Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 132, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

PERTURBATION OF THE FREE BOUNDARY IN ELLIPTIC PROBLEM WITH DISCONTINUITIES

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Abstract. We study the discontinuous elliptic problem

$$\begin{aligned} -\Delta u &= \lambda H(u-\mu) \quad \text{in } \Omega, \\ u &= h \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a regular bounded domain of $\mathbb{R}^n$, H is the Heaviside function, λ, μ are a positive real parameters and h is a given function. We prove the existence of solutions, and characterize the free boundaries $\{x \in \Omega : u(x) = \mu\}$ using the perturbation of the boundary condition and smooth boundary of the domain.

1. Introduction

Partial differential equations with discontinuous nonlinearities arise in models from many concrete problems in mathematical physics like those of combustion theory, porous media, plasma physics. In this article, we study the existence of solutions for the problem

$$\begin{aligned} -\Delta u &= \lambda H(u-\mu) \quad \text{in } \Omega, \\ u &= h \quad \text{on } \partial\Omega, \end{aligned} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^n$, H is the Heaviside function

$$H(t) = \begin{cases} 1 & \text{if } t \geq 0 \\ 0 & \text{if } t < 0, \end{cases}$$

h is a given function, and λ, μ are a positive real parameters.

This problem can be reformulated as an equivalent free boundary problem: Find $u \in C^2(\Omega \setminus \partial w) \cap C^1(\overline{\Omega})$ such that

$$\begin{aligned} -\Delta u &= \lambda \quad \text{in } w, \\ -\Delta u &= 0 \quad \text{in } \Omega \setminus w, \\ u &= h \quad \text{on } \partial\Omega, \end{aligned} \tag{1.2}$$

where $w = \{x \in \Omega : u(x) > \mu\}$ and ∂w is the free boundary to be determined. The set $\{x \in \Omega : u(x) = \mu\}$, dividing the domain Ω into two (or more) regions where $-\Delta u = \lambda$ or $\Delta u = 0$ is satisfied in the classical sense.

2010 *Mathematics Subject Classification.* 34R35, 35J25.
Key words and phrases. Perturbation; implicit function theorem; free boundary problem.

Submitted April 14, 2016. Published June 7, 2016.

To the best of our knowledge, no investigation has been devoted to establishing the existence of solutions to such problem when Ω is a general domain using perturbation methods. Here, we give a positive answer to this study. The problem (1.1) was investigated in the variational context when $h = 0$ by Ambrosetti and Badial [3] and the extension to the results for p-Laplacian operator ($p > 1$) has been studied by Arcoya and Calahorrano in [5]. Further works with the Neumann boundary conditions can be found in [11, 14].

When h does not vanish identically, and the domain Ω is the unit ball of $\mathbb{R}^n$, the existence of positive solutions and the behavior of the corresponding free boundaries with respect to h has been discussed by Alexander in [1]. In the case $n = 2$, Alexander and Fleishman [2] studied the problem (1.1) when Ω is the unit square. Recently, the authors in [6] and [7] studied the more general following problem when Ω is the unit ball of $\mathbb{R}^n$,

$$\begin{gathered} -\Delta u = f(u)H(u-\mu) \quad \text{in } \Omega, \\ u = h \quad \text{on } \partial\Omega, \end{gathered}$$

where f is a given function.

The methods used hinge on the parametrization of the free boundary which is the unknown of our problem. This technique reduces the study to solve a nonlinear integral equation and allows us to obtain positive result regarding the solvability of the equation of the free boundary under perturbation of the boundary without the additional requirement of the regularity of the boundary of the initial domain. This will be the approach of this paper.

In general, the natural way to deal with the discontinuous elliptic problems are the variational methods, see for instance [9, 10]. But one soon realizes that the study entails serious difficulties mainly to characterize the variation of the free boundary . Our goal is different, having a solution of the given problem, we study the effect on the solutions under perturbations of the boundary conditions and a smooth boundary of the domain Ω. We point out also that in this work, we can not use the symmetry of solutions as in [1, 6, 7]. Hence, we have uses the perturbation techniques to overcome the encountered difficulties. For more works on the discontinuous elliptic problems, we invite the reader to consult [1, 3, 5, 6, 7, 15] and the references given here.

Because the nonlinearity has a discontinuity at $u = \mu$, so, a suitable concept of solution is needed. We say that a function $u \in W^{2,p}(\Omega), (p > 1)$ is a solution of problem (1.1) if $-\Delta u = \lambda H(u - \mu)$ a.e in Ω and the trace of u on $\partial\Omega$ is equal to h.

We need the following assumptions in this work.

- (H1) Assuming that the boundary $\partial\Omega$ of the domain Ω can be parameterized as $R + \beta(\theta)$ where $\beta \in C^2(S), \theta \in S$ and $R > 0$ where S is the unit sphere.
- (H2) For $n = 2$,
$$\frac{\lambda}{\mu} > \frac{4e}{R^2}.$$
and for $n \geq 3$,
$$\frac{\lambda}{\mu} > M_n(R) := \frac{1}{R^2} \frac{n(n-2)}{\left(\frac{2}{n}\right)^{\frac{2}{n-2}} - \left(\frac{2}{n}\right)^{\frac{n}{n-2}}}$$
- (H3) The function h is small enough with $0 \leq \|h\|_\infty < \mu$ where $\|h\|_\infty = \max_{x\in\partial\Omega} |h(x)|$.

This article is organized as follows: Section 2 collects some known results for the problem in a ball giving only slight information on the different methods of proof. Section 3 contains the statement of the essential result. Section 4 provides an approach for studying the problem (1.1). In Section 5, we treat the regularity of free boundary and finally, we give an appendix which contains some useful results.

2. Existence results in a ball

In this section, we consider the problem

$$\begin{gathered} -\Delta u = \lambda H(u-\mu) \quad \text{in } B(0,R), \\ u = h_0 \quad \text{on } \partial B(0,R), \end{gathered} \tag{2.1}$$

We assume that the function h_0 satisfies (H3).

Theorem 2.1 ([7]). *Suppose that there exists $\mu > 0$ such that*

$$\begin{gathered} \frac{\lambda}{\mu} > \frac{4e}{R^2}, \quad \textit{for } n = 2, \\ \frac{\lambda}{\mu} > M_n(R), \quad \textit{for } n \geq 3, \end{gathered}$$

then the problem (2.1) *has at least two positive solutions and the free boundaries are analytic hypersurfaces.*

Remark 2.2. In [7], we have treat the case $n \geq 3$ and the more general problem

$$\begin{gathered} -\Delta u = f(u)H(u-\mu) \quad \text{in } B(0,1), \\ u = h_0 \quad \text{on } \partial B(0,1), \end{gathered}$$

There the assumptions are

- (H4) The function f is k-Lipstchitzian, non-decreasing, positive and there exist two strictly positive constants $k, \beta > 0$ such that $f(s) \leq ks + \beta$ with $k < \min\{\lambda_1, 1\}$, where λ_1 is the first eigenvalue of $-\Delta$ under homogeneous Dirichlet boundary conditions.
- (H5) The function f is differentiable and constant on the interval of the form $[0, c]$ where $c > \frac{\beta}{2n-k}$ and the function h_0 is small enough, $\|h\|_\infty < \mu$ where $\|h\|_\infty = \max_{x\in\partial B(0,1)} |h(x)|$.
- (H6) There exists $\mu > 0$ such that
$$\frac{f(\mu)}{\mu} > M_n = \frac{n(n-2)}{(\frac{2}{n})^{\frac{2}{n-2}} - (\frac{2}{n})^{\frac{n}{n-2}}}, \quad \text{for} \quad n \geq 3.$$

In this paper, we put $f(u) = \lambda$ to clarify the obtained the results. For $n = 2$, the treatment is similar with the adequate modification. For the convenience of the reader, we give some calculus related to the case $\Omega = B(0,R)$ in the appendix.

The proof of Theorem 2.1 is based on the transformation of our problem into an equivalent nonlinear integral equation which is solved with respect to the unknown free boundary $\{x \in \Omega : u(x) = \mu\}$ by the application of the implicit function theorem. We can find the complete proof in [7]. Here, we give some ideas of demonstration that will be used throughout this paper. We will denoted the solution of (2.1) by u_0 and we consider the problem called the reduced problem

$$\begin{gathered} -\Delta u_0 = \lambda H(u_0-\mu) \quad \text{in } B(0,R), \\ u_0 = 0 \quad \text{on } \partial B(0,R), \end{gathered} \tag{2.2}$$

We remark that since the Heaviside step function is monotone and not decreasing, the result of Gidas, Ni and Nirenberg [12] shows that all positive solutions of (2.2) are radial. Hence, we look for the free boundary in the form $\{(r_0,\theta),\theta\in S\}$ for some $r_0\in(0,R)$ and obtain all radial solutions of (2.2) by finding u_0,λ and r_0 so that the differential equation

$$\begin{gathered} r^{1-n}\frac{\partial}{\partial r}(r^{n-1}\frac{\partial u_0}{\partial r})=\lambda \quad 0<r<r_0 \\ r^{1-n}\frac{\partial}{\partial r}(r^{n-1}\frac{\partial u_0}{\partial r})=0 \quad r_0<r<R, \\ u_0(R)=0, \quad \frac{\partial u_0}{\partial r}(0)=0 \end{gathered} \tag{2.3}$$

is satisfied with the following transmission conditions on the free boundary

$$u_0(r_0)=\mu \quad \text{and} \quad \frac{\partial u_0}{\partial r}(r_0-0)=\frac{\partial u_0}{\partial r}(r_0+0),$$

where $\frac{\partial u_0}{\partial r}(r_0-0)$ denotes the left derivative of u and $\frac{\partial u_0}{\partial r}(r_0+0)$ denotes the right derivative at the value $r=r_0$.

Using assumption (H2), the resolution of (2.3) with the previous conditions gives two $r_0\in(0,R)$ says r_1,r_2. Hence, a simple calculus shows that the free boundaries are spheres with radii $r_1,r_2\in(0,R)$.

Remark 2.3. Let r_0 denote one of the values r_1 or r_2. Using hypothesis (H2), $r_0\neq Re^{-1/2}$ for $n=2$ and

$$r_0\neq\Big(\frac{2}{nR^{2-n}}\Big)^{\frac{1}{n-2}} \quad \text{for } n\geq 3.$$

See the Appendix.

Now, to study the problem

$$\begin{gathered} -\Delta u_0=\lambda H(u_0-\mu) \quad \text{in } B(0,R), \\ u_0=h_0 \quad \text{on } \partial B(0,R), \end{gathered} \tag{2.4}$$

we use the effect of perturbation on the solution in the boundary values. More precisely, when $h_0\neq 0$, we look for the free boundary in the form $r_0+b(\theta),\theta\in S$, where $b(\theta)$ is the perturbation caused by h_0. Let

$$w=\{(r,\theta)\in(0,R)\times S, 0\leq r<r_0+b(\theta),\theta\in S\}$$

Now, we denote by χ_w the characteristic function of w. In the following result, we formulate a nonlinear equation for the unknown function b and we prove that by solving it, we can solve the problem (2.4).

Proposition 2.4 ([7]). *Under assumption* (H2), *the problem*

$$\begin{gathered} -\Delta u=\lambda\chi_w(r,\theta) \quad \text{in } B(0,R), \\ u=h_0 \quad \text{on } \partial B(0,R) \end{gathered} \tag{2.5}$$

has a unique solution $u_0\in C^{1,\alpha}(\overline{\Omega},\mathbb{R})$ with $\alpha=1-\frac{n}{p}$. Moreover if $u_0(r_0+b(\theta),\theta)=\mu$ with $0\leq\|h\|_\infty<\mu$, then u_0 is a solution of (2.4).

As in [7], we conclude with problem (2.4) by showing the existence of the function b via the implicit function theorem. We can represent the solution of (2.5) in integral form, and we check the hypothesis of previous theorem. For more details, we refer the reader to [7]. In this article, we denote $r_0+b(\theta):=b_0(\theta)$, $\theta\in S$.

3. Perturbation of the free boundary

In this section, we are concerned with problem (1.1). We introduce some sets necessary for the study using the perturbation of boundary. We assume that Ω is diffeomorphic to a ball. Then, we can construct a curvilinear parametrisation (r, θ) in a neighborhood of any set in Ω, $r \in (0, R]$ and $\theta \in S$.

Let Ω_β be a perturbation of $B(0, R)$ in the sense that the boundary $\partial\Omega_\beta$ of every smooth open set Ω_β close to the ball $B(0, R)$ can be described by $R + \beta(\theta)$, $\theta \in S$ and $\beta \in C^2(S)$. Hence, from now, $\Omega = \Omega_\beta$, and $\Omega_0 = B(0, R)$. We denote also by Ω_β the admissible perturbation of the ball $B(0, R)$ and we define the set of admissible surfaces in Ω_β by

$$S_\beta = \{f \in C(S) : (f(\theta), \theta) \in \Omega_\beta \text{ for } \theta \in S\}.$$

Now, for a function $\psi \in S_\beta$, we define the set

$$\Omega_{\beta,\psi} = \{(r, \theta) \in \Omega_\beta, \quad r < \psi(\theta)\}.$$

We seek a solution in $W^{2,p}(\Omega_\beta), p > 1$, then the boundary value function h which is a trace of $W^{2,p}(\Omega_\beta)$ function will be taken in the set

$$H = \{h \in W^{2-\frac{1}{p},p}(\partial\Omega_\beta, \mathbb{R}), p > n\}.$$

We denote by $\chi_{\Omega_{\beta,\psi}}$ the characteristic function of $\Omega_{\beta,\psi}$, then, we have the following result.

Proposition 3.1. *Assume* (H1), (H2) *and that*

$$\frac{\lambda}{\mu} > \frac{4e}{R^2} \quad \textit{for } n = 2,$$
$$\frac{\lambda}{\mu} > M_n(R), \quad \textit{for } n \geq 3.$$

Then the problem

$$\begin{gathered} -\Delta u = \lambda \chi_{\Omega_{\beta,\psi}} \quad \textit{in } \Omega_\beta, \\ u = h \quad \textit{on } \partial\Omega_\beta \end{gathered} \tag{3.1}$$

has a unique solution $u \in C^{1,\alpha}(\overline{\Omega_\beta}, \mathbb{R})$ with $\alpha = 1 - \frac{n}{p}$. Moreover, if $u(\psi(\theta), \theta) = \mu$ with $0 \leq \|h\|_\infty < \mu$ then u is a solution of (1.1).

Proof. First, we see that $\lambda\chi_{\Omega_{\beta,\psi}} \in L^p(\Omega_\beta)$, $p > 1$. From [13, Theorem 9.15], there exists a unique solution of (3.1) in $W^{2,p}(\Omega_\beta)$. For $p > n$, $W^{2,p}(\Omega_\beta) \subset C^{1,\alpha}(\Omega_\beta, \mathbb{R})$ with $\alpha = 1 - \frac{n}{p}$. Now, we remark that u satisfies

$$\begin{gathered} -\Delta u = \lambda \quad \text{in } \Omega_{\beta,\psi}, \\ -\Delta u = 0 \quad \text{in } \Omega_\beta \setminus \Omega_{\beta,\psi}, \\ u = h \quad \text{on } \partial\Omega_\beta. \end{gathered}$$

If we prove the existence of a function ψ such that $u(\psi(\theta), \theta) = \mu$, then u will be a solution of

$$\begin{gathered} -\Delta u = \lambda \quad \text{in } \Omega_{\beta,\psi}, \\ u = \mu \quad \text{on } \partial\Omega_{\beta,\psi} \\ -\Delta u = 0 \quad \text{in } \Omega_\beta \setminus \Omega_{\beta,\psi}, \\ u = h \quad \text{on } \partial\Omega_\beta. \end{gathered}$$

In $\Omega_{\beta,b}$, the function u satisfies

$$\begin{gathered}-\Delta u=\lambda \quad \text{in } \Omega_{\beta,\psi},\\ u=\mu \quad \text{on } \partial\Omega_{\beta,\psi}\end{gathered}$$

The maximum principle implies

$$\min_{\Omega_{\beta,\psi}} u=\min_{\partial\Omega_{\beta,\psi}} u=\mu.$$

Hence, $u>\mu$ in $\Omega_{\beta,\psi}$. In $\Omega_\beta \setminus \overline{\Omega_{\beta,b}}$, we have

$$\begin{gathered}-\Delta u=0 \quad \text{in } \Omega_\beta \setminus \overline{\Omega_{\beta,\psi}},\\ u=\mu \quad \text{on } \partial\Omega_{\beta,\psi},\\ u=h \quad \text{on } \partial\Omega_\beta.\end{gathered}$$

As $0\le \|h\|_\infty<\mu$, then

$$\max_{\Omega_\beta\setminus\overline{\Omega_{\beta,\psi}}} u=\max_{\partial\Omega_{\beta,\psi}} u=\mu$$

and consequently, $u<\mu$ in $\Omega_\beta\setminus\Omega_{\beta,\psi}$. Therefore, the function u satisfies

$$\begin{gathered}-\Delta u=\lambda H(u-\mu) \quad \text{in } \Omega_\beta,\\ u=h \quad \text{on } \partial\Omega_\beta.\end{gathered}$$

To conclude with the existence of solutions of problem (1.1), we need only to show the existence of the function ψ satisfying the equation

$$u(\psi(\theta),\theta)=\mu.$$

Now, since the solution u depend on the domain Ω_β, then we can not use the local methods directly to prove the existence of ψ. The variation of the domain Ω_β suggests to use an adequate transformation which maps the changing domain into a fixed domain and solves the governing equations in the mapped domain. To exclude this difficulties, we proceed as the following. To each admissible perturbation (Ω_0,Ω_β) of the domain Ω_0 correspond a transformation T_β of the domain Ω_β onto the initial domain Ω_0. $T_\beta:\Omega_\beta\to\Omega_0$,

$$(r,\theta)\to(\overline{r},\theta)=(r+r\frac{\beta}{R},\theta)$$

where (r,θ) is the coordinates in Ω_β and $(\overline{r},\theta)$ the coordinates in Ω_0. For a small β, the transformation T_β is a diffeomorphism of class C^2 of the domain Ω_β into Ω_0. The mapping T_β maps the class of admissible surfaces S_β into S_0. Hence,

$$T_\beta(\theta,\psi(\theta))=(\theta,f(\theta)) \quad \text{where} \quad f(\theta)\in S_0.$$

Using the relation

$$\overline{u}(T_\beta(r,\theta))=u(r,\theta),$$

problem (3.1) with $u(\psi(\theta),\theta)=\mu$ is equivalent to the problem

$$\begin{gathered}-L_\beta\overline{u}=\lambda\chi_{\Omega_{\beta,f}} \quad \text{in } \Omega_0,\\ \overline{u}=h \quad \text{on } \partial\Omega_0\end{gathered} \tag{3.2}$$

with the equation

$$\overline{u}(f(\theta),\theta)=\mu,$$

where L_β is a linear operator with continuous coefficients depending on β. The following lemma gives the exact expression of L_β. $\square$

Lemma 3.2. *The linear operator L_β is given by $L_\beta = \Delta + \delta_\beta$ where δ_β has the form*

$$\begin{aligned}\delta_\beta &= \frac{\beta}{R}(2+\frac{\beta}{R})\frac{\partial^2}{\partial \overline{r}^2} + \frac{(n-1)}{\overline{r}}\frac{\beta}{R}\frac{\partial}{\partial \overline{r}} \\ &\quad + \frac{1}{\overline{r}^2}\Big[a_{ij}(\theta)\Big[\frac{r}{R}\frac{\partial\beta}{\partial\theta_j}\Big(\frac{\partial^2}{\partial\overline{r}\partial\theta_i} + \frac{1}{R(1+\frac{\beta}{R})}\frac{\partial\beta}{\partial\theta_i}\frac{\partial}{\partial\overline{r}} + \frac{r}{R}\frac{\partial\beta}{\partial\theta_i}\frac{\partial^2}{\partial\overline{r}^2}\Big) \\ &\quad + \frac{r}{R}\frac{\partial^2\beta}{\partial\theta_j\partial\theta_i}\frac{\partial}{\partial\overline{r}} + \frac{r}{R}\frac{\partial\beta}{\partial\theta_i}\frac{\partial^2}{\partial\theta_j\partial\overline{r}}\Big] + b_i(\theta)\Big[\frac{r}{R}\frac{\partial\beta}{\partial\theta_i}\frac{\partial}{\partial\overline{r}}\Big]\Big]\end{aligned}$$

and

$$\Delta = \frac{\partial^2}{\partial\overline{r}^2} + \frac{(n-1)}{\overline{r}}\frac{\partial}{\partial\overline{r}} + \frac{1}{\overline{r}^2}\Delta_\theta$$

when Δ_θ is the Beltrami-Laplace operator.

Remark 3.3. This formulas are obtained from the Laplacian in polar coordiantes. More precisely, from

$$\Delta = \frac{\partial^2}{\partial r^2} + \frac{(n-1)}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\Delta_\theta \tag{3.3}$$

where $\Delta_\theta = a_{ij}(\theta)\frac{\partial^2}{\partial\theta_i\partial\theta_j} + b_i(\theta)\frac{\partial}{\partial\theta_i}$.

Proof of Lemma 3.2. First, we have

$$\begin{gathered}\overline{u}(\overline{r},\theta) = u(r,\theta) \quad \text{where } \overline{r} = r + r\frac{\beta}{R}, \\ \frac{\partial u}{\partial r} = \frac{\partial\overline{u}}{\partial\overline{r}}\frac{\partial\overline{r}}{\partial r} + \frac{\partial\overline{u}}{\partial\theta}\frac{\partial\theta}{\partial r} = \frac{\partial\overline{u}}{\partial\overline{r}} + \frac{\partial\overline{u}}{\partial\overline{r}}\frac{\beta}{R} := Q, \\ \begin{aligned}\frac{\partial^2 u}{\partial r} &= \frac{\partial Q}{\partial\overline{r}}\frac{\partial\overline{r}}{\partial r} + \frac{\partial Q}{\partial\theta}\frac{\partial\theta}{\partial r} \\ &= \frac{\partial^2\overline{u}}{\partial\overline{r}^2}(1+\frac{\beta}{R}) + \frac{\partial^2\overline{u}}{\partial\overline{r}^2}\frac{\beta}{R}(1+\frac{\beta}{R}) \\ &= \frac{\partial^2\overline{u}}{\partial\overline{r}^2} + \frac{\beta}{R}(2+\frac{\beta}{R})\frac{\partial^2\overline{u}}{\partial\overline{r}^2}.\end{aligned} \\ \frac{\partial u}{\partial\theta_i} = \frac{\partial\overline{u}}{\partial\overline{r}}\frac{\partial\overline{r}}{\partial\theta_i} + \frac{\partial\overline{u}}{\partial\theta_i}\frac{\partial\theta_i}{\partial\theta_i} = \frac{r}{R}\frac{\partial\beta}{\partial\theta_i}\frac{\partial\overline{u}}{\partial\overline{r}} + \frac{\partial\overline{u}}{\partial\theta_i} := Z\end{gathered}$$

$$\begin{aligned}\frac{\partial^2 u}{\partial\theta_j\partial\theta_i} &= \frac{\partial Z}{\partial\theta_j} = \frac{\partial Z}{\partial\overline{r}}\frac{\partial\overline{r}}{\partial\theta_j} + \frac{\partial Z}{\partial\theta_j}\frac{\partial\theta_i}{\partial\theta_j} \\ &= \frac{r}{R}\frac{\partial\beta}{\partial\theta_j}\frac{\partial}{\partial\overline{r}}\Big(\frac{\partial\overline{u}}{\partial\theta_i} + \frac{r}{R}\frac{\partial\beta}{\partial\theta_i}\frac{\partial\overline{u}}{\partial\overline{r}}\Big) + \frac{\partial Z}{\partial\theta_i} \\ &= \frac{r}{R}\frac{\partial\beta}{\partial\theta_j}\Big(\frac{\partial^2\overline{u}}{\partial\overline{r}\partial\theta_i} + \frac{1}{R(1+\frac{\beta}{R})}\frac{\partial\beta}{\partial\theta_i}\frac{\partial\overline{u}}{\partial\overline{r}} + \frac{r}{R}\frac{\partial\beta}{\partial\theta_i}\frac{\partial^2\overline{u}}{\partial\overline{r}^2}\Big) \\ &\quad + \frac{\partial^2\overline{u}}{\partial\theta_i\partial\theta_j} + \frac{r}{R}\frac{\partial^2\beta}{\partial\theta_j\partial\theta_i}\frac{\partial\overline{u}}{\partial\overline{r}} + \frac{r}{R}\frac{\partial\beta}{\partial\theta_i}\frac{\partial^2\overline{u}}{\partial\theta_j\partial\overline{r}}.\end{aligned}$$

If we replace the previous terms in the expression (3.3), we find the formulas of L_β. □

Thus, to solve problem (1.1), it is sufficient to prove the existence of function f such that

$$\overline{u}(f(\theta),\theta) = \mu.$$

In fact, using the implicit function theorem, we prove that the equation

$$\overline{u}(f(\theta),\theta)-\mu=0$$

is uniquely solvable in a given small neighborhood. This is the subject of the following section. (See Theorem 4.2 below).

4. Solvability in the neighborhood of smooth free boundary

To each function $f\in S_0$, we associate the solution $\overline{u}$ of problem (3.2). We have

$$\begin{gathered} -\Delta\overline{u}-\delta_\beta\overline{u}=\lambda\chi_{\Omega_{\beta,f}} \quad \text{in } \Omega_0,\\ \overline{u}=h \quad \text{on } \partial\Omega_0 \end{gathered} \tag{4.1}$$

The solution $\overline{u}$ corresponding to (4.1) has an integral representation which is well defined [6, Theorem 4.1], given by

$$\overline{u}(x)=\int_S P(x,y)h(y)dS-\lambda\int_{\Omega_0}G(x,y)\chi_{\Omega_{\beta,f}}dy-\int_{\Omega_0}G(x,y)\delta_\beta(\overline{u})(y)dy.$$

Now, we consider polar coordinates and we define the operator $J:H\times S_0\times D\to C(S,\mathbb{R})$ by

$$J(h,f,\beta)(\theta)=\overline{u}(f(\theta),\theta)-\mu,$$

where D is the neighborhood of zero in $C(S)$; i.e.,

$$\begin{aligned} J(h,f,\beta)(\theta)=&\int_{\partial\Omega_0}P(f(\theta),\theta,\theta')h(\theta')d\theta'-\lambda\int_{\Omega_0}G(f(\theta),\theta,r',\theta')\chi_{\Omega_{\beta,f}}(r',\theta')dr'd\theta'\\ &-\int_{\Omega_0}\delta_\beta(\overline{u})(r',\theta')G(f(\theta),\theta,r',\theta')dr'd\theta'-\mu. \end{aligned}$$

Then we have the following result.

Lemma 4.1. *The operator J is continuously differentiable with respect to the second variable.*

Proof. Let D_jJ denote the Frechet derivative of J, with respect to the variable of order j ($j=1,2,3$). Let $\varphi(\theta)$ be a small perturbation of $f(\theta)$, then

$$\begin{aligned} &J(h,f+\varphi,\beta)(\theta)-J(h,f,\beta)(\theta)-D_2J\varphi(\theta)\\ &=\int_{\partial\Omega_0}P(f(\theta)+\varphi(\theta),\theta,\theta')h(\theta')d\theta'\\ &\quad-\lambda\int_{\Omega_0}G(f(\theta)+\varphi(\theta),\theta,r',\theta')\chi_{\Omega_{f+\varphi}}(r,\theta')dr'd\theta'\\ &\quad-\int_{\Omega_0}l_\beta(\overline{u})(r',\theta')G(f(\theta)+\varphi(\theta),\theta,r',\theta')dr'd\theta'-\int_{\partial\Omega_0}P(f(\theta),\theta,\theta')h(\theta')d\theta'\\ &\quad+\lambda\int_{\Omega_0}G(f(\theta),\theta,r',\theta')\chi_{\Omega_f}(r',\theta')dr'd\theta'+\int_{\Omega_0}l_\beta(\overline{u})(r',\theta')G(f(\theta),\theta,r',\theta')dr'd\theta'\\ &\quad+\lambda\int_{\Omega_0}G(f(\theta)+\varphi(\theta),\theta,r',\theta')\chi_{\Omega_f}dr'd\theta'\\ &\quad-\lambda\int_{\Omega_0}G(f(\theta)+\varphi(\theta),\theta,r',\theta')\chi_{\Omega_f}dr'd\theta'\\ &\quad-\int_{\partial\Omega_0}\frac{\partial P}{\partial r}(f(\theta'),\theta,\theta')h(\theta')\varphi(\theta)d\theta' \end{aligned}$$

$$
\begin{aligned}
&+\lambda\int_S\int_0^f (r')^{n-1}\frac{\partial G}{\partial r}(f(\theta),\theta,r',\theta')\varphi(\theta)dr'd\theta'\\
&+\lambda\int_S f^{n-1}(\theta)G(f(\theta),\theta,f(\theta'),\theta')\varphi(\theta')d\theta'\\
&+\int_{\Omega_0} l_\beta(\overline{u})(r',\theta')\frac{\partial G}{\partial r}((f\theta),\theta,r',\theta')\varphi(\theta)d\theta'
\end{aligned}
$$

Hence,

$$
J(h,f+\varphi,\beta)(\theta)-J(h,f,\beta)(\theta)-D_2J\varphi(\theta)=I_1+I_2+I_3+I_4
$$

where

$$
\begin{aligned}
I_1=&\int_{\partial\Omega_0}P(f(\theta)+\varphi(\theta),\theta,\theta')h(\theta')d\theta'-\int_{\partial\Omega_0}P(f(\theta),\theta,\theta')h(\theta')d\theta'\\
&-\int_{\partial\Omega_0}\frac{\partial P}{\partial r}(f(\theta'),\theta,\theta')h(\theta')\varphi(\theta)d\theta'\\
I_2=&-\lambda\int_{\Omega_0}G(f(\theta)+\varphi(\theta),\theta,r',\theta')\chi_{\Omega_f}dr'd\theta'\\
&+\lambda\int_{\Omega_0}G(f(\theta),\theta,r',\theta')\chi_{\Omega_f}(r',\theta')dr'd\theta'\\
&+\lambda\int_S\int_0^f (r')^{n-1}\frac{\partial G}{\partial r}(f(\theta),\theta,r',\theta')\varphi(\theta)dr'd\theta'\\
I_3=&-\int_{\Omega_0} l_\beta(\overline{u})(r',\theta')G(f(\theta)+\varphi(\theta),\theta,r',\theta')dr'd\theta'\\
&+\int_{\Omega_0} l_\beta(\overline{u})(r',\theta')G(f(\theta),\theta,r',\theta')dr'd\theta'\\
&+\int_{\Omega_0} l_\beta(\overline{u})(r',\theta')\frac{\partial G}{\partial r}((f\theta),\theta,r',\theta')\varphi(\theta)d\theta'\\
I_4=&\lambda\int_{\Omega_0}G(f(\theta)+\varphi(\theta),\theta,r',\theta')[\chi_{\Omega_f}-\chi_{\Omega_{f+\varphi}}]dr'd\theta'\\
&+\lambda\int_S f^{n-1}(\theta)G(f(\theta),\theta,f(\theta'),\theta')\varphi(\theta')d\theta'\\
=&-\lambda\int_S d\theta'\Big[\int_f^{f+\varphi}(r')^{n-1}G(f(\theta)+\varphi(\theta),\theta,r',\theta')dr'\\
&-f^{n-1}(\theta)G(f(\theta),\theta,f(\theta'),\theta')\varphi(\theta')\Big]
\end{aligned}
$$

Using Taylor's theorem, we have $I_1,I_2,I_3,I_4=o(\|\varphi\|_\infty)$ when $\|\varphi\|_\infty\to+\infty$. Hence,

$$
D_2J(h,f,\beta)\varphi(\theta)=\frac{\partial\overline{u}}{\partial r}(f(\theta),\theta)\varphi(\theta)-\lambda\int_S f^{n-1}(\theta')G(f(\theta),\theta,f(\theta'),\theta')\varphi(\theta')d\theta'.
$$

Now, to solve the equation $J(h,f,\beta)(\theta)=0$, for $\theta\in S$, in the neighborhood of $(h_0,b_0,0)$. we need to know the invertibility of J. We recall the reader that we consider here a general domain and that causes some difficulties to build clear conditions so that the operator $D_2J(h_0,b_0,0)$ either invertible of S_0 into $C(S)$ or not. We refer to [6] to see explicit conditions such that the proper operator is

invertible when the domain is a ball.
Let the operator

$$D_2J(h_0,b_0,0)\varphi(\theta)=\frac{\partial u}{\partial r}(b_0(\theta),\theta)\varphi(\theta)-\lambda\int_S b_0^{n-1}(\theta')G(b_0(\theta),\theta,b_0(\theta'),\theta')\varphi(\theta')d\theta'.$$

First, note that u satisfies the equation

$$\begin{gathered}-\Delta u=\lambda H(u-\mu)\quad\text{in } B(0,R),\\ u=h_0\quad\text{on } \partial B(0,R).\end{gathered}$$

Then, when h_0 is small, the solution u is close in $C^{1,\alpha}$ to the solution of

$$\begin{gathered}-\Delta u=\lambda H(u-\mu)\quad\text{in } B(0,R),\\ u=0\quad\text{on } \partial B(0,R).\end{gathered}$$

Since, $\frac{\partial u}{\partial r}(r_0,\theta)<0$ for $r_0\in(0,R),\theta\in S$, then $\frac{\partial u}{\partial r}(b_0(\theta),\theta)<0$ for b_0 caused by the small perturbation h. For the other part, let

$$K\varphi(\theta)=\lambda\int_S b_0^{n-1}(\theta')G(b_0(\theta),\theta,b_0(\theta'),\theta')\varphi(\theta')d\theta'.$$

Let $L^2_{b_0}(S)$ the space of functions belonging to the space $L^2(S)$ with the inner product

$$\langle u,v\rangle=\int_S b_0^{n-1}u(\theta)v(\theta)d\theta. \tag{4.2}$$

We remark that the operator K is negative definite in the space $L^2_{b_0}(S)$, (The Green function G is negative). Since, the function b_0 is bounded in S, the inner product (4.2) is equivalent to the standard product in the space $L^2(S)$.

In [6], we have proved that the operator $D_2J(0,r_0,0)$ is invertible (note that $b_0=r_0$ when $h_0=0$ on $\partial\Omega$). In this case, we use the explicit eigenvalue of the operator K to conclude the nondegeneracy of the operator $D_2J(0,r_0,0)$. For more details, see [6].

Finally, using the fact that $D_2J(0,r_0,0)$ is invertible in $L^2_{r_0}(S)$, we can be sure that $D_2J(h_0,b_0,0)$ is invertible in $L^2_{b_0}(S)$ with the previous inner product. The preceding argument shows that $D_2J(h_0,b_0,0)$ is invertible in $L^2(S)$. Hence, using the implicit function theorem, we have the following result. ☐

Theorem 4.2. *Under assumptions* (H1)–(H3)*, there exist a neighborhood V of $(h_0,0)$ in $H\times C^2(S)$ and a continuous mapping $B:V\to C(S)$ such that*

(i) $B(h_0,0)=b_0$,
(ii) $J(h,B(h,\beta),\beta)=0$.

We recall that h_0 is a given function satisfying (H3) and $b_0(\theta):=r_0+b(\theta)$, where $r_0\in(0,1)$, $b\in C^{1,\alpha}(S),\alpha=1-\frac{n}{p}$ and $\theta\in S$.

Proof of Theorem 4.2. Because the operator J is invertible in the neighborhood of $(h_0,b_0,0)$ and $J(h_0,b_0,0)(\theta)=0$ for $\theta\in S$, then the implicit function theorem implies the existence of function B depending on h and β such that $B(h,\beta)$ satisfies $J(h,B(h,\beta),\beta)=0$. ☐

5. Regularity of the free boundary

Theorem 5.1. *If $\|h\|_\infty$ and $\|\beta\|_\infty$ are small enough, then the free boundary $\{x \in \Omega/u(x) = \mu\}$ is an analytic hypersurface.*

Proof. First, let $\overline{u}$ be a solution of (4.1) and let

$$\begin{aligned}\Gamma :&= \{x \in \Omega/\overline{u}(x) = \mu\}\\ &= \{(r,\theta) \in (0,R)\times S, \overline{u}(r,\theta) = \mu\}\\ &= \{(f(\theta),\theta),\ \text{for } \theta \in S\}.\end{aligned}$$

when β is sufficiently close to 0 in $C^2(S)$ and f is close to b_0 in $C(S)$, then the solution $\overline{u}$ is close to u in $C^{1,\alpha}(\Omega), \alpha \in (0,1)$. Since

$$\frac{\partial u}{\partial r}(b_0(\theta),\theta) \neq 0,$$

then

$$\frac{\partial \overline{u}}{\partial r}(f(\theta),\theta) \neq 0 \quad \text{for} \quad \theta \in S.$$

The implicit function theorem gives that $f \in C^{1,\alpha}(S)$. Hence, the free boundary Γ is an hypersurface of class $C^{1,\alpha}$. Now, we conclude that the free boundary Γ is analytic by the application of Hodograph transformation. This method was using by the author in [7]. □

Final remarks. (1) So far, the problem (1.1) has been studied for a few class of domain, in particular the ball for uses the notion of symmetry. Hence, the main results of this paper remain true when Ω is a ring shaped domain.

(2) The advantage of the perturbation procedure described in this paper is the the explicit formula of behavior of the free boundary with respect to h. For example, in dimension 2, taking $\Omega := \{(r,\theta)/r < 2 + \sin(2\theta), \theta \in S\}$ and $h(x) = \cos(x)$, we can give an explicit formulation of the solution of problem (1.1) and the shape of free boundary.

(3) When $n = 1$, the problem (1.1) becomes the following second order differential equation. For example, we can see easily that under a suitable conditions, the problem

$$\begin{gathered}-u'' = \lambda H(u-\mu) \quad \text{for } |x| \leq a,\\ u(\pm a) = 0, \quad (a \in \mathbb{R}^+).\end{gathered} \tag{5.1}$$

admits multiple solutions. See [15]. In the other part, the problem

$$\begin{gathered}-u'' = \lambda H(\mu-u) \quad \text{for } |x| \leq a,\\ u(\pm a) = 0, \quad (a \in \mathbb{R}^+).\end{gathered} \tag{5.2}$$

has a unique positive solution when the set $\{x \in \Omega, u(x) = \mu\}$ is a given segment included in $|x| \leq a$. In fact, the form of the discontinuity in the second member has a surprising effects on the set of solutions and their free boundaries. In higher dimensions, the result stays true also,we can see [8] for more details.

(4) The regularity of the free boundary is preserved after perturbations. One of the important questions is when will the free boundary develop singularities.

(5) It is also interesting to studied the bifurcation phenomenon. In the case when Ω is the unit ball, the study is in [7].

6. Appendix

In this section, we give the proof that the condition in assumption (H6) in Remark 2.2 becomes

$$\frac{\lambda}{\mu} > \frac{4e}{R^2}, \quad \text{for} \quad n = 2,$$
$$\frac{\lambda}{\mu} > M_n(R) := \frac{1}{R^2}\frac{n(n-2)}{(\frac{2}{n})^{\frac{2}{n-2}} - (\frac{2}{n})^{\frac{n}{n-2}}}, \quad \text{for } n \geq 3,$$

as in Theorem 2.1 when $\Omega = B(0,R)$ and $f(u) = \lambda$. Now, define a function u that satisfies

$$\begin{gathered} -\Delta u = \lambda H(u-\mu) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered} \tag{6.1}$$

Since $H(u-\mu)$ is monotone and not decreasing, the result of [12] shows that all positive solutions are radial. Hence, by the maximum principle, a positive radial solution u take on the value μ at only one value say r_0. Then we obtain all radial solutions of (2.2) by finding λ, u and r_0 so that the two problems

$$\begin{gathered} -r^{1-n}\frac{\partial}{\partial r}(r^{n-1}\frac{\partial u}{\partial r}) = \lambda \quad \text{for } 0 < r < r_0, \\ u'(0) = 0, \quad u(r_0) = \mu \end{gathered} \tag{6.2}$$

and

$$\begin{gathered} -r^{1-n}\frac{\partial}{\partial r}(r^{n-1}\frac{\partial u}{\partial r}) = 0 \quad \text{for } r_0 < r < R, \\ u(R) = 0, \quad u(r_0) = \mu \end{gathered} \tag{6.3}$$

are satisfied. When $n = 2$ and from the problems (6.2), (6.3), we have

$$u(r) = \begin{cases} \frac{\lambda}{4}(r_0^2 - r^2) + \mu & 0 < r \leq r_0, \\ \mu\frac{\ln(R/r)}{\ln(R/r_0)} & r_0 \leq r < R. \end{cases}$$

Since $u \in C^{1,\alpha}(\overline{\Omega})$, we have

$$\frac{\partial u}{\partial r}(r_0^-) = \frac{\partial u}{\partial r}(r_0^+).$$

We obtain

$$\frac{\lambda}{2}r_0 = \frac{\mu}{r_0(\ln(R/r_0))}$$

which implies

$$\frac{2\mu}{\lambda} = r_0^2\ln(R/r_0) := g(r_0). \tag{6.4}$$

Now, by considering the function $g(\rho)$ in $(0,R)$, we have that f has a maximum value $\frac{R^2}{2e}$ at $\rho = Re^{-\frac{1}{2}}$. Hence, the equation (6.4) has two roots when $\frac{\lambda}{\mu} > \frac{4e}{R^2}$.

Similarly, we treat the case $n \geq 3$. The solution u is given by

$$u(r) = \begin{cases} \frac{\lambda}{2n}(r_0^2 - r^2) + \mu & 0 < r \leq r_0, \\ \frac{\mu r^{2-n}}{r_0^{2-n} - R^{2-n}} - \frac{\mu R^{2-n}}{r_0^{2-n} - R^{2-n}} & r_0 \leq r < R. \end{cases}$$

The transmission conditions imply that

$$\frac{\partial u}{\partial r}(r_0^+) = \frac{\partial u}{\partial r}(r_0^-),$$

so

$$\frac{(2-n)\mu r_0^{1-n}}{r_0^{2-n}-R^{2-n}} = -\frac{\lambda r_0}{n}.$$

Hence,

$$\frac{\lambda}{\mu} = \frac{n(n-2)}{r_0^2 - r_0^n R^{2-n}} := g(r_0$$

It follows that the function g has a minimum value

$$M_n(R) = \frac{1}{R^2}\frac{n(n-2)}{\left(\frac{2}{n}\right)^{\frac{2}{n-2}} - \left(\frac{2}{n}\right)^{\frac{n}{n-2}}} \tag{6.5}$$

reached at the point $(\frac{2}{nR^{2-n}})^{\frac{1}{n-2}}$ which implies that the equation (6.5) has two roots when $\frac{\lambda}{\mu} > M_n(R)$.

Acknowledgments. The author would like to thank the professor B. Abdellaoui for the discussion and the suggestions about the problem in this paper. I want to thank also the anonymous referee for valuable suggestions and comments.

References

[1] R. Alexander; A discontinuous nonlinear eigenvalue/Free boundary problem, *Math. Meth in the Appl. Sci.*, **4**, (1982), 131–142.

[2] R. Alexander, B. Fleishman; Perturbation and bifurcation in a free boundary problem. *J. Differential Equations*, **45**, no. 1, (1982), 34–52.

[3] A. Ambrosetti, M. Badiale; The dual variational principle and elliptic problems with discontinuous nonlinearities, *J. Math. Anal. Appl*, **140**, No.2, (1989), 363–373.

[4] A. Ambrosetti, P.H. Rabinowitz; Dual variational methods in critical point theory and applications, *J. Funct. Anal*, **14**, (1973), 349-381.

[5] D. Arcoya, M. Calahorrano; Some discontinuous problems with quasilinear operator, *J. Math. Anal. Appl.* **187**, (1994), 1059–1072.

[6] S. Bensid, S. M. Bouguima; On a free boundary problem, *Nonlinear Anal. T.M.A*, **68**, (2008), 2328-2348.

[7] S. Bensid, S. M. Bouguima; Existence and multiplicity of solutions to elliptic problems with discontinuities and free boundary conditions, *Electron. J. Differential Equations.*, **56** (2010), 1-16 .

[8] S. Bensid, S. M. Bouguima; A note on discontinuous problem with a free boundary, *J. Egyptian Math. Soc.*, **19** (2011), 86-87.

[9] G. Bonanno, P. Candito; Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities. *J. Differential Equations.*, **244**, (2008), no. 12, 3031-3059.

[10] G. Bonanno, P. Candito, D. Motreanu; A coincidence point theorem for sequentially continuous mappings. *J. Math. Anal. Appl.*, **435**, (2016), no. 1, 606-615.

[11] P. Candito; Infinitely many solutions to the Neumann problem for elliptic equations involving the p-Laplacian and with discontinuous nonlinearities, *Proc. Edinb. Math. Soc.*, II. Ser., 45 (2) (2002), 397–409.

[12] B. Gidas, W. N. Ni, L. Nirenberg; Symmetry and related properties via maximum principle. *Commun. Math. Phys* **68**, (1979), 209-243.

[13] D. Gilbarg, N. Trudinger; *Elliptic Partial Differential Equations of Second Order*. Springer-Verlag, New York, 1983.

[14] S. A. Marano, D. Motreanu; Infinitely many critical points of non-differentiable functions and applications to a Neumann-type problem involving the p-Laplacian, *J. Differential Equations* 182, **1** (2002), 108–120.

[15] C. A. Stuart; Differential equations with discontinuous nonlinearities, *Arch. Ration. Mech. Anal.*, 63, **1** (1976), 59-75.

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