD_t^\alpha v(x,t) &= \frac{1}{\Gamma(1-\alpha)} \Big[\frac{\partial}{\partial t} \int_0^t \frac{v(x,\tau)}{(t-\tau)^\alpha} d\tau - \frac{v(x,0)}{t^\alpha} \Big] \quad \text{for } \alpha \in (0,1), \\
{}^cD_t^\alpha v(x,t) &= \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{v_{\tau\tau}(x,\tau)}{(t-\tau)^{\alpha-1}} d\tau \\
&= \frac{1}{\Gamma(2-\alpha)} \Big[\frac{\partial}{\partial t} \int_0^t \frac{u_\tau(x,\tau)}{(t-\tau)^{\alpha-1}} d\tau - \frac{u_t(x,0)}{t^{\alpha-1}} \Big] \quad \text{for } \alpha \in (1,2).
\end{aligned}
$$

Then

$$ {}^cD_t^\alpha v(x,t) = v_t^{(\alpha)}(x,t) - f_{1-\alpha}(t) v(x,0), \quad \alpha \in (0,1), \tag{2.1}$$

$$ {}^cD_t^\alpha v(x,t) = v_t^{(\alpha)}(x,t) - f_{1-\alpha}(t) v(x,0) - f_{2-\alpha}(t) v_t(x,0), \quad \alpha \in (1,2). \tag{2.2}$$

We denote ${}^cD_t^1 v = \frac{\partial v}{\partial t}$, and use the Mittag-Leffler function [3],

$$E_{\alpha,\mu}(x) = \sum_{p=0}^\infty \frac{z^p}{\Gamma(p\alpha + \mu)}.$$

The function $E_{\alpha,\mu}(-x)$ $(x > 0)$ is infinitely differentiable for $\alpha \in (0,2)$, $\mu \in \mathbb{R}$. It satisfies the estimate [18],

$$E_{\alpha,\mu}(-x) \le \frac{r_{\alpha,\mu}}{1+z}, \quad x > 0,$$

where $r_{\alpha,\mu}$ is a positive constant, and has the asymptotic behavior

$$E_{\alpha,\mu}(-x) = O\big(\frac{1}{x}\big), \quad x \to +\infty. \tag{2.3}$$

Let $X_k(x) = \sin kx$, $k \in \mathbb{N}$. Similarly to [23, p. 120], we denote by $\mathcal{D}'_{2\pi}(\mathbb{R})$ the space of periodic distributions, i.e., the space of $v \in \mathcal{D}'(\mathbb{R})$ such that

$$v(x+2\pi) = v(x) = -v(-x) \quad \forall x \in \mathbb{R}.$$

The formal series

$$\sum_{k=1}^{\infty} v_k X_k(x), \quad x \in \mathbb{R} \tag{2.4}$$

is the Fourier series of the distribution $v \in \mathcal{D}'_{2\pi}(\mathbb{R})$, and numbers

$$v_k = \frac{2}{\pi}(v, X_k)_{2\pi} = \frac{2}{\pi}(v, hX_k)$$

are its Fourier coefficients. Here $h(x)$ is even function from $\mathcal{D}(\mathbb{R})$ possessing the properties:

$$h(x) = \begin{cases} 1, & x \in (-\pi+\varepsilon, \pi-\varepsilon) \\ 0, & x \in \mathbb{R} \setminus (-\pi, \pi) \end{cases}$$

Note that $0 \le h(x) \le 1$, and

$$v_k = \frac{2}{\pi}\int_0^{\pi} v(x)X_k(x)dx \quad \text{for } v \in \mathcal{D}'_{2\pi}(\mathbb{R}) \cap L^1_{\rm loc}\,.$$

Then the series (2.4) is the classical Fourier series of v by the system X_k, $k \in \mathbb{N}$.

As is known (see [23, p. 123]), $\mathcal{D}'_{2\pi}(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R})$, the series (2.4) of $v \in \mathcal{D}'_{2\pi}(\mathbb{R})$ converges in $\mathcal{S}'(\mathbb{R})$ to v, and the Fourier coefficients (clearly defined) satisfy

$$|v_k| \le C_0(m)C(v,m)(1+k)^m \quad \forall k \in \mathbb{N}$$

for some $m \in \mathbb{Z}_+$ where $C_0(m)$, $C(v,m)$ are positive constants, the same for all $k \in \mathbb{N}$, in particular,

$$C(v,m) \;=\; \Big(\int_{\mathbb{R}} (1+x^2)^{-m/2}|v(x)|dx\Big)^{1/2}.$$

The number m is called the order of the distribution v. Note that the order of a regular periodic distribution is a non-positive number.

We assume that $\gamma \in \mathbb{R}$ and define

$$H^{\gamma}(\mathbb{R}) = \{v \in \mathcal{D}'_{2\pi}(\mathbb{R}) : \|v\|_{H^{\gamma}(\mathbb{R})} = \sup_{k\in\mathbb{N}} |v_k|(1+k)^{\gamma} < +\infty\}\,.$$

Note that functions in $H^{\gamma}(\mathbb{R})$ have the order $-\gamma$ in the sense of the above definition. Let $C\big([0,T];H^{\gamma}(\mathbb{R})\big)$ be the space of functions $v(x,t)$ that are continuous in $t \in [0,T]$, with values $v(\cdot,t) \in H^{\gamma}(\mathbb{R})$ endowed with the norm

$$\|v\|_{C\left([0,T];H^{\gamma}(\mathbb{R})\right)} = \max_{t\in[0,T]} \|v(\cdot,t)\|_{H^{\gamma}(\mathbb{R})}\,.$$

Let $C_{2,\alpha}\big([0,T];H^{\gamma}(\mathbb{R})\big) = \{v \in C\big([0,T];H^{2+\gamma}(\mathbb{R})\big) : {}^cD^{\alpha}v \in C\big([0,T];H^{\gamma}(\mathbb{R})\big)\}$ be the subspace endowed with the norm

$$\|v\|_{C_{2,\alpha}\left([0,T];H^{\gamma}(\mathbb{R})\right)} = \max\{\|v\|_{C\left([0,T];H^{2+\gamma}(\mathbb{R})\right)}, \|{}^cD^{\alpha}v\|_{C\left([0,T];H^{\gamma}(\mathbb{R})\right)}\}.$$

Note that $H^{\gamma+\varepsilon}(\mathbb{R}) \subset H^{\gamma}(\mathbb{R})$ for all $\varepsilon > 0$, $\gamma \in \mathbb{R}$.

3. Well posedness of the problem

In this section we study the problem

$$
{}^cD_t^\alpha u - u_{xx} = F_0(x,t), \quad (x,t) \in Q_T := \mathbb{R} \times (0,T], \tag{3.1}
$$

$$
u(x,0) = F_1(x), \quad u_t(x,0) = F_2(x), \quad x \in \mathbb{R}, \tag{3.2}
$$

$$
\int_0^{t_0} u(x,t)dt = \Phi(x), \quad x \in \mathbb{R},\ t_0 \in (0,T] \tag{3.3}
$$

where $\alpha \in (0,1) \cup (1,2)$, F_0, F_2, Φ are given functions, T is a given positive number, F_1 is an unknown function. The second condition in (3.2) is omitted when $\alpha \in (0,1)$.

We use the following assumptions:

(A1) $\gamma \in \mathbb{R}$, $\theta \in (0,1)$, $F_0 \in C\big([0,T]; H^{\gamma+2+2\theta}(\mathbb{R})\big)$, $F_2 \in H^{\gamma+2}(\mathbb{R})$; $F_0 \in C\big([0,T]; H^{\gamma+2}(\mathbb{R})$ and $F_2 = 0$ if $\alpha \in (0,1)$;

(A2) $\Phi \in H^{\gamma+4}(\mathbb{R})$ and, in addition, $E_{\alpha,2}(-k^2 t_0^\alpha) \neq 0$ for all $k \in \mathbb{N}$ if $\alpha \in (1,2)$.

We remark that $E_{\alpha,\mu}(-k^2 t^\alpha) > 0$ for all $t > 0$, and $\mu \geq \alpha$ if $\alpha \in (0,1)$ (see [18]). In the case $\alpha \in (1,2)$, the functions $E_{\alpha,1}(-z)$, $E_{\alpha,2}(-z)$ have a finite number of real positive zeroes [18], therefore, there exists a certain $t_0 \in (0,T]$ such that $E_{\alpha,2}(-k^2 t_0^\alpha) \neq 0$ for all $k \in \mathbb{N}$.

Note that the existence of a solution to the fractional Cauchy problem, which is classical in time and belongs to Bessel potential classes in space variables, was proved in [12], the existence and uniqueness theorems to the boundary-value problems for partial differential equations in Sobolev spaces were obtained by Yu. Berezansky, Ya. Roitberg, J.-L. Lions, E. Magenes, V. A. Mikhailets, A. A. Murach and others (see [16] and references therein).

Decompose the functions $F_0(x,t)$, $F_j(x)$, $j \in \{1,2\}$, $\Phi(x)$ in formal Fourier series by the system $X_k(x)$, $k \in \mathbb{N}$:

$$
\begin{gathered}
F_0(x,t) = \sum_{k=1}^\infty F_{0k}(t) X_k(x), \quad (x,t) \in Q_T, \\
F_j(x) = \sum_{k=1}^\infty F_{jk} X_k(x), \quad x \in \mathbb{R},\ j = 1,2, \\
\Phi(x) = \sum_{k=1}^\infty \Phi_k X_k(x), \quad x \in \mathbb{R}.
\end{gathered} \tag{3.4}
$$

Definition 3.1. A pair of functions

$$
(u, F_1) \in \mathcal{M}_{\alpha,\gamma} := C_{2,\alpha}\big([0,T]; H^\gamma(\mathbb{R})\big) \times H^{\gamma+2}(\mathbb{R})
$$

given by the series

$$
u(x,t) = \sum_{k=1}^\infty u_k(t) X_k(x), \quad (x,t) \in Q_T \tag{3.5}
$$

and (3.4) with $j = 1$, satisfying the equation (3.1) in $\mathcal{S}'(\mathbb{R})$ and the conditions (3.2), (3.3), is called a solution of the problem (3.1)–(3.3) under assumptions (A1), (A2).

Substituting the function (3.5) in (3.1) and conditions (3.2), (3.3), we obtain the problems

$$
{}^cD^\alpha u_k + k^2 u_k = F_{0k}(t),\ t \in (0,T], \quad u_k(0) = F_{1k},\ u_k'(0) = F_{2k}, \tag{3.6}
$$

$$\int_0^{t_0} u_k(t)dt = \Phi_k, \quad k \in \mathbb{N} \tag{3.7}$$

for the unknown $u_k(t)$, $k \in \mathbb{N}$. Given the link (2.2) between the derivatives in the sense of Riemann-Liouville and Caputo–Djrbashian, we write the problems (3.6) in the form

$$u_k^{(\alpha)} + k^2 u_k = F_{0k}(t) + f_{1-\alpha}(t)F_{1k} + f_{2-\alpha}(t)F_{2k}, \quad t \in (0,T],\ k \in \mathbb{N}. \tag{3.8}$$

So, the pairs $(u_k(t), F_{1k})$ ($k \in \mathbb{N}$) of the Fourier coefficients of the problem's solution satisfy the equations (3.8) and the conditions (3.7).

Theorem 3.2. *Assume that $\gamma \in \mathbb{R}$, $\theta \in (0,1)$, $F_0 \in C\big([0,T]; H^{\gamma+2\theta}(\mathbb{R})\big)$, $F_j \in H^{\gamma+2}(\mathbb{R})$, $j = 1,2$, if $\alpha \in (1,2)$; and $F_0 \in C\big([0,T]; H^{\gamma}(\mathbb{R})\big)$, $F_1 \in H^{\gamma+2}(\mathbb{R})$, $F_2 = 0$, if $\alpha \in (0,1]$. Then there exists a unique solution $u \in C_{2,\alpha}\big([0,T]; H^{\gamma}(\mathbb{R})\big)$ to the direct problem (3.1),(3.2). It is given by (3.5) where*

$$\begin{aligned} u_k(t) &= t^{\alpha-1}E_{\alpha,\alpha}(-k^2t^\alpha) * F_{0k}(t) + F_{1k}E_{\alpha,1}(-k^2t^\alpha) \\ &\quad + F_{2k}tE_{\alpha,2}(-k^2t^\alpha), \quad t \in [0,T],\ k \in \mathbb{N}. \end{aligned} \tag{3.9}$$

The solution depends continuously on the data (F_0, F_1, F_2), and the following inequality of coercivity holds:

$$\|u\|_{C_{2,\alpha}\big([0,T];H^\gamma(\mathbb{R})\big)} \le a_0\|F_0\|_{C\big([0,T];H^{\gamma+2\theta}(\mathbb{R})\big)} + \sum_{j=1}^{2} a_j\|F_j\|_{H^{\gamma+2}(\mathbb{R})}, \tag{3.10}$$

where a_j, $j \in \{0,1,2\}$ are positive constants independent of data, $F_2 = 0$ and $\theta = 0$ in (3.10) if $\alpha \in (0,1]$.

Proof. By the method of successive approximations one can find each solution (3.9) of equations (3.8). Uniqueness of a solution follows from the convolution properties in $\mathcal{D}'_+(\mathbb{R})$. Let us explain that $u \in C_{2,\alpha}\big([0,T]; H^\gamma(\mathbb{R})\big)$.

Assume that $\alpha \in (1,2)$, $u_{k0}(t) = t^{\alpha-1}E_{\alpha,\alpha}(-k^2t^\alpha) * F_{0k}(t)$. Then

$$\begin{aligned} |u_{k0}(t)| &\le \sup_{t\in(0,T]} |F_{0k}(t)| \int_0^t \tau^{\alpha-1}|E_{\alpha,\alpha}(-k^2\tau^\alpha)|d\tau \\ &\le r_{\alpha,\alpha} \sup_{t\in(0,T]} |F_{0k}(t)| \int_0^t \frac{\tau^{\alpha-1}d\tau}{1+k^2\tau^\alpha} \\ &= \frac{r_{\alpha,\alpha}}{\alpha k^2} \sup_{t\in(0,T]} |F_{0k}(t)| ln(1+k^2t^\alpha) \\ &\le \frac{r_{\alpha,\alpha}}{\alpha} \sup_{t\in(0,T]} |F_{0k}(t)| t^{s\alpha}k^{2s-2} \quad \forall s > 0. \end{aligned}$$

In what follows, $K_j = K_j(\alpha,\gamma)$, $j \in \{0,1,2\}$, will be positive constants. For $s = \theta$, using the inequality

$$k^\gamma \le c(\gamma)(1+k)^\gamma, \quad c(\gamma) = \begin{cases} 1, & \gamma \ge 0, \\ 2^{-\gamma}, & \gamma < 0, \end{cases}$$

one obtains

$$(1+k)^{\gamma+2}|u_{k0}(t)| \le K_0 \sup_{t\in(0,T]} |F_{0k}(t)|(1+k)^{\gamma+2\theta}, \quad t \in [0,T],\ k \in \mathbb{N}.$$

Using the boundedness of $E_{\alpha,j}(-k^2t^\alpha)$, $j=1,2$, one obtains

$$t^{j-1}|F_{jk}E_{\alpha,j}(-k^2t^\alpha)|(1+k)^{\gamma+2} \le K_j|F_{jk}|(1+k)^{\gamma+2}, \quad t\in[0,T],\ j\in\{1,2\},\ k\in\mathbb{N}.$$

So, the function (3.5) belongs to $C\big([0,T];H^\gamma(\mathbb{R})\big)$ and the following inequality of coercivity follows from (3.5) and (3.9):

$$\|u\|_{C([0,T];H^{\gamma+2}(\mathbb{R}))} \le \hat{a}_0\|F_0\|_{C\left([0,T];H^{\gamma+2\theta}(\mathbb{R})\right)} + \sum_{j=1}^{2}\hat{a}_j\|F_j\|_{H^{\gamma+2}(\mathbb{R})}, \tag{3.11}$$

where $\hat{a}_j$, $j\in\{0,1,2\}$ are positive constants independent of the data in this problem.

From (3.6), it follows the existence of continuous derivatives ${}^cD^\alpha u_k(t)$, $t\in(0,T]$, and the following estimates:

$$|{}^cD^\alpha u_k(t)| \le k^2|u_k(t)| + |F_{0k}(t)|, \quad k\in\mathbb{N},$$

$$\begin{aligned}
&\|{}^cD^\alpha u\|_{C([0,T];H^\gamma(\mathbb{R}))}\\
&= \max_{t\in[0,T]}\sup_{k\in\mathbb{N}}|{}^cD^\alpha u_k(t)|(1+k)^\gamma\\
&\le \max_{t\in[0,T]}\Big[\sup_{k\in\mathbb{N}}k^2|u_k(t)|(1+k)^\gamma + \sup_{k\in\mathbb{N}}|F_{0k}(t)|(1+k)^{\gamma+2\theta}(1+k)^{-2\theta}\Big]\\
&\le \|u\|_{C([0,T];H^{\gamma+2}(\mathbb{R}))} + \|F_0\|_{C\left([0,T];H^{\gamma+2\theta}(\mathbb{R})\right)}.
\end{aligned}$$

So, $u\in C_{2,\alpha}\big([0,T];H^\gamma(0,l)\big)$. By using the last inequality and (3.11), we obtain (3.10).

In the case $\alpha\in(0,1)$, we have $E_{\alpha,\mu}(-x)>0$ for $x>0$, $\mu\ge\alpha$ and

$$\begin{aligned}
\int_0^t \tau^{\alpha-1}E_{\alpha,\alpha}(-k^2\tau^\alpha)d\tau &= \int_0^t\sum_{p=0}^\infty\frac{(-1)^pk^{2p}\tau^{p\alpha+\alpha-1}d\tau}{\Gamma(p\alpha+\alpha)}\\
&= \sum_{p=0}^\infty\frac{(-1)^pk^{2p}t^{p\alpha+\alpha}}{\Gamma(p\alpha+\alpha+1)}\\
&= E_{\alpha,\alpha+1}(-k^2t^\alpha).
\end{aligned}$$

Then $|u_{k0}(t)|\le K_0\sup_{t\in(0,T]}|F_{0k}(t)|$ and

$$(1+k)^{\gamma+2}|u_{k0}(t)| \le K_0\sup_{t\in(0,T]}|F_{0k}(t)|(1+k)^{\gamma+2}, \quad t\in[0,T],\ k\in\mathbb{N}.$$

Father, as for $\alpha\in(1,2)$, we obtain that $u\in C_{2,\alpha}\big([0,T];H^\gamma(\mathbb{R})\big)$.

The inequality (3.10) implies that a solution of the problem is unique and depends continuously on the data. □

Theorem 3.3. *Assume that* (A1) *and* (A2) *hold. Then there exists s unique solution* $(u,F_1)\in\mathcal{M}_{\alpha,\gamma}$ *of the inverse problem* (3.1)–(3.3). *It is given by the Fourier series* (3.5) *and* (3.4) *with* $j=1$ *where*

$$F_{1k} = \frac{\Phi_k - \int_0^{t_0}\big[t^{\alpha-1}E_{\alpha,\alpha}(-k^2t^\alpha)*F_{0k}(t) + F_{2k}tE_{\alpha,2}(-k^2t^\alpha)\big]dt}{\int_0^{t_0}E_{\alpha,1}(-k^2t^\alpha)dt}, \quad k\in\mathbb{N}. \tag{3.12}$$

The solution depends continuously on the data F_0, F_2, Φ and the following inequality of coercivity holds:

$$\begin{aligned}&\|u\|_{C_{2,\alpha}\big([0,T];H^{\gamma}(\mathbb{R})\big)}+\|F_1\|_{H^{\gamma+2}(\mathbb{R})}\\&\le b_0\|F_0\|_{C\big([0,T];H^{\gamma+2+2\theta}(\mathbb{R})\big)}+b_1\|\Phi\|_{H^{\gamma+4}(\mathbb{R})}+b_2\|F_2\|_{H^{\gamma+2}(\mathbb{R})},\end{aligned}\tag{3.13}$$

where b_j, $j\in\{0,1,2\}$ are positive constants independent of data (F_0, F_2, Φ) and $F_2=0$, $\theta=0$ in (3.13) *if $\alpha\in(0,1)$.*

Proof. Using (3.9), we write the conditions (3.7) as follows

$$\begin{aligned}&\int_0^{t_0}\big[t^{\alpha-1}E_{\alpha,\alpha}(-k^2t^\alpha)*F_{0k}(t)+F_{1k}E_{\alpha,1}(-k^2t^\alpha)\\&+F_{2k}tE_{\alpha,2}(-k^2t^\alpha)\big]dt=\Phi_k,\quad k\in\mathbb{N}.\end{aligned}$$

So, to find the unknown coefficients F_{1k}, one obtains

$$\begin{aligned}&F_{1k}\int_0^{t_0}E_{\alpha,1}(-k^2t^\alpha)dt\\&=\Phi_k-\int_0^{t_0}\big[t^{\alpha-1}E_{\alpha,\alpha}(-k^2t^\alpha)*F_{0k}(t)+F_{2k}tE_{\alpha,2}(-k^2t^\alpha)\big]dt,\quad k\in\mathbb{N}.\end{aligned}\tag{3.14}$$

Note that

$$\begin{aligned}\int_0^{t_0}E_{\alpha,1}(-k^2t^\alpha)dt&=\frac{1}{\alpha k^{2/\alpha}}\int_0^{k^2{t_0}^\alpha}E_{\alpha,1}(-z)z^{\frac1\alpha-1}dz\\&=\frac{1}{\alpha k^{2/\alpha}}\int_0^{k^2{t_0}^\alpha}\sum_{p=0}^\infty\frac{(-1)^pz^{p+\frac1\alpha-1}dz}{\Gamma(p\alpha+1)}\\&=\frac{1}{k^{2/\alpha}}\sum_{p=0}^\infty\frac{(-1)^pz^{p+\frac1\alpha}}{\Gamma(p\alpha+1)(p\alpha+1)}\Big|_{z=k^2{t_0}^\alpha}\\&=\frac{1}{k^{2/\alpha}}\sum_{p=0}^\infty\frac{(-1)^p(k^2{t_0}^\alpha)^{p+\frac1\alpha}}{\Gamma(p\alpha+2)}\\&=t_0E_{\alpha,2}(-k^2{t_0}^\alpha),\quad k\in\mathbb{N}.\end{aligned}$$

From (3.14), according to the assumption (A2), we find the expressions (3.12) of the unknown Fourier coefficients F_{1k}, $k\in\mathbb{N}$. Let us show that the founded solution belongs to $\mathcal{M}_{\alpha,\gamma}$.

For $F_0\in C\big([0,T];H^{\gamma+2\theta}(\mathbb{R})\big)$, in the proof of the theorem 3.2, the estimates

$$\begin{gathered}|u_{k0}(t)|\le K_0\sup_{t\in(0,T]}|F_{0k}(t)|k^{2\theta-2}\quad\text{if }\alpha\in(1,2),\\|u_{k0}(t)|\le K_0\sup_{t\in(0,T]}|F_{0k}(t)|\quad\text{if }\alpha\in(0,1)\end{gathered}$$

were obtained.

Given that the functions $E_{\alpha,\mu}(-k^2t^\alpha)$ ($\mu\in\{1,\alpha,2\}$) have the same behavior (2.3) for large k and given the formulas (3.12) into account, one obtains

$$\begin{aligned}(1+k)^{\gamma+2}|F_{1k}|\le c_0\Big[&\sup_{t\in(0,T]}|F_{0k}(t)|(1+k)^{\gamma+2\theta+2}+|\Phi_k|(1+k)^{\gamma+2}\\&+|F_{2k}|(1+k)^{\gamma+2}\Big],\quad\alpha\in(1,2),\end{aligned}$$

$$
\begin{aligned}
(1+k)^{\gamma+2}|F_{1k}| \le c_0\Big[&\sup_{t\in(0,T]}|F_{0k}(t)|(1+k)^{\gamma+2}+|\Phi_k|(1+k)^{\gamma+2}\\
&+|F_{2k}|(1+k)^{\gamma+2}\Big],\quad \alpha\in(0,1),\ k\in\mathbb{N}
\end{aligned}
$$

where c_0 is a positive constant. So, under the assumptions, $F_1 \in H^{\gamma+2}(\mathbb{R})$. As in the proof of theorem 3.2 we obtain the inequality (3.13). This inequality implies that a solution of the problem is unique and depends continuously on the data. $\square$

The uniqueness of the solution of the inverse problem (3.1)–(3.3) is obtained without any conditions on the data, for all $t_0 \in (0,T]$, in the case $\alpha \in (0,1)$ and only under assumption on t_0 in the case $\alpha \in (1,2)$.

The obtained result can be transferred to the case of the boundary value problem (with homogeneous boundary conditions) to equations

$$
{}^cD_t^\alpha u - A(x,D)u = F_0(x,t), \quad (x,t)\in\Omega\times(0,T]
$$

on bounded domain $\Omega \subset \mathbb{R}^n$ where $A(x,D)$ is an elliptic differential expression with infinitely differentiable coefficients, and when the corresponding Sturm-Liouville problem has positive eigenvalues.

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