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PULLBACK ATTRACTORS FOR NONCLASSICAL DIFFUSION DELAY EQUATIONS ON UNBOUNDED DOMAINS WITH NON-AUTONOMOUS DETERMINISTIC AND STOCHASTIC FORCING TERMS

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Abstract. In this article, we prove the existence of pullback attractor in $C([-h,0];H^1(\mathbb{R}^N))$ for a stochastic nonclassical diffusion equations on unbounded domains with non-autonomous deterministic and stochastic forcing terms, and the pullback asymptotic compactness of the random dynamical system is established by a tail-estimates method.

1. Introduction

The study of pullback attractor for infinite dimensional dynamical systems has attracted much attention and has made a lot of progress in recent decades; see, for instance,[4, 6, 7, 11, 12, 13, 14, 16, 17, 20, 21, 22, 26, 27] and the references therein.

In this article, we focus on the asymptotic behavior of solutions to the stochastic nonclassical diffusion equation with delays

$$\mathrm{d}u - \mathrm{d}(\Delta u) - \Delta u \mathrm{d}t + \lambda u \mathrm{d}t = f(x, u(x, t-\rho(t)))\mathrm{d}t + g(t,x)\mathrm{d}t + \sum_{j=1}^{m} h_j \mathrm{d}w_j, \quad (1.1)$$

for $x \in \mathbb{R}^N$ and $t > \tau$; with the initial condition

$$u(x, t+\tau) = u_\tau(x), \quad t \in [-h, 0],\ x \in \mathbb{R}^N, \quad (1.2)$$

where $g \in L^2_{Loc}(\mathbb{R}, L^2(\mathbb{R}^N))$, $h_j (j = 1, \dots, m)$ are given functions, and $\{w_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on a probability space which will be specified in Section 2, and f is a nonlinear function containing some memory effects during a fixed interval of time of length $h > 0$, ρ being an adequate given delay function.

Nonclassical diffusion equations arise as models to describe physical phenomena such as non-Newtonian flow, soil mechanics, heat conduction, etc. (see Aifantis [1, 2], Kuttler and Aifantis [18, 19] and references therein). Aifantis et al [1, 2] pointed out that the classical reaction-diffusion equation

$$u_t - \Delta u = g(u),$$

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does not contain each aspect of the reaction-diffusion problem, and it neglects viscidity, elasticity, and pressure of medium in the process of solid diffusion. The authors obtained a diffusion theory similar to Fick's classical model for solute in an undisturbed solid matrix, obtaining a hyperbolic equation

$$u_t + D_1 u_{tt} = D_2 \Delta u,$$

where D_1 and D_2 are positive constants. Assigning viscosity to the diffusing substance, they arrived at the following equation

$$u_t + D_1 u_{tt} = D_2 \Delta u + D_3 \Delta u_t.$$

and neglecting the inertia term, finally obtained the nonclassical parabolic equation

$$u_t = D_2 \Delta u + D_3 \Delta u_t.$$

where D_3 is also a positive constant.

For the nonclassical diffusion equations without delay

$$u_t - \Delta u_t - \Delta u = f(u) + g,$$

the long-time behavior, especially the uniform attractor and pullback attractor have been extensively studied by several authors, see for example [3, 23, 29, 30, 31].

For the case with the variable delay term

$$u_t - \Delta u_t - \Delta u = f(x, u(x, t - \rho(t))) + g(t),$$

Hu and Wang [15] proved the existence of pullback attractors in bounded domain $\Omega \subset \mathbb{R}^N$. To our best knowledge, the dynamics for stochastic nonclassical diffusion equations with non-autonomous deterministic and stochastic forcing terms has not been considered by any predecessors, even for the bounded case.

In this article, we focus on the existence of pullback attractor for (1.1) in unbounded domain. There are some barriers encountered. On the one hand, (1.1) contains the term $-\Delta u_t$, it is different from the usual reaction-diffusion equation in [27, 28]. For example, the reaction diffusion equation has some smoothing effect, e.g., although the initial data only belongs to a weaker topology space, the solution will belong to a stronger topology space with higher regularity. However, for (1.1), if the initial data u_τ belongs to $C([-h, 0]; H^1(\mathbb{R}^N))$, then the solution $u(t, x)$ is always in $C([-h, 0]; H^1(\mathbb{R}^N))$ and has no higher regularity because of $-\Delta u_t$, it will cause some difficulties. On the other hand, the unbounded domain also brings some difficulties since the embeddings are no longer compact, so the asymptotic compactness of solutions can not be obtained by the standard method. Thirdly, note that (1.1) is a nonclassical diffusion equations with variable delay and stochastic forcing terms, hence this problem is not only stochastic but also non-autonomous, and this cause an additional difficulty, because one has to take into consideration not only the random parameter but also the time shift when defining the non-autonomous non-compact dynamical system.

This article is organized as follows: In Section 1, we have expounded on research progress as regards our research problem, and given some assumptions. In Section 2, we introduce some notations and functions spaces, and we recall some useful results on pullback attractors and non-autonomous non-compact dynamical systems. In Section 3, we prove the existence of pullback attractor for (1.1) in $C([-h, 0]; H^1(\mathbb{R}^N))$.

2. Preliminaries

In this section, we introduce some notation and functions spaces, and we recall some useful results on pullback attractors and non-autonomous non-compact dynamical systems in [26]. The attractors theory for random systems with only stochastic terms can be found in [4, 6, 7, 12, 13, 14] and the references therein.

Let $\mathcal{Q}$ be a nonempty set, $(\Omega, \mathcal{F}, P)$ be a probability space, and (X, d) be a Polish space with Borel σ-algebra $\mathcal{B}(X)$. The Hausdorff semi-distance between two nonempty subsets A and B of X is defined by

$$d(A, B) = \sup\{d(a, B) : a \in A\},$$

where $d(a, B) = \inf\{d(a, b) : b \in B\}$. Denote by $\mathcal{N}_r(A)$ the open r-neighborhood $\{y \in X : d(y, A) < r\}$ of radius $r > 0$ of a subset A of X.

Let 2^X be the collection of all subsets of X. Assume that there are two groups $\{\sigma_t\}_{t\in\mathbb{R}}$ and $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $\mathcal{Q}$ and Ω, respectively. In the sequel, we will call both $(\mathcal{Q}, \{\theta_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ parametric dynamical systems.

Definition 2.1. Let $(\mathcal{Q}, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ be parametric dynamical systems. A mapping $\Phi : \mathbb{R}^+ \times \mathcal{Q} \times \Omega \times X \to X$, is called a continuous cocycle on X over $(\mathcal{Q}, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $q \in \mathcal{Q}$, $\omega \in \Omega$ and t, $\tau \in \mathbb{R}^+$, the following conditions are satisfied:

- (i) $\Phi(\cdot, q, \cdot, \cdot) : \mathbb{R}^+ \times \Omega \times X \to X$ is $(\mathcal{B}(\mathbb{R}^+) \times \mathcal{F} \times \mathcal{B}(X), \mathcal{B}(X))$-measurable;
- (ii) $\Phi(0, q, \cdot, \cdot)$ is the identity on X;
- (iii) $\Phi(t + \tau, q, \omega, \cdot) = \Phi(t, \sigma_\tau q, \theta_\tau\omega, \cdot) \circ \Phi(\tau, q, \omega, \cdot)$;
- (iv) $\Phi(t, q, \omega, \cdot) : X \to X$ is continuous.

Let D be a family of some subsets of X which is parameterized by

$$(q, \omega) \in \mathcal{Q} \times \Omega : D = \{D(q, \omega) \subseteq X : q \in \mathcal{Q}, \omega \in \Omega\}.$$

Then we can associated with D a set-valued map $f_D : \mathcal{Q} \times \Omega \to 2^X$ such that

$$f_D(q, \omega) = D(q, \omega), \quad \text{for all } q \in \mathcal{Q} \text{ and } \omega \in \Omega.$$

Clearly, $D = f_D(\mathcal{Q} \times \Omega)$. In the sequel, we use $\mathcal{D}$ to denote a collection of some families of nonempty subsets of X:

$$\mathcal{D} = \{D = \{\emptyset \neq D(q, \omega) \subseteq X : q \in \mathcal{Q}, \omega \in \Omega\} : f_D \text{ satisfies some conditions}\}.$$

Note that a family D belongs to $\mathcal{D}$ if and only if the corresponding map f_D satisfies certain conditions rather than the image $f_D(\mathcal{Q} \times \Omega)$ of f_D.

Definition 2.2. A collection $\mathcal{D}$ of some families of nonempty subsets of X is said to be neighborhood closed if for each

$$D = \{D(q, \omega) : q \in \mathcal{Q}, \omega \in \Omega\} \in \mathcal{D},$$

there exists a positive number ε depending on D such that the family

$$\{B(q, \omega) : B(q, \omega) \text{ is a nonempty subsets of } N_\varepsilon(D(q, \omega)) \text{ for all } q \in \mathcal{Q}, \omega \in \Omega\} \in \mathcal{D}.$$

Definition 2.3. (i) A set-valued mapping $K : \mathcal{Q} \times \Omega \to 2^X$ is called measurable with respect to $\mathcal{F}$ in Ω if the value $K(q, \omega)$ is a closed nonempty subsets of X for all $q \in \mathcal{Q}$, $\omega \in \Omega$, and the mapping $\omega \in \Omega \to d(x, K(q, \omega))$ is $(\mathcal{F}, \mathcal{B}(\mathbb{R}))$-measurable for every fixed $x \in X$ and $q \in \mathcal{Q}$.

(ii) Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X and $K = \{K(q, \omega) : q \in \mathcal{Q}, \omega \in \Omega\} \in \mathcal{D}$. Then K is called $\mathcal{D}$-pullback absorbing set for Φ if for all $q \in \mathcal{Q}$, $\omega \in \Omega$ and for every $B = \{B(q,\omega) : q \in \mathcal{Q}, \omega \in \Omega\} \in \mathcal{D}$, there exists $T = T(B, q, \omega) > 0$ such that

$$\Phi(t, \sigma_{-t}q, \theta_{-t}\omega, B(\sigma_{-t}q, \theta_{-t}\omega)) \subseteq K(q,\omega), \quad \forall t \geq T.$$

(iii) Let $\mathcal{D}$ be a collection of some families of nonempty subsets of X. Then Φ is said to be $\mathcal{D}$-pullback asymptotically compact in X if for for all $q \in \mathcal{Q}$, $\omega \in \Omega$, the sequence

$$\{\Phi(t_n, \sigma_{-t_n}q, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \text{ has a convergent subsequence in } X$$

whenever $t_n \to \infty (n \to \infty)$, and $x_n \in B(\sigma_{-t_n}q, \theta_{-t_n}\omega)$ with $\{B(q,\omega) : q \in \mathcal{Q}, \omega \in \Omega\} \in \mathcal{D}$.

Definition 2.4. Let $\mathcal{D}$ be a collection of families of nonempty subsets of X and $A = \{A(q,\omega) : q \in \mathcal{Q}, \omega \in \Omega\} \in \mathcal{D}$. Then $\mathcal{A}$ is said to be a $\mathcal{D}$-pullback attractor for Φ if

(i) $\mathcal{A}$ is measurable with respect to the $\mathbb{P}$-completion of $\mathcal{F}$ in Ω, and $A(q,\omega)$ is compact for all $q \in \mathcal{Q}$, $\omega \in \Omega$.

(ii) $\mathcal{A}$ is invariant, that is, for every $q \in \mathcal{Q}$, $\omega \in \Omega$,

$$\Phi(t, q, \omega, \mathcal{A}(q,\omega)) = \mathcal{A}(\sigma_t q, \theta_t\omega) \quad \forall t \geq 0.$$

(iii) $\mathcal{A}$ attracts every set in $\mathcal{D}$, that is, for every $B = \{B(q,\omega) : q \in \mathcal{Q}, \omega \in \Omega\} \in \mathcal{D}$, for every $q \in \mathcal{Q}$, $\omega \in \Omega$,

$$\lim_{t\to\infty} d(\Phi(t, \sigma_{-t}q, \theta_{-t}\omega, B(\sigma_{-t}q, \theta_{-t}\omega)), \mathcal{A}(q,\omega)) = 0. \tag{2.1}$$

Definition 2.5. Let $\mathcal{D}$ be a collection of families of nonempty subsets of X. A mapping $\phi : \mathbb{R} \times \mathcal{Q} \times \Omega \to X$ is called a complete orbit of Φ if for every $\tau \in \mathbb{R}$, $t \geq 0$, $q \in \mathcal{Q}$ and $\omega \in \Omega$, the following holds:

$$\Phi(t, \sigma_\tau q, \theta_\tau\omega, \phi(\tau, q, \omega) = \phi(t + \tau, q, \omega). \tag{2.2}$$

If, in addition, there exists $D = \{D(q,\omega) : q \in \mathcal{Q}, \omega \in \Omega\} \in \mathcal{D}$, such that $\phi(t, q, \omega)$ belongs to $D(\sigma_t q, \theta_t\omega)$ for every $t \in \mathbb{R}$, $q \in \mathcal{Q}$ and $\omega \in \Omega$, then ϕ is called a $\mathcal{D}$-complete orbit of Φ.

Definition 2.6. Let $B = \{B(q,\omega) : q \in \mathcal{Q}, \omega \in \Omega\} \in \mathcal{D}$ be a family of nonempty subsets of X. For every $q \in \mathcal{Q}$ and $\omega \in \Omega$, let

$$\Theta(B, q, \omega) = \cap_{\tau\geq 0}\overline{\cup_{t\geq\tau}\Phi(t, \sigma_{-t}q, \theta_{-t}\omega, B(\sigma_{-t}q, \theta_{-t}\omega))}.$$

Then the family $\{\Theta(B, q, \omega) : q \in \mathcal{Q}, \omega \in \Omega\}$ is called the Θ-limit set of B and is denoted by $\Theta(B)$.

Theorem 2.7. *Let $\mathcal{D}$ be a neighborhood closed collection of some families of nonempty subsets of X and Φ be a continuous cocycle on X over $(\mathcal{Q}, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. Then Φ has a $\mathcal{D}$-pullback attractor $\mathcal{A}$ in $\mathcal{D}$ if and only if Φ is $\mathcal{D}$-pullback asymptotically compact in X and Φ has a closed measurable (with respect to the $\mathcal{P}$-completion of $\mathcal{F}$) $\mathcal{D}$-pullback absorbing set K in $\mathcal{D}$. The $\mathcal{D}$-pullback attractor $\mathcal{A}$ is unique and is given by, for each $q \in \mathcal{Q}$ and $\omega \in \Omega$,*

$$\begin{aligned}\mathcal{A}(q,\omega) &= \Theta(K, q, \omega) = \cup_{B\in\mathcal{D}}\Theta(B, q, \omega)\\ &= \{\phi(0, q, \omega) : \phi \text{ is a } \mathcal{D}\text{-complete orbit of } \Phi\}.\end{aligned}$$

Let X be a Banach space with norm $\|\cdot\|_X$. Let $h > 0$ be a given positive number, which will denote the delay time, and let C_X denote the Banach space with the norm

$$\|\psi\|_{C_X} := \sup_{s\in[-h,0]} \|\psi(s)\|_X.$$

Given $\tau \in \mathbb{R}$, $T > \tau$ and $u : [\tau - h, T) \to X$, for each $t \in [\tau, T)$ we denote by $u_t : [-h, 0] \to X$ denote the function defined on by $u_t(s) = u(t + s), s \in [-h, 0]$.

The following theorem (see [27, 28]) will be used to verify the pullback asymptotically compactness of the cocycle Φ on X.

Theorem 2.8. *Let $\mathcal{D}$ be a collection of families of nonempty subsets of X and Φ be a continuous cocycle on X over $(\mathcal{Q}, \{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. Suppose that for any fixed $q \in \mathcal{Q}$, $\omega \in \Omega$ and for every $B = \{B(q, \omega) : q \in \mathcal{Q}, \omega \in \Omega\} \in \mathcal{D}$ and any $\varepsilon > 0$, there exists $\tau_0 = \tau_0(B, q, \omega, \varepsilon) > 0$, a finite-dimensional subspace X_ε of X and a $\delta > 0$ such that*

(1) *for each fixed $s \in [-h, 0]$,*

$$\| \cup_{s\geq\tau_0} \cup_{u_t(\cdot)\in\Phi(t,\sigma_{-t}q,\theta_{-t}\omega,B(\sigma_{-t}q,\theta_{-t}\omega))} Pu(t + s)\|_X \quad \textit{is bounded;}$$

(3) *for all $t \geq \tau_0$, $u_t(\cdot) \in \Phi(t, \sigma_{-t}q, \theta_{-t}\omega, B(\sigma_{-t}q, \theta_{-t}\omega))$, $s_1, s_2 \in [-h, 0]$ with $|s_2 - s_1| < \delta$,*

$$\|P(u(t + s_1) - u(t + s_2))\|_X < \varepsilon;$$

(4) *for all $s \geq \tau_0$, $u_t(\cdot) \in \Phi(t, \sigma_{-t}q, \theta_{-t}\omega, B(\sigma_{-t}q, \theta_{-t}\omega))$,*

$$\sup_{s\in[-h,0]} \|(I - P)u(t + s)\|_X < \varepsilon,$$

where $P : X \to X_\varepsilon$ is the canonical projector. Then Φ is $\mathcal{D}$-pullback ω-limit compact in C_X.

With the usual notation, hereafter let $|u|$ be the modular (or absolute value) of u, $|\cdot|$ be the norm of $L^2(\mathbb{R}^N) = H$, $\|\cdot\|$ be the norm of $H^1(\mathbb{R}^N) = V$. Let C the arbitrary positive constant, which may be different from line to line and even in the same line.

3. Existence of the pullback attractors

In this section, we prove the existence of the pullback attractors for the non-classical diffusion equation with both non-autonomous deterministic and stochastic forcing terms:

$$\mathrm{d}u - \mathrm{d}(\Delta u) - \Delta u\mathrm{d}t + \lambda u\mathrm{d}t = f(x, u(x, t - \rho(t)))\mathrm{d}t + g(x, t)\mathrm{d}t + \sum_{j=1}^{m} h_j \frac{\mathrm{d}w_j}{\mathrm{d}t}, \quad (3.1)$$

for $x \in \mathbb{R}^N$ and $t > 0$, with the initial condition

$$u(x, t + \tau) = u_\tau(x, t), \quad t \in [-h, 0],\ x \in \mathbb{R}^N, \quad (3.2)$$

where $g \in L^2_{Loc}(\mathbb{R}, L^2(\mathbb{R}^N)) \cap \mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^N))$, $h_j (j = 1, \dots, m)$ are given functions, and $\{w_j\}_{j=1}^m$ are independent two-sided real-valued Wiener processes on a probability space which will be specified below, and f is a nonlinear function satisfy the following conditions:

(H1) there exist a positive constant α_2, a positive scalar function $\alpha_1 \in L^2(\mathbb{R}^N)$ such that the functions $f \in \mathcal{C}(\mathbb{R}^N \times \mathbb{R}; \mathbb{R})$, $\rho \in \mathcal{C}(\mathbb{R}; [0, h])$ satisfy

$$|f(x,v)|^2 \le |\alpha_1(x)|^2 + \alpha_2^2 |v|^2, \quad \forall x \in \mathbb{R}^N,\ v \in \mathbb{R}; \tag{3.3}$$

$$|\rho'(t)| \le \rho_* < 1, \quad \forall t \in \mathbb{R}; \tag{3.4}$$

(H2) there exists a $L > 0$ such that

$$|f(x,u) - f(x,v)| \le L|u-v|, \quad \forall x \in \mathbb{R}^N, \quad u, v \in \mathbb{R}; \tag{3.5}$$

(H3) the external force $g(x,t)$ belongs to $L^2_{\rm loc}(\mathbb{R}, L^2(\mathbb{R}^N)) \cap \mathcal{C}(\mathbb{R}, L^2(\mathbb{R}^N))$ such that

$$\int_{-\infty}^{\tau} e^{\lambda r} |g(r,\cdot)|^2_{L^2(\mathbb{R}^N)} \mathrm{d}r < \infty, \quad \forall \tau \in \mathbb{R}, \tag{3.6}$$

which implies

$$\lim_{k\to\infty} \int_{-\infty}^{\tau} \int_{|x|\ge k} e^{\lambda r} |g(r,\cdot)|^2 \mathrm{d}r \mathrm{d}x = 0, \quad \forall \tau \in \mathbb{R}. \tag{3.7}$$

Next, we consider the probability space $(\Omega, \mathcal{F}, P)$ where

$$\Omega = \{\omega = (\omega_1, \omega_2, \dots, \omega_m) \in \mathcal{C}(\mathbb{R}, \mathbb{R}^m) : \omega(0) = 0\},$$

$\mathcal{F}$ is the Borel σ -algebra induced by the compact-open topology of Ω, and P the corresponding Wiener measure on $(\Omega, \mathcal{F})$. Then we will identify ω with

$$W(t) \equiv (w_1(t), w_2(t), \dots, w_m(t)) = \omega(t) \quad for\ t \in \mathbb{R}.$$

Define the time shift by

$$\theta_t \omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega,\ t \in \mathbb{R}.$$

Then $(\Omega, \mathcal{F}, P, (\theta_t)_t \in \mathbb{R})$ is a metric dynamical system.

Given $j = 1, \dots, m$, considering the Ornstein-Uhlenbeck equation

$$\mathrm{d}z_j + z_j \mathrm{d}t = \mathrm{d}w_j(t). \tag{3.8}$$

One may easily check that a solution to (3.8) is given by

$$z_j(t) = z_j(\theta_t \omega_j) \equiv -\int_{-\infty}^{0} e^{\tau} (\theta_t \omega_j)(\tau) \mathrm{d}\tau, t \in \mathbb{R}.$$

Note that the random variable $|z_j(\omega_j)|$ is tempered and $z_j(\theta_t \omega_j)$ is $P-a.e.$ continuous. Therefore, it follows from [4, Proposition 4.3.3] that there exists a tempered function $r(\omega) > 0$ such that

$$\sum_{j=1}^{m} (|z_j(\omega_j)|^2 + |z_j(\omega_j)|^p) \le r(\omega), \tag{3.9}$$

where $r(\omega)$ satisfies, for P-a.e. $\omega \in \Omega$,

$$r(\theta_t \omega) \le e^{\frac{1}{2}|t|} r(\omega), \quad t \in \mathbb{R}, \tag{3.10}$$

Then it follows from (3.9)–(3.10) that, for P-a.e. $\omega \in \Omega$,

$$\sum_{j=1}^{m} (|z_j(\theta_t \omega_j)|^2 + |z_j(\theta_t \omega_j)|^p) \le e^{\frac{1}{2}|t|} r(\omega), t \in \mathbb{R}. \tag{3.11}$$

Putting $z(\theta_t\omega)=\sum_{j=1}^m h_j z_j(\theta_t\omega_j)$, we have

$$\mathrm{d}z+z\mathrm{d}t=\sum_{j=1}^m h_j\mathrm{d}\omega_j. \tag{3.12}$$

3.1. **Well-posedness.** Put $z_j(\theta_t)=(I-\Delta)^{-1}h_jy_j(\theta_t\omega)(j=1,\dots,m)$, where Δ is the Laplacian. By (3.12) we find that

$$\mathrm{d}z-\mathrm{d}(\Delta z)+(z-\Delta z)\mathrm{d}t=\sum_{j=1}^m h_j z_j(\theta_t\omega_j).$$

Let $v(t,\omega)=u(t,\omega)-z(\theta_t\omega)$ where $u(t,\omega)$ is a solution of (3.1)–(3.2). Then $v(t,\omega)$ satisfies

$$v_t-\Delta v_t-\Delta v+\lambda v=f(x,v(t-\rho(t))+z(\theta_{t-\rho(t)}\omega))+g(x,t)+(1-\lambda)\Delta z(\theta_t\omega). \tag{3.13}$$

with the initial value condition

$$v(x,t+\tau)=v_\tau(x,t)=u_\tau(x,t)-z(\theta_{t+\tau}\omega),\quad t\in[-h,0],\quad x\in\mathbb{R}^N. \tag{3.14}$$

Using the standard Galerkin approximation method, we can obtain the following result concerning the existence of solutions, see e.g. [23, 24, 29, 31].

Theorem 3.1. *Under assumptions* (H1)–(H3)*, for P-a.e. $\omega\in\Omega$ and any $v_0\in C_V$, there is a solution $v(\cdot,\tau,\omega,v_\tau)$ of satisfying*

$$v(\cdot,\tau,\omega,v_\tau)\in\mathcal{C}([\tau-h,\tau+T];V)\cap L^2(\tau,\tau+T;V).$$

Theorem 3.2. *Under assumptions* (H1)–(H3)*, then the solutions of* (3.13)–(3.14) *are unique, and the solutions depend continuously on the initial data in C_V for any $t\geq\tau$ and P-a.e. $\omega\in\Omega$.*

Proof. Assume that $v_\tau^1,v_\tau^2\in C_V$, we consider the solutions $v^1(\cdot),v^2(\cdot)$ for (3.13)–(3.14) corresponding to the initial data v_τ^1, v_τ^2.

Let $w=v^1-v^2$, we infer that

$$\begin{aligned}&w_t-\Delta w_t-\Delta w+\lambda w\\&=f(x,v^1(t-\rho(t))+z(\theta_{t-\rho(t)}\omega))-f(x,v^2(t-\rho(t))+z(\theta_{t-\rho(t)}\omega)).\end{aligned} \tag{3.15}$$

Multiplying (3.15) by w, and using (H2), we obtain

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(|w|^2+|\nabla w|^2)+|\nabla w|^2+\lambda|w|^2\leq L|w|^2. \tag{3.16}$$

In particular, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}(|w|^2+|\nabla w|^2)\leq 2L(|w|^2+|\nabla w|^2)\leq C\|w^t\|_{C_V}^2. \tag{3.17}$$

Integrating (3.17) from τ to t, we infer

$$|w(t)|^2+|\nabla w(t)|^2\leq C\|v_\tau^1-v_\tau^2\|_{C_V}^2+C\int_\tau^t\|w^t\|_{C_V}^2. \tag{3.18}$$

Hence

$$\sup_{s\in[-h,0]}(|w(t)|^2+|\nabla w(t)|^2)=\|w^t\|_{C_V}^2\leq C\|v_\tau^1-v_\tau^2\|_{C_V}+C\int_\tau^t\|w^t\|_{C_V}^2. \tag{3.19}$$

The Gronwall lemma implies that for any $t\geq\tau$,

$$\|v_t^1-v_t^2\|_{C_V}^2\leq C\|v_\tau^1-v_\tau^2\|_{C_V}^2e^{C(t-\tau)}, \tag{3.20}$$

and thus the conclusion follows immediately. □

For $t \geq \tau$, $v(\cdot,\tau,\omega,v_\tau)$ is $(\mathcal{F},\mathcal{B}(C_V))$-measurable in $\omega\in\Omega$ and continuous in v_τ with respect to the norm of C_V. Let

$$u_t(\cdot,\tau,\omega,v_\tau)=v_t(\cdot,\tau,\omega,v_\tau)+z(\theta_{t+\cdot}\omega)$$

with $u_\tau(\cdot)=v_\tau(\cdot)+z(\theta_{t+\cdot}\omega)$. Furthermore, we find that u is continuous in both $t\geq\tau$ and $u_\tau\in C_V$ and is $(\mathcal{F},\mathcal{B}(C_V))$-measurable in $\omega\in\Omega$. It follows from (3.13) that u is a solution of (3.1)-(3.2).

Now, we define a mapping $\Phi:\mathbb{R}^+\times\mathbb{R}\times\Omega\times C_V\to C_V$ by

$$\Phi(t,\tau,\omega,u_\tau)=u_{t+\tau}(\cdot,\tau,\theta_{-\tau}\omega,u_\tau)=v_{t+\tau}(\cdot,\tau,\theta_{-\tau}\omega,v_\tau)+z(\theta_{t+\cdot}\omega), \tag{3.21}$$

for all $(t,\tau,\omega,\psi)\in\mathbb{R}^+\times\mathbb{R}\times\Omega\times C_V$, where $v_\tau(\cdot)=u_\tau(\cdot)-z(\theta_\tau,\omega)$.

Then Φ satisfies conditions (i)–(iv) in Definition 2.1. Therefore, Φ is a continuous cocycle on C_V over over $(\mathbb{R},\{\sigma_t\}_{t\in\mathbb{R}})$ and $(\Omega,\mathcal{F},P,\{\theta_t\}_{t\in\mathbb{R}})$.

3.2. **Pullback absorbing sets.** In this subsection, we prove the existence of pullback absorbing set in C_V of the cocycle Φ associated with (3.1)–(3.2).

Assume that $D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ is a family of bounded nonempty subsets of C_V satisfying, for every $\tau\in\mathbb{R}$ and $\omega\in\Omega$,

$$\lim_{s\to-\infty}e^{\alpha s}\|D(\tau+s,\theta_s\omega)\|^2_{C_V}=0, \tag{3.22}$$

where $0<\alpha<1$. Denote by $\mathcal{D}_\alpha$ the collection of all families of bounded nonempty subsets of C_V which fulfill conditions (3.22), i.e.,

$$D_\alpha=\{D=\{D(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}:D \text{ satisfies (3.22)}\}.$$

Obviously D_α is neighborhood closed.

Lemma 3.3. *Assume that* (H1)–(H3) *hold and*

$$\lambda>\frac{2+\alpha}{2-\alpha}+\frac{8\alpha_2^2e^{\alpha h}}{(2-\alpha)(1-\rho^*)},$$

where λ is a large positive constant, and $0<\alpha<1$. Then there exists a closed measurable set $K=\{K(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ in $\mathcal{D}_\alpha$ for the cocycle Φ associated with (3.1)–(3.2).

Proof. Multiplying (3.1) by $v+v_t$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}((1+\lambda)|v|^2+2|\nabla v|^2)+|\nabla v|^2+\lambda|v|^2+|v_t|^2+|\nabla v_t|^2\\&\leq(f(x,v(t-\rho(t))+z(\theta_{t-\rho(t)}\omega)),v+v_t)+(g(x,t),v+v_t)\\&\quad+((1-\lambda)\Delta z(\theta_t\omega),v+v_t).\end{aligned}\tag{3.23}$$

Using (1.2) and Young's inequality, for $\varepsilon_1,\varepsilon_2,\varepsilon_3>0$, we infer that

$$\begin{aligned}&(f(x,v(t-\rho(t))+z(\theta_{t-\rho(t)}\omega)),v+v_t)\\&\leq 2\varepsilon_1|v|^2+2\varepsilon_1|v_t|^2+\frac{\alpha_2^2}{4\varepsilon_1}|v(t-\rho(t)+z(\theta_{t-\rho(t)}\omega)|^2+\frac{1}{4\varepsilon_1}|\alpha_1|^2\\&\leq 2\varepsilon_1|v|^2+2\varepsilon_1|v_t|^2+\frac{\alpha_2^2}{2\varepsilon_1}|v(t-\rho(t)|^2+\frac{\alpha_2^2}{2\varepsilon_1}|z(\theta_{t-\rho(t)}\omega)|^2+\frac{1}{4\varepsilon_1}|\alpha_1|^2,\end{aligned}\tag{3.24}$$

$$(g(x,t),v+v_t)\leq 2\varepsilon_2|v|^2+2\varepsilon_2|v_t|^2+\frac{1}{4\varepsilon_2}|g(x,t)|^2. \tag{3.25}$$

$$((1-\lambda)\Delta z(\theta_t\omega), v+v_t) \le 2\varepsilon_3|v|^2 + 2\varepsilon_3|v_t|^2 + \frac{C_\lambda}{\varepsilon_3}|\Delta z(\theta_t\omega)|^2. \tag{3.26}$$

It follows from (3.23)–(3.26) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}((1+\lambda)|v|^2 + 2|\nabla v|^2) + |\nabla v|^2 + (1-2\varepsilon_1-2\varepsilon_2-2\varepsilon_3)|v_t|^2 \\ &+ (\lambda - 2\varepsilon_1 - 2\varepsilon_2 - 2\varepsilon_3)|v|^2 + |\nabla v_t|^2 \\ &\le \frac{\alpha_2^2}{2\varepsilon_1}|v(t-\rho(t))|^2 + \frac{1}{4\varepsilon_2}|g(x,t)|^2 + \frac{1}{4\varepsilon_1}|\alpha_1|^2 + \frac{\alpha_2^2}{2\varepsilon_1}|z(\theta_{t-\rho(t)}\omega)|^2 \\ &\quad + \frac{C_\lambda}{\varepsilon_3}|\Delta z(\theta_t\omega)|^2. \end{aligned} \tag{3.27}$$

Multiplying (3.27) by $e^{\alpha t}(0<\alpha<1)$, we infer that

$$\begin{aligned} &\frac{d}{dt}(e^{\alpha t}((1+\lambda)|v|^2 + 2|\nabla v|^2)) \\ &= \alpha e^{\alpha t}((1+\lambda)|v|^2 + 2|\nabla v|^2) + e^{\alpha t}\frac{d}{dt}((1+\lambda)|v|^2 + 2|\nabla v|^2) \\ &\le e^{\alpha t}(\alpha(1+\lambda) + 4\varepsilon_1 + 4\varepsilon_2 + 4\varepsilon_3 - 2\lambda)|v|^2 + e^{\alpha t}(2\alpha-2)|\nabla v|^2 \\ &\quad + (4\varepsilon_1 + 4\varepsilon_2 + 4\varepsilon_3 - 2)e^{\alpha t}|v_t|^2 - 2e^{\alpha t}|\nabla v_t|^2 + \frac{\alpha_2^2 e^{\alpha t}}{\varepsilon_1}|v(t-\rho(t))|^2 \\ &\quad + \frac{e^{\alpha t}}{2\varepsilon_2}|g(x,t)|^2 + \frac{e^{\alpha t}}{2\varepsilon_1}|\alpha_1|^2 + \frac{\alpha_2^2}{\varepsilon_1}e^{\alpha t}|z(\theta_{t-\rho(t)}\omega)|^2 + \frac{C_\lambda}{\varepsilon_3}e^{\alpha t}|\Delta z(\theta_t\omega)|^2. \end{aligned} \tag{3.28}$$

Now integrating (3.28) from $\tau-t$ to $\tau+s$, where $s\in[-h,0]$, we obtain

$$\begin{aligned} &e^{\alpha(\tau+s)}((1+\lambda)|v(\tau+s,\tau-t,\omega,v_{\tau-t})|^2 + 2|\nabla v(\tau+s,\tau-t,\omega,v_{\tau-t})|^2)) \\ &+ \int_{\tau-t}^{\tau+s} e^{\alpha r}|\nabla v_r(r,\tau-t,\omega,v_{\tau-t})|^2 \mathrm{d}r \\ &\le e^{\alpha(\tau-t)}((1+\lambda)|v(\tau-t,\tau-t,\omega,v_{\tau-t})|^2 + 2|\nabla v(\tau-t,\tau-t,\omega,v_{\tau-t})|^2)) \\ &\quad + (\alpha(1+\lambda) + 4\varepsilon_1 + 4\varepsilon_2 + 4\varepsilon_3 - 2\lambda)\int_{\tau-t}^{\tau+s} e^{\alpha r}|v(r,\tau-t,\omega,v_{\tau-t})|^2 \mathrm{d}r \\ &\quad + (4\varepsilon_1 + 4\varepsilon_2 + 4\varepsilon_3 - 2)\int_{\tau-t}^{\tau+s} e^{\alpha r}|v_r(r,\tau-t,\omega,v_{\tau-t})|^2 \mathrm{d}r \\ &\quad + (2\alpha-2)\int_{\tau-t}^{\tau+s} e^{\alpha r}|\nabla v(r,\tau-t,\omega,v_{\tau-t})|^2 \mathrm{d}r \\ &\quad + \frac{\alpha_2^2}{\varepsilon_1}\int_{\tau-t}^{\tau+s} e^{\alpha r}|v(r-\rho(r),\tau-t,\omega,v_{\tau-t})|^2 \mathrm{d}r + \frac{|\alpha_1|^2}{\varepsilon_1}e^{\alpha(\tau+s)} \\ &\quad + \frac{\alpha_2^2}{\varepsilon_1}\int_{\tau-t}^{\tau+s} e^{\alpha r}|z(\theta_{r-\rho(r)}\omega)|^2 \mathrm{d}r + \frac{1}{\varepsilon_2}\int_{\tau-t}^{\tau+s} e^{\alpha r}|g(x,r)|^2 \mathrm{d}r \\ &\quad + \frac{C_\lambda}{\varepsilon_3}\int_{\tau-t}^{\tau+s} e^{\alpha r}|\Delta z(\theta_r\omega)|^2 \mathrm{d}r. \end{aligned} \tag{3.29}$$

Noting that $\rho(s) \in [0,h]$ and the fact $\frac{1}{1-\rho'(s)} \leq \frac{1}{1-\rho^*}$ for all $s \in \mathbb{R}$. Setting $r' = r - \rho(r)$, we arrive at

$$\begin{aligned} &\frac{\alpha_2^2}{\varepsilon_1}\int_{\tau-t}^{\tau+s} e^{\alpha r}|v(r-\rho(r),\tau-t,\omega,v_{\tau-t})|^2 \mathrm{d}r \\ &\leq \frac{\alpha_2^2}{\varepsilon_1}\int_{\tau-t-h}^{\tau+s} \frac{e^{\alpha h}e^{\alpha r'}}{1-\rho^*}|v(r',\tau-t,\omega,v_{\tau-t})|^2 dr' \\ &= \frac{\alpha_2^2 e^{\alpha h}}{\varepsilon_1(1-\rho^*)}\Big(\int_{\tau-t-h}^{\tau-t} e^{\alpha r'}|u(r',\tau-t,\omega,u_{\tau-t}) - z(\theta_{r'}\omega)|^2 dr' \\ &\quad + \int_{\tau-t}^{\tau+s} e^{\alpha r'}|v(r',\tau-t,\omega,v_{\tau-t})|^2 dr'\Big) \\ &\leq \frac{2\alpha_2^2 e^{\alpha h+\alpha\tau}\|u_{\tau-t}\|_{C_V}^2}{\varepsilon_1(1-\rho^*)}e^{-\alpha t} + \frac{2\alpha_2^2 e^{\alpha h}}{\varepsilon_1(1-\rho^*)}\int_{\tau-t-h}^{\tau-t} e^{\alpha r'}|z(\theta_{r'}\omega)|^2 dr' \\ &\quad + \frac{\alpha_2^2 e^{\alpha h}}{\varepsilon_1(1-\rho^*)}\int_{\tau-t}^{\tau+s} e^{\alpha r'}|v(r',\tau-t,\omega,v_{\tau-t})|^2 dr', \end{aligned} \tag{3.30}$$

and similarly,

$$\begin{aligned} &\frac{\alpha_2^2}{\varepsilon_1}\int_{\tau-t}^{\tau+s} e^{\alpha r}|z(\theta_{r-\rho(r)}\omega)|^2 \mathrm{d}r \\ &\leq \frac{\alpha_2^2 e^{\alpha h}}{\varepsilon_1(1-\rho^*)}\int_{\tau-t-h}^{\tau-t} e^{\alpha r'}|z(\theta_{r'}\omega|^2 dr' + \frac{\alpha_2^2 e^{\alpha h}}{\varepsilon_1(1-\rho^*)}\int_{\tau-t}^{\tau+s} e^{\alpha r'}|z(\theta_{r'}\omega|^2 dr'. \end{aligned} \tag{3.31}$$

Note that $s \in [-h,0]$, it follows from (3.29)–(3.31) that

$$\begin{aligned} &e^{\alpha(\tau+s)}((1+\lambda)|v(\tau+s,\tau-t,\omega,v_{\tau-t})|^2 + 2|\nabla v(\tau+s,\tau-t,\omega,v_{\tau-t})|^2) \\ &+ \int_{\tau-t}^{\tau+s} e^{\alpha r}|\nabla v_r(r,\tau-t,\omega,v_{\tau-t})|^2 \mathrm{d}r \\ &\leq e^{\alpha(\tau-t)}((1+\lambda)|v(\tau-t,\tau-t,\omega,v_{\tau-t})|^2 + 2|\nabla v(\tau-t,\tau-t,\omega,v_{\tau-t})|^2) \\ &\quad + (\alpha(1+\lambda) + 4\varepsilon_1 + 4\varepsilon_2 + 4\varepsilon_3 - 2\lambda \\ &\quad + \frac{\alpha_2^2 e^{\alpha h}}{\varepsilon_1(1-\rho^*)})\int_{\tau-t}^{\tau+s} e^{\alpha r}|v(r,\tau-t,\omega,v_{\tau-t})|^2 \mathrm{d}r \\ &\quad + (4\varepsilon_1 + 4\varepsilon_2 + 4\varepsilon_3 - 2)\int_{\tau-t}^{\tau+s} e^{\alpha r}|v_r(r,\tau-t,\omega,v_{\tau-t})|^2 \mathrm{d}r \\ &\quad + (2\alpha-2)\int_{\tau-t}^{\tau+s} e^{\alpha r}|\nabla v(r,\tau-t,\omega,v_{\tau-t})|^2 \mathrm{d}r \\ &\quad + \frac{|\alpha_1|^2}{\varepsilon_1}e^{\alpha(\tau+s)} + \frac{1}{\varepsilon_2}\int_{\tau-t}^{\tau} e^{\alpha r}|g(x,r)|^2 \mathrm{d}r + \frac{C_\lambda}{\varepsilon_3}\int_{\tau-t}^{\tau} e^{\alpha r}|\Delta z(\theta_r\omega)|^2 \mathrm{d}r \\ &\quad + \frac{2\alpha_2^2 e^{\alpha h+\alpha\tau}\|u_{\tau-t}\|_{C_V}^2}{\varepsilon_1(1-\rho^*)}e^{-\alpha t} + \frac{4\alpha_2^2 e^{\alpha h}}{\varepsilon_1(1-\rho^*)}\int_{\tau-t-h}^{\tau-t} e^{\alpha r'}|z(\theta_{r'}\omega)|^2 \mathrm{d}r'. \end{aligned} \tag{3.32}$$

Choosing $\varepsilon_1 = \frac{1}{8}$, $0 < \varepsilon_2 < \frac{1}{8}$, $0 < \varepsilon_3 < \frac{1}{8}$, and noting that

$$\lambda > \frac{2+\alpha}{2-\alpha} + \frac{8\alpha_2^2 e^{\alpha h}}{(2-\alpha)(1-\rho^*)},$$

then

$$\alpha(1+\lambda)+4\varepsilon_1+4\varepsilon_2+4\varepsilon_3+\frac{\alpha_2^2e^{\alpha h}}{\varepsilon_1(1-\rho^*)}-2\lambda<0,$$
$$4\varepsilon_1+4\varepsilon_2+4\varepsilon_3-2<0.$$

Replacing ω with $\theta_{-\tau}\omega$, we arrive at

$$\begin{aligned}
&(1+\lambda)|v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2+2|\nabla v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\\
&\le e^{\alpha h}e^{-\alpha t}((1+\lambda)|v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2+2|\nabla v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2)\\
&\quad+C+Ce^{-\alpha\tau}\int_{-\infty}^{\tau}e^{\alpha r}|g(x,r)|^2\mathrm{d}r+C\int_{\tau-t}^{\tau}e^{\alpha(r-\tau)}|\Delta z(\theta_{r-\tau}\omega)|^2\mathrm{d}r\\
&\quad+C\|u_{\tau-t}\|_{C_V}^2e^{-\alpha t}+C\int_{\tau-t-h}^{\tau-t}e^{\alpha(r-\tau)}|z(\theta_{r-\tau}\omega)|^2dr.
\end{aligned}\tag{3.33}$$

Note that $z(\theta_t\omega)=\sum_{j=1}^m h_jz_j(\theta_t\omega_j)$ and $h_j\in H^2(\mathbb{R}^N)$, we infer that

$$\begin{aligned}
\int_{\tau-t-h}^{\tau-t}e^{\alpha(r-\tau)}|z(\theta_{r-\tau}\omega)|^2\mathrm{d}r&=\int_{-t-h}^{-t}e^{\alpha s}|z(\theta_s\omega)|^2\mathrm{d}s\\
&\le\int_{-t-h}^{-t}e^{\alpha s}\sum_{j=1}^m|h_jz_j(\theta_s\omega_j)|^2\mathrm{d}s\\
&\le C\int_{-\infty}^0e^{\alpha r}\sum_{j=1}^m|z_j(\theta_r\omega_j)|^2\mathrm{d}r,
\end{aligned}\tag{3.34}$$

and

$$\begin{aligned}
\int_{\tau-t}^{\tau}e^{\alpha(r-\tau)}|\Delta z(\theta_{r-\tau}\omega)|^2\mathrm{d}r&\le\int_{-t}^0e^{\alpha s}\sum_{j=1}^m|\Delta h_jz_j(\theta_s\omega_j)|^2\mathrm{d}s\\
&\le C\int_{-\infty}^0e^{\alpha r}\sum_{j=1}^m|z_j(\theta_r\omega_j)|^2\mathrm{d}r.
\end{aligned}\tag{3.35}$$

Then, it follows from (3.33)–(3.35) that

$$\begin{aligned}
&(1+\lambda)|v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2+2|\nabla v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\\
&\le e^{\alpha h}e^{-\alpha t}((1+\lambda)|v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\\
&\quad+2|\nabla v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2)+C+Ce^{-\alpha\tau}\int_{-\infty}^{\tau}e^{\alpha r}|g(x,r)|^2\mathrm{d}r\\
&\quad+C\|u_{\tau-t}\|_{C_V}^2e^{-\alpha t}+C\int_{-\infty}^0e^{\alpha r}\sum_{j=1}^m|z_j(\theta_r\omega_j)|^2\mathrm{d}r.
\end{aligned}\tag{3.36}$$

Note that for each $\tau\in\mathbb{R}$, $t\ge0$, $s\in[-h,0]$ and $\omega\in\Omega$,

$$u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})=v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})+z(\theta_s\omega),$$

where

$$u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})=v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})+z(\theta_{-t}\omega).$$

Then

$$
\begin{aligned}
&|u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2+|\nabla u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2\\
&\le 2|v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2+2|\nabla v(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2\\
&\quad+2|z(\theta_s\omega)|^2+2|\nabla z(\theta_s\omega)|^2\\
&\le Ce^{-\alpha t}(|v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2+|\nabla v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2)\\
&\quad+C+Ce^{-\alpha\tau}\int_{-\infty}^{\tau}e^{\alpha r}|g(x,r)|^2\mathrm{d}r+C\|u_{\tau-t}\|_{C_V}^2e^{-\alpha t}\\
&\quad+C\int_{-\infty}^{0}e^{\alpha r}\sum_{j=1}^{m}|z_j(\theta_r\omega_j)|^2\mathrm{d}r+2|z(\theta_s\omega)|^2+2|\nabla z(\theta_s\omega)|^2\\
&\le Ce^{-\alpha t}(|u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})-z(\theta_{-t}\omega)|^2\\
&\quad+|\nabla u(\tau-t,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})-\nabla z(\theta_{-t}\omega)|^2)+C\\
&\quad+Ce^{-\alpha\tau}\int_{-\infty}^{\tau}e^{\alpha r}|g(x,r)|^2\mathrm{d}r+C\|u_{\tau-t}\|_{C_V}^2e^{-\alpha t}\\
&\quad+C\int_{-\infty}^{0}e^{\alpha r}\sum_{j=1}^{m}|z_j(\theta_r\omega_j)|^2\mathrm{d}r+2|z(\theta_s\omega)|^2+2|\nabla z(\theta_s\omega)|^2\\
&\le C\|u_{\tau-t}\|_{C_V}^2e^{-\alpha t}+C+Ce^{-\alpha\tau}\int_{-\infty}^{\tau}e^{\alpha r}|g(x,r)|^2\mathrm{d}r\\
&\quad+C\int_{-\infty}^{0}e^{\alpha r}\sum_{j=1}^{m}|z_j(\theta_r\omega_j)|^2\mathrm{d}r+2|z(\theta_s\omega)|^2+2|\nabla z(\theta_s\omega)|^2\\
&\quad+Ce^{-\alpha t}|z(\theta_{-t}\omega)|^2+Ce^{-\alpha t}|\nabla z(\theta_{-t}\omega)|^2.
\end{aligned}
\tag{3.37}
$$

Note that $z(\theta_t\omega)=\sum_{j=1}^m h_jz_j(\theta_t\omega_j)$ and $h_j\in H^2(\mathbb{R}^N)$, we infer that

$$
\begin{aligned}
&Ce^{-\alpha t}|z(\theta_{-t}\omega)|^2+Ce^{-\alpha t}|\nabla z(\theta_{-t}\omega)|^2\\
&\le Ce^{-\alpha t}\sum_{j=1}^{m}|h_j|^2|z_j(\theta_{-t}\omega_j)|^2+Ce^{-\alpha t}\sum_{j=1}^{m}|\nabla h_j|^2|z_j(\theta_{-t}\omega_j)|^2\\
&\le Ce^{-\alpha t}\sum_{j=1}^{m}|z_j(\theta_{-t}\omega_j)|^2,
\end{aligned}
\tag{3.38}
$$

and

$$
\begin{aligned}
&\sup_{s\in[-h,0]}|z(\theta_s\omega)|^2+\sup_{s\in[-h,0]}|\nabla z(\theta_s\omega)|^2\\
&\le\sup_{s\in[-h,0]}\sum_{j=1}^{m}|h_j|^2|z_j(\theta_s\omega_j)|^2+\sup_{s\in[-h,0]}\sum_{j=1}^{m}|\nabla h_j|^2|z_j(\theta_s\omega_j)|^2\\
&\le\sup_{s\in[-h,0]}C\sum_{j=1}^{m}|z_j(\theta_s\omega_j)|^2.
\end{aligned}
\tag{3.39}
$$

It follows from (3.37)-(3.39) that

$$
\begin{aligned}
&|u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2+|\nabla u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2\\
&\le Ce^{-\alpha t}\|u_{\tau-t}\|_{C_V}^2+C+Ce^{-\alpha\tau}\int_{-\infty}^{\tau}e^{\alpha r}|g(x,r)|^2\mathrm{d}r\\
&\quad+C\int_{-\infty}^{0}e^{\alpha r}\sum_{j=1}^{m}|z_j(\theta_r\omega_j)|^2\mathrm{d}r+Ce^{-\alpha t}\sum_{j=1}^{m}|z_j(\theta_{-t}\omega_j)|^2\\
&\quad+\sup_{s\in[-h,0]}C\sum_{j=1}^{m}|z_j(\theta_s\omega_j)|^2.
\end{aligned}\tag{3.40}
$$

Note that $u_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$ and $r(\omega)$ is tempered. Then

$$
\limsup_{t\to\infty}e^{-\alpha t}\|u_{\tau-t}\|_{C_V}^2\le\limsup_{t\to\infty}e^{-\alpha t}\|D(\tau-t,\theta_{-t}\omega)\|_{C_V}^2=0. \tag{3.41}
$$

Therefore, there exists $T=T(\tau,\omega,D)>0$, such that for all $t\ge T$, we have

$$
Ce^{-\alpha t}\|u_{\tau-t}\|_{C_V}^2+Ce^{-\alpha t}\sum_{j=1}^{m}|z_j(\theta_{-t}\omega_j)|^2\le C. \tag{3.42}
$$

Hence, for all $t\ge T$, we infer that

$$
\begin{aligned}
&\|u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\|_{C_V}^2\\
&\le Ce^{-\alpha\tau}\int_{-\infty}^{\tau}e^{\alpha r}|g(x,r)|^2\mathrm{d}r+C\int_{-\infty}^{0}e^{\alpha r}\sum_{j=1}^{m}|z_j(\theta_r\omega_j)|^2\mathrm{d}r\\
&\quad+\sup_{s\in[-h,0]}C\sum_{j=1}^{m}|z_j(\theta_s\omega_j)|^2+C,
\end{aligned}\tag{3.43}
$$

where $u_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$.

Given $\tau\in\mathbb{R}$ and $\omega\in\Omega$, let

$$
K(\tau,\omega)=\{u\in C_V:\|u\|_{C_V}^2\le R(\tau,\omega)\},
$$

where $R(\tau,\omega)$ is the constant given by the right-side of (3.43). Clearly, for each $\tau\in\mathbb{R}$, $R(\tau,\cdot):\Omega\to\mathbb{R}$ is $(\mathcal{F},\mathcal{B}(\mathbb{R}))$-measurable, and satisfies

$$
\lim_{r\to-\infty}e^{\alpha r}\|K(\tau+r,\theta_r\omega)\|_{C_V}^2\le\lim_{r\to-\infty}e^{\alpha r}R(\tau+r,\theta_r\omega)=0. \tag{3.44}
$$

In other words, $K=\{K(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}$ belongs to $\mathcal{D}_\alpha$. For each $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $D\in\mathcal{D}_\alpha$, according to (3.43), there exists $T=T(\tau,\omega,D)>0$, such that for all $t\ge T$, we arrive at

$$
\Phi(t,\tau-t,\theta_{-\tau}\omega,D(\tau-t,\theta_{-\tau}\omega))\subseteq K(\tau,\omega),
$$

that is, K is a closed measurable $\mathcal{D}_\alpha$-pullback absorbing set in $\mathcal{D}_\alpha$ for Φ. This completes the proof. $\square$

Corollary 3.4. *The proof of Lemma 3.3 implies that*

$$
\begin{aligned}
&\int_{\tau-t}^{\tau} e^{\alpha r}|v(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\mathrm{d}r+\int_{\tau-t}^{\tau} e^{\alpha r}|v_r(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\mathrm{d}r\\
&+\int_{\tau-t}^{\tau} e^{\alpha r}|\nabla v_r(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\mathrm{d}r\\
&+\int_{\tau-t}^{\tau} e^{\alpha r}|\nabla v(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\mathrm{d}r\\
&\le Ce^{-\alpha\tau}\int_{-\infty}^{\tau} e^{\alpha r}|g(x,r)|^2\mathrm{d}r+C\int_{-\infty}^{0} e^{\alpha r}\sum_{j=1}^{m}|z_j(\theta_r\omega_j)|^2\mathrm{d}r\\
&\quad+\sup_{s\in[-h,0]} C\sum_{j=1}^{m}|z_j(\theta_s\omega_j)|^2+C,
\end{aligned}
\tag{3.45}
$$

for all $t\ge T$.

3.3. **Estimates on the exterior of a ball.** We now establish the following skillfull estimates, and these estimates are crucial for proving the pullback asymptotically compact.

Lemma 3.5. *Assume the hypotheses Lemma 3.3, and let* $\tau\in\mathbb{R}$, $\omega\in\Omega$ *and* $D\in\mathcal{D}_\alpha$. *Then for every* $\epsilon>0$, *there exists* $T=T(\tau,\omega,\epsilon,D)>0$ *and* $R=R(\tau,\omega,\epsilon)>0$ *such that for all* $t\ge T$,

$$
\begin{aligned}
&\sup_{\theta\in[-h,0]}\int_{\Omega_R^C}(|v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\\
&+|\nabla v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2)dx\le C\epsilon,
\end{aligned}
\tag{3.46}
$$

where $\Omega_R^C=\{x\in\mathbb{R}^N||x|\ge R\}$, $v_{\tau-t}(\cdot,\tau,\theta_{-\tau}\omega,v_\tau)=u_{\tau-t}(\cdot,\tau,\theta_{-\tau}\omega,u_\tau)-z(\theta_{-t+\cdot}\omega)$ *and* $u_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$.

Proof. Choose a smooth function $\xi(\cdot)$ with

$$
\xi(s)=\begin{cases}0, & 0\le s\le 1,\\ 1, & s\ge 2,\end{cases}
\tag{3.47}
$$

where $0\le\xi(s)\le 1$, $s\in\mathbb{R}^+$, and with a constant c such that $|\xi'(s)|\le c$ for $s\in\mathbb{R}^+$.

Multiplying (3.1) by $\xi^2(\frac{|x|^2}{K^2})v$ and integrating on $\mathbb{R}^N$, we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|\nabla v|^2+|v|^2)-\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})v\Delta v+\lambda\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v|^2\\
&=\int_{\mathbb{R}^N} f(x,v(t-\rho(t))+z(\theta_{t-\rho(t)}\omega)\xi^2(\frac{|x|^2}{K^2})v\\
&\quad+\int\frac{4x}{K^2}\xi(\frac{|x|^2}{K^2})\xi'(\frac{|x|^2}{K^2})v\nabla v_t+\int_{\mathbb{R}^N}(g+(1-\lambda)\Delta z(\theta_t\omega))\xi^2(\frac{|x|^2}{K^2})v\,.
\end{aligned}
\tag{3.48}
$$

Next, we bound each term in (3.48) one by one as follows

$$
\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})v\Delta v=-\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\nabla v|^2-\int_{\mathbb{R}^N}\frac{4x}{K^2}\xi(\frac{|x|^2}{K^2})\xi'(\frac{|x|^2}{K^2})v\nabla v.
\tag{3.49}
$$

$$
\begin{aligned}
|\int_{\mathbb{R}^N} \frac{4x}{K^2}\xi(\frac{|x|^2}{K^2})\xi'(\frac{|x|^2}{K^2})v\nabla v| &\le \frac{c}{K}\int_{K\le|x|\le\sqrt{2}K}|v||\nabla v| \\
&\le \frac{c}{K}\int_{\mathbb{R}^N}|v||\nabla v| \\
&\le \frac{c}{K}|v||\nabla v| \\
&\le \frac{c}{K}(|v|^2+|\nabla v|^2),
\end{aligned}
\tag{3.50}
$$

and

$$
\begin{aligned}
|\int_{\mathbb{R}^N} \frac{4x}{K^2}\xi(\frac{|x|^2}{K^2})\xi'(\frac{|x|^2}{K^2})v\nabla v_t| &\le \frac{c}{K}\int_{K\le|x|\le\sqrt{2}K}|v||\nabla v_t| \\
&\le \frac{c}{K}\int_{\mathbb{R}^N}|v||\nabla v_t| \\
&\le \frac{c}{K}|v||\nabla v_t| \\
&\le \frac{c}{K}(|v|^2+|\nabla v_t|^2).
\end{aligned}
\tag{3.51}
$$

From (1.2), using Young's inequality, for $\nu,\mu>0$, we have

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N} f(x,v(t-\rho(t)))+z(\theta_{t-\rho(t)}\omega)\xi^2(\frac{|x|^2}{K^2})v\Big| \\
&\le \frac{\nu}{2}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v|^2+\frac{1}{2\nu}\int_{\mathbb{R}^N}|f(x,v(t-\rho(t)))+z(\theta_{t-\rho(t)}\omega)\xi^2(\frac{|x|^2}{K^2})| \\
&\le \frac{\nu}{2}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v|^2+\frac{\alpha_2^2}{\nu}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v(t-\rho(t))|^2 \\
&\quad+\frac{\alpha_2^2}{\nu}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|z(\theta_{t-\rho(t)}\omega)|^2+\frac{1}{2\nu}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\alpha_1|^2,
\end{aligned}
\tag{3.52}
$$

and

$$
\begin{aligned}
&\Big|\int_{\mathbb{R}^N}(g+(1-\lambda)\Delta z(\theta_t\omega))\xi^2(\frac{|x|^2}{K^2})v\Big| \\
&\le \frac{\mu}{2}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v|^2+\frac{1}{\mu}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|g(x,t)|^2 \\
&\quad+\frac{C_\lambda}{\mu}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\Delta z(\theta_t\omega)|^2.
\end{aligned}
\tag{3.53}
$$

It follows from (3.48)–(3.53) that

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|\nabla v|^2+|v|^2)+\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\nabla v|^2 \\
&+(\lambda-\frac{\nu}{2}-\frac{\mu}{2})\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v|^2 \\
&\le \frac{c}{K}(|v|^2+|\nabla v|^2+|\nabla v_t|^2)+\frac{\alpha_2^2}{\nu}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v(t-\rho(t))|^2 \\
&\quad+\frac{\alpha_2^2}{\nu}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|z(\theta_{t-\rho(t)}\omega)|^2 +\frac{1}{2\nu}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\alpha_1|^2 \\
&\quad+\frac{1}{\mu}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|g(x,t)|^2+\frac{C_\lambda}{\mu}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\Delta z(\theta_t\omega)|^2.
\end{aligned}
\tag{3.54}
$$

Thus

$$\begin{aligned}
&\frac{d}{dt}(e^{\alpha t}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|\nabla v|^2+|v|^2)) \\
&= \alpha e^{\alpha t}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|\nabla v|^2+|v|^2)+e^{\alpha t}\frac{d}{dt}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|\nabla v|^2+|v|^2) \\
&\le (\alpha-2)e^{\alpha t}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\nabla v|^2+(\mu+\nu+\alpha-2\lambda)e^{\alpha t}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v|^2 \\
&\quad+\frac{2\alpha_2^2}{\nu}e^{\alpha t}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v(t-\rho(t))|^2+\frac{2}{\mu}e^{\alpha t}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|g(x,t)|^2 \\
&\quad+\frac{C_\lambda}{\mu}e^{\alpha t}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\Delta z(\theta_t\omega)|^2+\frac{2}{\nu}e^{\alpha t}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\alpha_1|^2 \\
&\quad+\frac{C}{K}e^{\alpha t}(|v|^2+|\nabla v|^2+|\nabla v_t|^2)+\frac{2\alpha_2^2}{\nu}e^{\alpha t}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|z(\theta_{t-\rho(t)}\omega)|^2.
\end{aligned}\tag{3.55}$$

Integrating (3.55) from $\tau-t$ to $\tau+s$, where $s\in[-h,0]$, and note that $\alpha<1$, thus

$$\begin{aligned}
&e^{\alpha(\tau+s)}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|v(\tau+s,\tau-t,\omega,v_{\tau-t})|^2+|\nabla v(\tau+s,\tau-t,\omega,v_{\tau-t})|^2) \\
&\le e^{\alpha(\tau-t)}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|v(\tau-t,\tau-t,\omega,v_{\tau-t})|^2+|\nabla v(\tau-t,\tau-t,\omega,v_{\tau-t})|^2) \\
&\quad+(\mu+\nu+\alpha-2\lambda)\int_{\tau-t}^{\tau+s}e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v(r,\tau-t,\omega,v_{\tau-t})|^2 \\
&\quad+\frac{2\alpha_2^2}{\nu}\int_{\tau-t}^{\tau+s}e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v(r-\rho(r),\tau-t,\omega,v_{\tau-t}))|^2 \\
&\quad+\frac{2}{\mu}\int_{\tau-t}^{\tau+s}e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|g(x,r)|^2+\frac{C_\lambda}{\mu}\int_{\tau-t}^{\tau+s}e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\Delta z(\theta_r\omega)|^2 \\
&\quad+\frac{2}{\nu}\int_{\tau-t}^{\tau+s}e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\alpha_1|^2+\frac{2\alpha_2^2}{\nu}\int_{\tau-t}^{\tau+s}e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|z(\theta_{r-\rho(r)}\omega)|^2 \\
&\quad+\frac{C}{K}\int_{\tau-t}^{\tau+s}e^{\alpha r}(|v(r,\tau-t,\omega,v_{\tau-t})|^2+|\nabla v(r,\tau-t,\omega,v_{\tau-t})|^2 \\
&\quad+|\nabla v_r(r,\tau-t,\omega,v_{\tau-t})|^2).
\end{aligned}\tag{3.56}$$

Noting that $\rho(s) \in [0,h]$ and the fact $\frac{1}{1-\rho'(s)} \leq \frac{1}{1-\rho^*}$ for all $s \in \mathbb{R}$. Setting $r' = r - \rho(r)$, similar to (3.30)-(3.31), we arrive at

$$
\begin{aligned}
&\frac{2\alpha_2^2}{\nu}\int_{\tau-t}^{\tau+s} e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v(r-\rho(r),\tau-t,\omega,v_{\tau-t}))|^2\\
&+\frac{2\alpha_2^2}{\nu}\int_{\tau-t}^{\tau+s} e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|z(\theta_{r-\rho(r)}\omega)|^2\\
&\leq C\|u_{\tau-t}\|_{C_V}^2 e^{-\alpha(t-\tau)} + C\int_{\tau-t-h}^{\tau-t} e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|z(\theta_r\omega)|^2 dr \\
&\quad+\frac{2\alpha_2^2 e^{\alpha h}}{\varepsilon_1(1-\rho^*)}\int_{\tau-t}^{\tau+s} e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v(r,\tau-t,\omega,v_{\tau-t})|^2 dr\\
&\quad+C\int_{\tau-t}^{\tau+s} e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|z(\theta_r\omega)|^2 dr.
\end{aligned} \tag{3.57}
$$

Choosing $0<\mu<1/8$, $0<\nu<1/8$, and noting that

$$
\lambda > \frac{2+\alpha}{2-\alpha} + \frac{8\alpha_2^2 e^{\alpha h}}{(2-\alpha)(1-\rho^*)},
$$

we have

$$
\mu+\nu+\alpha+\frac{16\alpha_2^2 e^{\alpha h}}{(1-\rho^*)} - 2\lambda < 0.
$$

Thus,

$$
\begin{aligned}
&e^{\alpha(\tau+s)}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|v(\tau+s,\tau-t,\omega,v_{\tau-t})|^2+|\nabla v(\tau+s,\tau-t,\omega,v_{\tau-t})|^2)\\
&+C\int_{\tau-t}^{\tau+s} e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v(r,\tau-t,\omega,v_{\tau-t})|^2\\
&\leq e^{\alpha(\tau-t)}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|v(\tau-t,\tau-t,\omega,v_{\tau-t})|^2+|\nabla v(\tau-t,\tau-t,\omega,v_{\tau-t})|^2)\\
&\quad+\frac{2}{\mu}\int_{\tau-t}^{\tau} e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|g(x,r)|^2+\frac{C_\lambda}{\mu}\int_{\tau-t}^{\tau} e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\Delta z(\theta_r\omega)|^2\\
&\quad+\frac{2}{\nu}\int_{\tau-t}^{\tau} e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\alpha_1|^2\\
&\quad+\frac{C}{K}\int_{\tau-t}^{\tau} e^{\alpha r}(|v(r,\tau-t,\omega,v_{\tau-t})|^2+|\nabla v(r,\tau-t,\omega,v_{\tau-t})|^2\\
&\quad+|\nabla v_r(r,\tau-t,\omega,v_{\tau-t})|^2)\\
&\quad+C\|u_{\tau-t}\|_{C_V}^2 e^{-\alpha(t-\tau)} + C\int_{\tau-t-h}^{\tau-t} e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|z(\theta_r\omega)|^2\\
&\quad+C\int_{\tau-t}^{\tau} e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|z(\theta_r\omega)|^2.
\end{aligned} \tag{3.58}
$$

Multiplying by $e^{-\alpha(\tau+s)}$, and replacing ω by $\theta_{-\tau}\omega$, note that $s\in[-h,0]$, we have

$$
\begin{aligned}
&\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\\
&+|\nabla v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2)\\
&+C\int_{\tau-t}^{\tau+s}e^{\alpha(r-\tau)}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|v(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\\
&\le Ce^{-\alpha t}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\\
&\quad+|\nabla v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2)\\
&\quad+Ce^{-\alpha\tau}\int_{\tau-t}^{\tau}e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|g(x,r)|^2\\
&\quad+Ce^{-\alpha\tau}\int_{\tau-t}^{\tau}e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|\Delta z(\theta_{r-\tau}\omega)|^2+|z(\theta_{r-\tau}\omega)|^2)\\
&\quad+Ce^{-\alpha\tau}\int_{\tau-t}^{\tau}e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\alpha_1|^2+C\|u_{\tau-t}\|_{C_V}^2e^{-\alpha t}\\
&\quad+\frac{C}{K}e^{-\alpha\tau}\int_{\tau-t}^{\tau}e^{\alpha r}(|v(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\\
&\quad+|\nabla v(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2+|\nabla v_r(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2)\\
&\quad+C\int_{\tau-t-h}^{\tau-t}e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|z(\theta_{r-\tau}\omega)|^2.
\end{aligned}
\tag{3.59}
$$

Note that $z(\theta_t\omega)=\sum_{j=1}^m h_jz_j(\theta_t\omega_j)$ and $h_j\in H^2(\mathbb{R}^N)$. Hence, given $\varepsilon>0$, there is $K^*=K^*(\epsilon,\omega)$ such that for all $K\ge K^*$, we have

$$
\int_{|x|\ge K}(|h_j(x)|^2+|\nabla h_j(x)|^2+|\Delta h_j(x)|^2)\le\frac{\epsilon}{r(\omega)},\quad j=1,\dots,m,
\tag{3.60}
$$

where $r(\omega)$ is the tempered function in (3.9). Then

$$
\begin{aligned}
&Ce^{-\alpha\tau}\int_{\tau-t}^{\tau}e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|\Delta z(\theta_{r-\tau}\omega)|^2+|z(\theta_{r-\tau}\omega)|^2)\\
&\le Ce^{-\alpha\tau}\int_{\tau-t}^{\tau}e^{\alpha r}\sum_{j=1}^m\int_{|x|\ge K}(|h_j(x)|^2|z_j(\theta_{r-\tau}\omega_j)|^2\\
&\quad+|\Delta h_j(x)|^2|z_j(\theta_{r-\tau}\omega_j)|^2)\\
&\le C\int_{-\infty}^{0}e^{\alpha r}\sum_{j=1}^m\int_{|x|\ge K}(|h_j(x)|^2|z_j(\theta_r\omega_j)|^2+|\Delta h_j(x)|^2|z_j(\theta_r\omega_j)|^2)\\
&\le\frac{\epsilon}{r(\omega)}\int_{-\infty}^{0}e^{\alpha r}\sum_{j=1}^m|z_j(\theta_r\omega_j)|^2,
\end{aligned}
\tag{3.61}
$$

and

$$
C\int_{\tau-t-h}^{\tau-t}e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|z(\theta_{r-\tau}\omega)|^2\le\frac{\epsilon}{r(\omega)}\int_{-\infty}^{0}e^{\alpha r}\sum_{j=1}^m|z_j(\theta_r\omega_j)|^2.
\tag{3.62}
$$

By (3.36), we arrive at

$$\begin{aligned} &Ce^{-\alpha t}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\\ &+|\nabla v(\tau-t,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2)\\ &\le Ce^{-\alpha t}\|u_{\tau-t}\|_{C_V}^2+Ce^{-\alpha t}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|z(\theta_{-t}\omega)|^2\le C\epsilon. \end{aligned} \tag{3.63}$$

By (H1) and (H3), we obtain

$$\begin{aligned} Ce^{-\alpha\tau}\int_{\tau-t}^{\tau}e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|g(x,r)|^2 &\le Ce^{-\alpha\tau}\int_{-\infty}^{\tau}e^{\alpha r}\int_{|x|\ge K}|g(x,r)|^2\\ &\le C\epsilon, \end{aligned} \tag{3.64}$$

and

$$Ce^{-\alpha\tau}\int_{\tau-t}^{\tau}e^{\alpha r}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})|\alpha_1|^2\le C\epsilon. \tag{3.65}$$

By (3.45), when K and t are large enough, we infer

$$\begin{aligned} &\frac{C}{K}e^{-\alpha\tau}\int_{\tau-t}^{\tau}e^{\alpha r}(|v(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2+|\nabla v(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\\ &+|\nabla v_r(r,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2)\\ &\le\frac{C}{K}(\int_{-\infty}^{\tau}e^{\alpha r}|g(x,r)|^2\mathrm{d}r+\int_{-\infty}^{0}e^{\alpha r}\sum_{j=1}^{m}|z_j(\theta_r\omega_j)|^2\mathrm{d}r\\ &\quad+\sup_{s\in[-h,0]}\sum_{j=1}^{m}|z_j(\theta_s\omega_j)|^2+C)\le C\epsilon. \end{aligned} \tag{3.66}$$

Thanks to (3.59)–(3.66), when K and t are sufficiently large, we complete the proof. □

Lemma 3.6. *Under the assumptions of Lemma 3.3, let $\tau\in\mathbb{R}$, $\omega\in\Omega$ and $D\in\mathcal{D}_\alpha$. Then for every $\epsilon>0$, there exist $T^*=T(\tau,\omega,\epsilon,D)>0$ and $R_1=R(\tau,\omega,\epsilon)>0$ such that for all $t\ge T^*$, we have*

$$\begin{aligned} &\sup_{\theta\in[-h,0]}\int_{\Omega_{R_1}^C}(|u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2\\ &+|\nabla u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2)dx\le C\epsilon, \end{aligned} \tag{3.67}$$

where $\Omega_{R_1}^C=\{x\in\mathbb{R}^N||x|\ge R\}$ and $u_{\tau-t}\in D(\tau-t,\theta_{-t}\omega)$.

Proof. Noting that

$$v(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})=u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})-z(\theta_s\omega),$$

by Lemma 3.5, we infer that

$$\begin{aligned}
&\sup_{\theta\in[-h,0]}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2\\
&+|\nabla u(\tau+s,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})|^2)dx\\
&\le 2\sup_{\theta\in[-h,0]}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2\\
&\quad+|\nabla v(\tau+s,\tau-t,\theta_{-\tau}\omega,v_{\tau-t})|^2)dx\\
&\quad+2\sup_{\theta\in[-h,0]}\int_{\mathbb{R}^N}\xi^2(\frac{|x|^2}{K^2})(|z(\theta_s\omega)|^2+|\nabla z(\theta_s\omega)|^2)dx\\
&\le C\epsilon.
\end{aligned} \tag{3.68}$$

This completes the proof. □

3.4. **Pullback attractors.** We denote $\Omega_K=\{x\in\mathbb{R}^N||x|<K\}$, and let

$$\chi_{\Omega_K}=\begin{cases}1, & x\in\Omega_K,\\ 0, & x\in\Omega_K^C,\end{cases}\qquad \chi_{\Omega_K}^C=\begin{cases}1, & x\in\Omega_K^C,\\ 0, & x\in\Omega_K.\end{cases}$$

We decompose (3.1) as follows:

$$u(x,t)=Y(x,t)+W(x,t),$$

where $Y(x,t)=u(x,t)\chi_{\Omega_K}$ and $W(x,t)=u(x,t)\chi_{\Omega_K}^C$ satisfying the following equations, respectively:

$$\begin{gathered}
\frac{dY}{dt}-\Delta\frac{dY}{dt}-\Delta Y+\lambda Y\\
=f(x,Y(t-\rho(t)))\chi_{\Omega_K}+g(x,t)\chi_{\Omega_K}+\sum_{j=1}^m h_j\chi_{\Omega_K}\frac{dw_j}{dt},\\
Y(t+\tau,x)=u(t+\tau,x)\chi_{\Omega_K}=u_\tau(t,x)\chi_{\Omega_K}=Y_\tau(t,x),\\
t\in[-h,0],\ x\in\mathbb{R}^N,
\end{gathered} \tag{3.69}$$

and

$$\begin{gathered}
\frac{dW}{dt}-\Delta\frac{dW}{dt}-\Delta w+\lambda W\\
=f(x,u(t-\rho(t)))-f(x,Y(t-\rho(t)))\chi_{\Omega_K}+g(x,t)\chi_{\Omega_K}^C\\
+\sum_{j=1}^m h_j\chi_{\Omega_K^C}\frac{dw_j}{dt},\\
W(t+\tau,x)=u(t+\tau,x)\chi_{\Omega_K^C}=u_\tau(t,x)\chi_{\Omega_K^C}=W_\tau(t,x),\\
t\in[-h,0],x\in\mathbb{R}^N.
\end{gathered} \tag{3.70}$$

By Theorem 3.1, there exists a solution $Y(t)$ to (3.69), and (3.70) has a solution $W(t):=u(t)-Y(t)$. Similar to the proof of Lemma 3.5, we obtain the following lemma.

Lemma 3.7. *Under the assumptions of Lemma 3.3, for any fixed $\tau\in\mathbb{R}$, $\omega\in\Omega$, $D\in D_\alpha$, any $\epsilon>0$, when K and t large enough, the solution of (3.70) with ω*

replaced by $\theta_{-\tau}\omega$ *satisfies*

$$\begin{aligned} \sup_{\theta\in[-h,0]} \int_{\Omega_K^C} (|W(\tau+s,\tau-t,\theta_{-\tau}\omega,W_{\tau-t})|^2 \\ +|\nabla W(\tau+s,\tau-t,\theta_{-\tau}\omega,W_{\tau-t})|^2)dx \le \epsilon. \end{aligned} \tag{3.71}$$

Now, we decompose (3.13) as follows:

$$v(x,t)=y(x,t)+w(x,t),$$

where $y(x,t)=v(x,t)\chi_{\Omega_K}$ and $w(x,t)=v(x,t)\chi_{\Omega_K}^C$ satisfying the following equations, respectively:

$$\begin{gathered} \frac{dy}{dt}-\Delta\frac{dy}{dt}-\Delta y+\lambda y=f(x,y(t-\rho(t))+z(\theta_{t-\rho(t)}\omega)\chi_{\Omega_K})\chi_{\Omega_K}+g(x,t)\chi_{\Omega_K} \\ +(1-\lambda)\Delta z(\theta_t\omega)\chi_{\Omega_K}, \\ y(t+\tau,x)=v(t+\tau,x)\chi_{\Omega_K}=v_\tau(t,x)\chi_{\Omega_K}=y_\tau(t,x), \\ t\in[-h,0], x\in\mathbb{R}^N, \end{gathered} \tag{3.72}$$

and

$$\begin{aligned} &\frac{dw}{dt}-\Delta\frac{dw}{dt}-\Delta w+\lambda w \\ &=f(x,v(t-\rho(t))+z(\theta_{t-\rho(t)}\omega))-f(x,y(t-\rho(t)) \\ &\quad+z(\theta_{t-\rho(t)}\omega)\chi_{\Omega_K})\chi_{\Omega_K}+g(x,t)\chi_{\Omega_K}^C+(1-\lambda)\Delta z(\theta_t\omega)\chi_{\Omega_K}^C, \\ &\quad w(t+\tau,x)=v(t+\tau,x)\chi_{\Omega_K^C}=v_\tau(t,x)\chi_{\Omega_K^C}=w_\tau(t,x), \\ &\qquad t\in[-h,0],\ x\in\mathbb{R}^N. \end{aligned} \tag{3.73}$$

We consider the operator $A=-\Delta$ with Dirichlet boundary conditions. Since A is self-adjoint, positive operator and has a compact inverse, there exists a complete set of eigenvectors $\{\omega_i'\}_{i=1}^\infty$ in $L^2(\Omega_K)$, the corresponding eigenvalues $\{\lambda_i\}_{i=1}^\infty$ satisfy

$$A\omega_i'=\lambda_i\omega_i',\quad 0<\lambda_1\le\lambda_2\le\dots\le\lambda_i\le\dots\to+\infty,\quad i\to+\infty.$$

We set $V_m=\operatorname{span}\{\omega_1',\omega_2',\dots,\omega_m'\}$, P_m is the orthogonal projection onto V_m.

Let $Y=Y_1+Y_2$, where $Y_1=P_mY$ and $Y_2=(I-P_m)Y$, and let $y=y^1+y^2$, where $y^1=P_my$ and $y^2=(I-P_m)y$, we decompose (3.69) as follows:

$$\begin{gathered} \frac{dy^1}{dt}-\Delta\frac{dy^1}{dt}-\Delta y^1+\lambda y^1=P_mf(x,y^1(t-\rho(t))+z(\theta_{t-\rho(t)}\omega)\chi_{\Omega_K})\chi_{\Omega_K} \\ +P_mg(x,t)\chi_{\Omega_K}+(1-\lambda)P_m\Delta z(\theta_t\omega)\chi_{\Omega_K}, \\ y^1(t+\tau,x)=P_mv(t+\tau,x)\chi_{\Omega_K}=P_mv_\tau(t,x)\chi_{\Omega_K}=y^1_\tau(t,x), \\ t\in[-h,0],\ x\in\mathbb{R}^N, \end{gathered} \tag{3.74}$$

and

$$\begin{aligned} &\frac{dy^2}{dt} - \Delta \frac{dy^2}{dt} - \Delta y^2 + \lambda y^2 \\ &= f(x, y(t-\rho(t)) + z(\theta_{t-\rho(t)}\omega)\chi_{\Omega_K})\chi_{\Omega_K} \\ &\quad - P_m f(x, y^1(t-\rho(t)) + z(\theta_{t-\rho(t)}\omega)\chi_{\Omega_K})\chi_{\Omega_K} \\ &\quad + (I-P_m)g(x,t)\chi_{\Omega_K} + (1-\lambda)(I-P_m)\Delta z(\theta_t\omega)\chi_{\Omega_K}, \\ y^2(t+\tau, x) &= (I-P_m)v(t+\tau,x)\chi_{\Omega_K} = (I-P_m)v_\tau(t,x)\chi_{\Omega_K} = y^2_\tau(t,x), \\ &\quad t\in[-h,0],\ x\in\mathbb{R}^N, \end{aligned} \tag{3.75}$$

Lemma 3.8. *Under the assumptions of Lemma 3.3, for any fixed $\tau \in \mathbb{R}$, $\omega \in \Omega$, $D \in D_\alpha$, any $\epsilon > 0$, there exist $T^* > 0$ and a finite-dimensional subspace $P(H_0^1(\Omega_K))$ of $H_0^1(\Omega_K)$ and $\delta > 0$ such that*

(1) *for all $t \geq T^*$, $u_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) \in \Phi(t, \tau-t, \theta_{-t}\omega, D(\tau-t, \theta_{-t}\omega))$, $s_1, s_2 \in [-h, 0]$ with $|s_2 - s_1| < \delta$,*
$$|P(Y(\tau+s_1) - Y(\tau+s_2))|_{H_0^1(\Omega_K)} < \epsilon;$$
(2) *for all $t \geq T^*$, $u_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) \in \Phi(t, \tau-t, \theta_{-t}\omega, D(\tau-t, \theta_{-t}\omega))$,*
$$\sup_{s\in[-h,0]} |(I-P)Y(\tau+s)|_{H_0^1(\Omega_K)} < \epsilon,$$

where $P : H_0^1(\Omega_K) \to P(H_0^1(\Omega_K))$ is the canonical projector.

Proof. We divide the proof into two steps.

Step 1. We consider the functional differential system (3.75). Taking the product of (3.75) with $-\Delta y^2$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}(|\Delta y^2|^2 + |\nabla y^2|^2) + |\Delta y^2|^2 + \lambda|\nabla y^2|^2 \\ &= (f(x, y(t-\rho(t)) + z(\theta_{t-\rho(t)}\omega)\chi_{\Omega_K})\chi_{\Omega_K}, -\Delta y^2) \\ &\quad - (P_m f(x, y^1(t-\rho(t)) + z(\theta_{t-\rho(t)}\omega)\chi_{\Omega_K})\chi_{\Omega_K}, -\Delta y^2) \\ &\quad + ((I-P_m)g(x,t)\chi_{\Omega_K}, -\Delta y^2) \\ &\quad + (1-\lambda)((I-P_m)\Delta z(\theta_t\omega)\chi_{\Omega_K}, -\Delta y^2), \end{aligned} \tag{3.76}$$

By Young's inequality and (H1), we have

$$\begin{aligned} &|(f(x, y(t-\rho(t)) + z(\theta_{t-\rho(t)}\omega)\chi_{\Omega_K})\chi_{\Omega_K}, -\Delta y^2)| \\ &+ |(P_m f(x, y^1(t-\rho(t)) + z(\theta_{t-\rho(t)}\omega)\chi_{\Omega_K})\chi_{\Omega_K}, -\Delta y^2)| \\ &\leq \frac{1}{4}|\Delta y^2|^2 + 4|\alpha_1|^2 + 4\alpha_2^2|v(t-\rho(t)) + z(\theta_{t-\rho(t)}\omega)|^2 \\ &\leq \frac{1}{4}|\Delta y^2|^2 + 4|\alpha_1|^2 + 8\alpha_2^2\|v^t\|^2_{C_V} + 8\alpha_2^2|z(\theta_{t-\rho(t)}\omega)|^2, \end{aligned} \tag{3.77}$$

and

$$\begin{aligned} &((I-P_m)g(x,t)\chi_{\Omega_K}, -\Delta y^2) + (1-\lambda)((I-P_m)\Delta z(\theta_t\omega)\chi_{\Omega_K}, -\Delta y^2) \\ &\leq \frac{1}{4}|\Delta y^2|^2 + 2|g(x,t)|^2 + C_\lambda|\Delta z(\theta_t\omega)|^2. \end{aligned} \tag{3.78}$$

From (3.76)–(3.78) it follows that

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(|\Delta y^2|^2+|\nabla y^2|^2)+\frac{1}{2}|\Delta y^2|^2+\lambda|\nabla y^2|^2\\&\le|\alpha_1|^2+8\alpha_2^2\|v^t\|^2_{C_V}+8\alpha_2^2|z(\theta_{t-\rho(t)}\omega)|^2+2|g(x,t)|^2+C_\lambda|\Delta z(\theta_t\omega)|^2.\end{aligned}\tag{3.79}$$

In particular,

$$\begin{aligned}&\frac{d}{dt}(|\Delta y^2|^2+|\nabla y^2|^2)+|\Delta y^2|^2+|\nabla y^2|^2\\&\le 2|\alpha_1|^2+16\alpha_2^2\|v^t\|^2_{C_V}+16\alpha_2^2|z(\theta_{t-\rho(t)}\omega)|^2+4|g(x,t)|^2+C_\lambda|\Delta z(\theta_t\omega)|^2.\end{aligned}\tag{3.80}$$

Applying the Gronwall Lemma in the interval $[\tau-t,\tau+s]$, and replacing ω by $\theta_{-\tau}\omega$, we have

$$\begin{aligned}&\sup_{s\in[-h,0]}(|\nabla y^2(\tau+s,\tau-t,\theta_{-\tau}\omega,y^2_{\tau-t})|^2+|\Delta y^2(\tau+s,\tau-t,\theta_{-\tau}\omega,y^2_{\tau-t})|^2)\\&\le e^{-(t-h)}(|\nabla y^2(\tau-t,\tau-t,\theta_{-\tau}\omega,y^2_{\tau-t})|^2+|\Delta y^2(\tau-t,\tau-t,\theta_{-\tau}\omega,y^2_{\tau-t})|^2)\\&\quad+C\sup_{s\in[-h,0]}\int_{\tau-t}^{\tau+s}e^{-(\tau+s-r)}(|\alpha_1|^2+\alpha_2^2\|v^r\|^2_{C_V}+\alpha_2^2|z(\theta_{r-\rho(t)-\tau}\omega)|^2\\&\quad+|g(r)|^2+|\Delta z(\theta_{r-\tau}\omega)|^2)dr.\end{aligned}\tag{3.81}$$

Note that Ω_K is a bounded domain, using Poincaré inequality, then for any $\epsilon>0$, we can choose t and m large enough such that

$$\frac{1}{\lambda_{m+1}}\sup_{s\in[-h,0]}\int_{\tau-t}^{\tau+s}e^{-(\tau+s-r)}|\alpha_1|^2\le\frac{C|\alpha_1|^2}{\lambda_{m+1}}\le\epsilon.\tag{3.82}$$

Note that $z(\theta_t\omega)=\sum_{j=1}^m h_jz_j(\theta_t\omega_j)$ and $h_j\in H^2(\mathbb{R}^N)$. Then

$$\begin{aligned}&\frac{\alpha_2^2}{\lambda_{m+1}}\sup_{s\in[-h,0]}\int_{\tau-t}^{\tau+s}e^{-(\tau+s-r)}|z(\theta_{r-\rho(t)-\tau}\omega)|^2dr\\&\le\frac{\alpha_2^2}{\lambda_{m+1}}\sup_{s\in[-h,0]}\int_{\tau-t}^{\tau+s}e^{-(\tau+s-r)}e^{-\frac{\lambda}{2}(r-\rho(r)-\tau)}r(\omega)dr\\&\le\frac{Cr(\omega)}{\lambda_{m+1}}\le\epsilon,\end{aligned}\tag{3.83}$$

and

$$\begin{aligned}&\frac{1}{\lambda_{m+1}}\sup_{s\in[-h,0]}\int_{\tau-t}^{\tau+s}e^{-(\tau+s-r)}|\Delta z(\theta_{r-\tau}\omega)|^2dr\\&\le\frac{1}{\lambda_{m+1}}\sup_{s\in[-h,0]}\int_{-t}^{s}e^{r'-s}\sum_{j=1}^m|\Delta h_jz_j(\theta_{r'}\omega_j)|^2dr'\\&\le\frac{1}{\lambda_{m+1}}\sup_{s\in[-h,0]}\int_{-t}^{s}e^{r'-s}e^{-\frac{1}{2}\lambda r'}r(\omega)dr'\\&\le\frac{Cr(\omega)}{\lambda_{m+1}}\le\epsilon.\end{aligned}\tag{3.84}$$

Note that $g\in C(\mathbb{R},H)$. By (H3),

$$\frac{1}{\lambda_{m+1}}\sup_{s\in[-h,0]}\int_{\tau-t}^{\tau+s}e^{-(\tau+s-r)}|g(r)|^2dr\le\epsilon.\tag{3.85}$$

By (3.36), and similar to the argument in (3.82)–(3.84), when m and t large enough, we obtain

$$\frac{1}{\lambda_{m+1}}\sup_{s\in[-h,0]}\int_{\tau-t}^{\tau+s}e^{-(\tau+s-r)}\alpha_2^2\|v^r\|_{C_V}^2\,dr\le\epsilon, \tag{3.86}$$

$$\begin{aligned}&\frac{1}{\lambda_{m+1}}e^{-(t-h)}(|\nabla y^2(\tau-t,\tau-t,\theta_{-\tau}\omega,y^2_{\tau-t})|^2\\&+|\Delta y^2(\tau-t,\tau-t,\theta_{-\tau}\omega,y^2_{\tau-t})|^2)\le\epsilon.\end{aligned} \tag{3.87}$$

It follows from (3.81)–(3.87) that

$$\sup_{s\in[-h,0]}|y^2(\tau+s,\tau-t,\theta_{-\tau}\omega,y^2_{\tau-t})|^2_{H_0^1(\Omega_K)}\le\epsilon. \tag{3.88}$$

Since $z(\theta_t\omega)=\sum_{j=1}^m h_jz_j(\theta_t\omega_j)$ and $h_j\in H^2(\mathbb{R}^N)$, by (3.9)–(3.11), (3.88), we obtain

$$\begin{aligned}&\sup_{s\in[-h,0]}|Y_2(\tau+s,\tau-t,\theta_{-\tau}\omega,y^2_{\tau-t})|^2_{H_0^1(\Omega_K)}\\&\le\sup_{s\in[-h,0]}|y^2(\tau+s,\tau-t,\theta_{-\tau}\omega,y^2_{\tau-t})|^2_{H_0^1(\Omega_K)}\\&\quad+\sup_{s\in[-h,0]}|(I-P_m)z(\theta_s\omega)\chi_{\Omega_K}|^2_{H_0^1(\Omega_K)}\le\epsilon.\end{aligned} \tag{3.89}$$

Step 2. We consider the functional differential system (3.74). Noting that $|\Delta y^1|^2\le\lambda_m|\nabla y^1|^2\le\lambda_m^2|y^1|^2$. Without loss of generality, we assume that $s_1,s_2\in[-h,0]$ with $0<s_1-s_2<1$. Then

$$\begin{aligned}&|y^1(\tau+s_1,\tau-t,\theta_{-\tau}\omega,y^1_{\tau-t})-y^1(\tau+s_2,\tau-t,\theta_{-\tau}\omega,y^1_{\tau-t})|_{H_0^1(\Omega)}\\&\le\sqrt{\lambda_m}|y^1(\tau+s_1,\tau-t,\theta_{-\tau}\omega,y^1_{\tau-t})-y^1(\tau+s_2,\tau-t,\theta_{-\tau}\omega,y^1_{\tau-t})|\\&\le\sqrt{\lambda_m}\int_{\tau+s_2}^{\tau+s_1}|\frac{dy^1(T,\tau-t,\theta_{-\tau}\omega,y^1_{\tau-t})}{dT}|dT\\&\le\sqrt{\lambda_m}\int_{\tau+s_2}^{\tau+s_1}(|\Delta\frac{dy^1(T,\tau-t,\theta_{-\tau}\omega,y^1_{\tau-t})}{dT}|+|\Delta y^1(T,\tau-t,\theta_{-\tau}\omega,y^1_{\tau-t})|\\&\quad+\lambda|y^1(T,\tau-t,\theta_{-\tau}\omega,y^1_{\tau-t})|+\big|P_mf(x,y^1(T-\rho(T),\tau-t,\theta_{-\tau}\omega,y^1_{\tau-t})\\&\quad+z(\theta_{T-\rho(T)-\tau}\omega)_{\chi_{\Omega_K}})_{\chi_{\Omega_K}}\big|\\&\quad+|P_mg(x,T)\chi_{\Omega_K}|+C_\lambda|P_m\Delta z(\theta_{T-\tau}\omega)\chi_{\Omega_K}|)dT\\&\quad=:I_1+I_2+I_3+I_4+I_5+I_6.\end{aligned} \tag{3.90}$$

For I_5, note that $g\in C(\mathbb{R},H)$, and τ is fixed, we have

$$I_5=\sqrt{\lambda_m}\int_{\tau+s_2}^{\tau+s_1}|P_mg(x,T)|dT\le C(s_1-s_2). \tag{3.91}$$

For I_2 and I_3, by (H3), (3.36) and (3.43), we have

$$\begin{aligned}
I_2 + I_3 &\le \sqrt{\lambda_m}\int_{\tau+s_2}^{\tau+s_1} (|\Delta y^1(T,\tau-t,\theta_{-\tau}\omega, y^1_{\tau-t})| \\
&\quad + \lambda|y^1(T,\tau-t,\theta_{-\tau}\omega, y^1_{\tau-t})|)dT \\
&\le C_{\lambda_m,\lambda}\int_{\tau+s_2}^{\tau+s_1} (|\nabla y^1(T,\tau-t,\theta_{-\tau}\omega, y^1_{\tau-t})| \\
&\quad + |y^1(T,\tau-t,\theta_{-\tau}\omega, y^1_{\tau-t})|)dT \\
&\le C_{\lambda_m,\lambda}\int_{\tau+s_2}^{\tau+s_1} (|\nabla y^1(T,\tau-t,\theta_{-\tau}\omega, y^1_{\tau-t})|^2 \\
&\quad + |y^1(T,\tau-t,\theta_{-\tau}\omega, y^1_{\tau-t})|^2)dT + C(s_1-s_2) \\
&\le C(e^{-\alpha s_1} - e^{-\alpha s_2}) + C(s_1 - s_2).
\end{aligned} \tag{3.92}$$

Similar to (3.82)-(3.83), we have

$$\begin{aligned}
I_6 &\le C_{\lambda_m,\lambda}\int_{\tau+s_2}^{\tau+s_1} |\Delta z(\theta_{T-\tau}\omega)|dT \\
&\le C_{\lambda_m,\lambda}\int_{\tau+s_2}^{\tau+s_1} e^{-\frac{1}{2}(T-\tau)}r(\omega)dT \\
&\le Cr(\omega)(e^{-\frac{1}{2}s_2} - e^{-s_1/2},
\end{aligned} \tag{3.93}$$

and

$$\begin{aligned}
\int_{\tau+s_2}^{\tau+s_1} |\Delta z(\theta_{T-\rho(T)-\tau}\omega)|dT &\le \int_{\tau+s_2}^{\tau+s_1} e^{-\frac{1}{2}(T-h-\tau)}r(\omega)dT \\
&\le Cr(\omega)(e^{-\frac{1}{2}s_2} - e^{-s_1/2}.
\end{aligned} \tag{3.94}$$

For I_4, by (H1) and (3.36), (3.42), (3.43) and (3.94), we have

$$\begin{aligned}
I_4 &\le \sqrt{\lambda_m}\int_{\tau+s_2}^{\tau+s_1} |P_m f(x, y^1(T-\rho(T),\tau-t,\theta_{-\tau}\omega, y^1_{\tau-t}) \\
&\quad + z(\theta_{T-\rho(T)-\tau}\omega)_{\chi\Omega_K})|dT \\
&\le \sqrt{\lambda_m}\int_{\tau+s_2}^{\tau+s_1} (|f(x, y^1(T-\rho(T),\tau-t,\theta_{-\tau}\omega, y^1_{\tau-t}) \\
&\quad + z(\theta_{T-\rho(T)-\tau}\omega)_{\chi\Omega_K})|^2 + C)dT \\
&\le \sqrt{\lambda_m}\int_{\tau+s_2}^{\tau+s_1} (|\alpha_1|^2 + 2\alpha_2^2\|v_T\|^2_{C_V} + 2\alpha_2^2|z(\theta_{T-\rho(T)-\tau}\omega)|^2 + C)dT \\
&\le C(e^{-\alpha s_1} - e^{-\alpha s_2}) + C(s_1 - s_2).
\end{aligned} \tag{3.95}$$

Now, it only remains to estimate the bound of

$$I_1 = \sqrt{\lambda_m}\int_{\tau+s_2}^{\tau+s_1} |\Delta \frac{dy^1(T,\tau-t,\theta_{-\tau}\omega, y^1_{\tau-t})}{dT}|.$$

Taking the inner product of (3.74) with y^1, $-\Delta y^1$, y^1_T and $-\Delta y^1_T$, respectively, we obtain

$$\begin{aligned}&\frac{d}{dT}(|y^1|^2+|\nabla y^1|^2)+2|\nabla y^1|^2+2\lambda|y^1|^2\\&=2(P_mf(x,y^1(T-\rho(T))+z(\theta_{T-\rho(T)}\omega)\chi_{\Omega_K})\chi_{\Omega_K},y^1)\\&\quad+2(P_mg(x,T)\chi_{\Omega_K},y^1)+2(1-\lambda)(P_m\Delta z(\theta_T\omega)\chi_{\Omega_K},y^1),\end{aligned}\tag{3.96}$$

$$\begin{aligned}&\frac{d}{dT}(|\nabla y^1|^2+|\Delta y^1|^2)+2|\Delta y^1|^2+2\lambda|\nabla y^1|^2\\&=2(P_mf(x,y^1(T-\rho(T))+z(\theta_{T-\rho(T)}\omega)\chi_{\Omega_K})\chi_{\Omega_K},-\Delta y^1)\\&\quad+2(P_mg(x,T)\chi_{\Omega_K},-\Delta y^1)+2(1-\lambda)(P_m\Delta z(\theta_T\omega)\chi_{\Omega_K},-\Delta y^1),\end{aligned}\tag{3.97}$$

$$\begin{aligned}&\frac{d}{dT}|\nabla y^1|^2+\lambda\frac{d}{dT}|y^1|^2+2|\nabla y^1_T|^2+2|y^1_T|^2\\&=2(P_mf(x,y^1(T-\rho(T))+z(\theta_{T-\rho(T)}\omega)\chi_{\Omega_K})\chi_{\Omega_K},y^1_T)\\&\quad+2(P_mg(x,T)\chi_{\Omega_K},y^1_T)+2(1-\lambda)(P_m\Delta z(\theta_T\omega)\chi_{\Omega_K},y^1_T),\end{aligned}\tag{3.98}$$

and

$$\begin{aligned}&\frac{d}{dT}|\Delta y^1|^2+\lambda\frac{d}{dT}|\nabla y^1|^2+2|\Delta y^1_T|^2+2|\nabla y^1_T|^2\\&=2(P_mf(x,y^1(T-\rho(T))+z(\theta_{T-\rho(T)}\omega)\chi_{\Omega_K})\chi_{\Omega_K},-\Delta y^1_T)\\&\quad+2(P_mg(x,T)\chi_{\Omega_K},-\Delta y^1_T)+2(1-\lambda)(P_m\Delta z(\theta_T\omega)\chi_{\Omega_K},-\Delta y^1_T).\end{aligned}\tag{3.99}$$

Now, computing (3.98) + (3.99) − (3.97) − λ(3.96), we obtain

$$\begin{aligned}&|\Delta y^1_T|^2+2|\nabla y^1_T|^2+|y^1_T|^2-|\Delta y^1|-2\lambda|\nabla y^1|^2-\lambda^2|y^1|^2\\&=(P_mf(x,y^1(T-\rho(T))+z(\theta_{T-\rho(T)}\omega)\chi_{\Omega_K})\chi_{\Omega_K},y^1_T-\Delta y^1_T\\&\quad+\Delta y^1-\lambda y^1)\\&\quad+(P_mg(x,T)\chi_{\Omega_K},y^1_T-\Delta y^1_T+\Delta y^1-\lambda y^1)\\&\quad+(1-\lambda)(P_m\Delta z(\theta_T\omega)\chi_{\Omega_K},y^1_T-\Delta y^1_T+\Delta y^1-\lambda y^1)\\&\le 4|P_mf(x,y^1(T-\rho(T))+z(\theta_{T-\rho(T)}\omega)\chi_{\Omega_K})\chi_{\Omega_K}|^2\\&\quad+4|P_mg(x,T)\chi_{\Omega_K}|^2\\&\quad+C_\lambda|P_m\Delta z(\theta_T\omega)\chi_{\Omega_K}|^2+\frac{3}{4}(|y^1_T|^2+|\Delta y^1_T|^2+|\Delta y^1|^2+\lambda|y^1|^2).\end{aligned}\tag{3.100}$$

By Young's inequality, we have

$$\begin{aligned}|\Delta y^1_T|^2&\le 4|P_mf(x,y^1(T-\rho(T))+z(\theta_{T-\rho(T)}\omega)\chi_{\Omega_K})\chi_{\Omega_K}|^2\\&\quad+4|P_mg(x,T)\chi_{\Omega_K}|^2+C_\lambda|P_m\Delta z(\theta_T\omega)\chi_{\Omega_K}|^2+C|\Delta y^1|^2.\end{aligned}\tag{3.101}$$

Then, by (3.101), (3.95), (3.91) and (3.93), using Poincaré inequality, we infer that

$$\begin{aligned}I_1&=\sqrt{\lambda_m}\int_{\tau+s_2}^{\tau+s_1}|\Delta\frac{dy^1(T,\tau-t,\theta_{-\tau}\omega,y^1_{\tau-t})}{dT}|\\&\le C(e^{-\alpha s_1}-e^{-\alpha s_2})+C(s_1-s_2)+Cr(\omega)(e^{-\frac12 s_2}-e^{-\frac12 s_1}).\end{aligned}\tag{3.102}$$

From (3.90)–(3.93), (3.95) and (3.102), it follows that for all $t\ge h$, $u_\tau(\cdot,\tau-t,\theta_{-\tau}\omega,u_{\tau-t})\in\Phi(t,\tau-t,\theta_{-t}\omega,D(\tau-t,\theta_{-t}\omega))$, we have

$$|y^1(\tau+s_1,\tau-t,\theta_{-\tau}\omega,y^1_{\tau-t})-y^1(\tau+s_2,\tau-t,\theta_{-\tau}\omega,y^1_{\tau-t})|_{H_0^1(\Omega_K)}\to 0 \tag{3.103}$$

as $s_2 \to s_1$.

Note that $z(\theta_t\omega)$ is continuous with respect t, we deduce that for all $t \geq h$, $u_\tau(\cdot, \tau - t, \theta_{-\tau}\omega, u_{\tau-t}) \in \Phi(t, \tau - t, \theta_{-t}\omega, D(\tau - t, \theta_{-t}\omega))$,

$$\begin{aligned}
&|Y^1(\tau + s_1, \tau - t, \theta_{-\tau}\omega, Y^1_{\tau-t}) - Y^1(\tau + s_2, \tau - t, \theta_{-\tau}\omega, Y^1_{\tau-t})|_{H^1_0(\Omega_K)} \\
&\leq |y^1(\tau + s_1, \tau - t, \theta_{-\tau}\omega, y^1_{\tau-t}) - y^1(\tau + s_2, \tau - t, \theta_{-\tau}\omega, y^1_{\tau-t})|_{H^1_0(\Omega_K)} \\
&\quad + |P_m(z(\theta_{s_1}\omega)_{\chi_{\Omega_K}} - z(\theta_{s_2}\omega)_{\chi_{\Omega_K}}|_{H^1_0(\Omega_K)} \to 0 \quad \text{as } s_2 \to s_1.
\end{aligned} \tag{3.104}$$

This completes the proof. ◻

Now, we state our main result.

Theorem 3.9. *Under the assumptions of Lemma 3.3, the cocycle Φ associated with* (1.1)–(1.2) *has an unique D_α-pullback attractor $\mathcal{A} \in D_\alpha$ in $C([-h, 0]; H^1(\mathbb{R}^N))$.*

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