Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 131, pp. 1–14.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ASYMPTOTIC DISTRIBUTION OF EIGENVALUES AND EIGENFUNCTIONS FOR A MULTI-POINT DISCONTINUOUS STURM-LIOUVILLE PROBLEM

KADRIYE AYDEMIR, OKTAY SH. MUKHTAROV

Abstract. In this article we study a class of generalized BVP' s consisting of discontinuous Sturm-Liouville equation on finite number disjoint intervals, with usual boundary conditions and supplementary transmission conditions at finite number interior points. The asymptotic behaviors of the eigenvalues and eigenfunctions are discussed. By modifying some techniques of classical Sturm-Liouville theory and suggesting own approaches we find asymptotic formulas for the eigenvalues and eigenfunctions.

1. Introduction

The study of eigenvalue problems for boundary-value problems (BVPs) is a topic of great interest. The problem of finding eigenvalues and eigenfunctions and studying their behavior plays a crucial role in modern mathematics. These investigations are of utmost importance for theoretical and applied problems in mechanics, physics, physical chemistry, biophysics, mathematical economics, theory of systems and their optimization, theory of random processes, and many other branches of natural science. In many cases eigenvalue problems model important physical processes. For example, the bound state energies of the hydrogen atom can be computed as the eigenvalues of a singular eigenvalue problem. Eigenvalues appear in many other places. Electric fields in cyclotrons, a special form of particle accelerators, have to oscillate in a precise manner, in order to accelerate the charged particles that circle around its center. The solutions of the Schrödinger equation from quantum physics and quantum chemistry have solutions that correspond to vibrations of the, say, molecule it models. The eigenvalues correspond to energy levels that molecule can occupy. Many characteristic quantities in science are eigenvalues: decay factors, frequencies, norms of operators (or matrices), singular values, condition numbers. Very often those problems arise due to the use of the method of separation of variables for the solution of classical partial differential equations. For instance, let us consider the one-dimensional wave equation

$$\rho_0 u_{tt} = (k u_x)_x \tag{1.1}$$

2010 *Mathematics Subject Classification.* 34B24, 34L20.
Key words and phrases. Sturm-Liouville problems; eigenvalue; eigenfunction; asymptotic distribution.

Submitted February 20, 2016. Published June 6, 2016.

with boundary conditions

$$u(0,t)=0, \quad u(L,t)=0 \tag{1.2}$$

for the longitudinal displacement $u(x;t)$ of a string of length L with mass-density $\rho_0(x)$ and stiffness $k(x)$, both of which we assume are smooth, strictly positive functions on $0 \le x \le L$. Looking for separable time-periodic solutions of (1.1)-(1.2) we get the Sturm-Liouville eigenvalue problem

$$-(k\varphi')' = \lambda\rho_0\varphi, \quad \varphi(0)=0, \quad \varphi(L)=0, \tag{1.3}$$

where $\lambda=\omega^2$ is a constant frequency and $\varphi(x)$ is a function of the spatial variable only. In recent years, boundary value problems various nonstandard for Sturm-Liouville equations have attracted extensive attention due to their intrinsic mathematical challenges and their applications in physics, engineering, biology, medicine and so on. For example, Sturm-Liouville equations with boundary conditions linearly or nonlinearly dependent on the spectral parameter were addressed by many authors (see [1, 6, 13, 16, 17, 18] and the references therein.). Such problems often arise from physical problems, for example, vibration of a string, quantum mechanics and geophysics. Because of its significance, a great deal of work has been done in the theory of Sturm-Liouville equations [11, 22, 28]. However, apart from classical Sturm-Liouville problems, also discontinuous Sturm-Liouville equations occur in applications, with or without the eigenvalue parameter in the boundary conditions. In this study we shall investigate a new class of BVP's which consist of the Sturm-Liouville equation

$$\tau(u) := -a(x)u''(x) + q(x)u(x) = \lambda u(x) \tag{1.4}$$

on finite number disjoint intervals $\Omega = \cup_{i=1}^{n+1}(\xi_{i-1},\xi_i)$, where $0=\xi_0<\xi_1<\dots<\xi_{n+1}=\pi$, together with boundary conditions (BCs) at end points $x=0,\pi$,

$$\tau_\alpha(u) := \cos\alpha u(0) + \sin\alpha u'(0) = 0, \tag{1.5}$$

$$\tau_\beta(u) := \cos\beta u(\pi) + \sin\beta u'(\pi) = 0, \tag{1.6}$$

and the transmission conditions at interior points $\xi_k \in (0,\pi)$, $k=1,2,\dots n$

$$\begin{aligned}\tau_{2k-1}(u) = \delta'_{2k-1}u'(\xi_k+0) + \delta_{2k-1}u(\xi_k+0)&\\ + \gamma'_{2k-1}u'(\xi_k-0) + \gamma_{2k-1}u(\xi_k-0) = 0,&\end{aligned} \tag{1.7}$$

$$\tau_{2k}(u) = \delta'_{2k}u'(\xi_k+0) + \delta_{2k}u(\xi_k+0) + \gamma'_{2k}u'(\xi_k-0) + \gamma_{2k}u(\xi_k-0) = 0, \tag{1.8}$$

where $a(x)=a_i^2>0$ for $x \in \Omega_i := (\xi_{i-1},\xi_i)$, $i=1,2,\dots n+1$, the potential $q(x)$ is real-valued function which continuous in each of the intervals (ξ_{i-1},ξ_i), and has a finite limits $q(\xi_i \mp 0)$, λ is a complex spectral parameter, δ_k, δ'_k, γ_k and γ'_k $(k=1,2,\dots 2n)$ are real numbers. The problems with transmission conditions has become an important area of research in recent years because of the needs of modern technology, engineering and physics. Many of the mathematical problems encountered in the study of boundary-value-transmission problem cannot be treated with the usual techniques within the standard framework of boundary value problem (see [5]). Note that some special cases of this problem arise after an application of the method of separation of variables to a varied assortment of physical problems. For example, some boundary value problems with transmission conditions arise in heat and mass transfer problems [12], in vibrating string problems when the string loaded additionally with point masses [21], in diffraction problems [25]. Also some problems with transmission conditions which arise in mechanics (thermal conduction problems for a thin laminated plate) were studied in [23]. Although the boundary value problems of Sturm-Liouville equation with spectral parameter have been studied in many literature, only few papers can be found in the literature on the Sturm-Liouville boundary value problems with transmission conditions. The spectral analysis and some properties of the regular symmetric (self-adjoint) boundary value transmission problems (BVTPs) have been studied in [2, 3, 4, 5, 7, 8, 9, 10, 15, 24, 26, 27]. In these direct problems there are at most two transmission points [9, 26, 27]. In [9] Kadakal et al. gived an operator-theoretic formulation for considered problem. In [26] and [27], Wang et al. studied such self-adjoint BCs when the definition of the maximum operator involves positive multiples, for the case of $k = 2$, and obtained some non-obvious examples. M. Shahriari et al. have investigated the inverse problem with a finite number of transmission points [20]. Hence, there is a gap in the regular and singular (selfadjoint and nonselfadjoint) multi-interval problems. Some of those approaches can be satisfactory in the case where only the first few eigenvalues are desired. But their usefulness becomes questionable and the accurate computation of eigenvalues tends to be a challenging problem in the cases where one wishes to compute a large number of eigenvalues. Since many applications in quantum physics, quantum chemistry, science and industry are connected to Sturm-Liouville problems (SLPs), their solution has drummed up interest of several researches. Many codes have been developed to solve regular and singular problems.

We want emphasize that the boundary value problem studied here differs from the standard boundary value problems in that it contains transmission conditions in a finite number interior point. Moreover the coefficient functions may have discontinuity at finite interior point. Naturally, eigenfunctions of this problem may have discontinuity at the finite inner point of the considered interval.

2. Notation and prerequisite results

For further consideration we need the next Lemma.

Lemma 2.1. *Let $q(x)$ be continuous in each of (ξ_{i-1}, ξ_i) and $f(\lambda), g(\lambda)$ be given entire functions. Then the initial value problem*

$$-a(x)u'' + q(x)u = \lambda u, \quad x \in \Omega_i = (\xi_{i-1}, \xi_i), \tag{2.1}$$

$$u(\xi_{i-1} + 0, \lambda) = f_i(\lambda), \quad \frac{\partial u(\xi_{i-1} + 0, \lambda)}{\partial x} = g_i(\lambda) \tag{2.2}$$

has a unique solution $u(x) = u_i(x, \lambda)$. For each $x \in (\xi_{i-1}, \xi_i)$, $u_i(x, \lambda)$ is an entire function of $\lambda \in \mathbb{C}$(i=1,2,... n).

Proof. It is sufficiend to transform the initial value problem (2.1)–(2.2) to the equivalent integral equation

$$\begin{aligned} u(x) &= f(\lambda)\cos(\sqrt{\lambda}(x - a_1)) + \frac{1}{\sqrt{\lambda}} g(\lambda)\sin(\sqrt{\lambda}(x - a_1)) \\ &\quad + \frac{1}{\sqrt{\lambda}} \int_{a_i}^{x} \sin(\sqrt{\lambda}(x - a_1)) q(y) u(y) dy, \end{aligned}$$

and then employ the analogue technique as in proof of [22, Theorem 5.1]. $\square$

With a view to constructing the characteristic function $\omega(\lambda)$ we shall define two fundamental solutions

$$\phi(x,\lambda)=\begin{cases}\phi_1(x,\lambda), & x\in[0,\xi_1)\\ \phi_2(x,\lambda), & x\in(\xi_1,\xi_2)\\ \phi_3(x,\lambda), & x\in(\xi_2,\xi_3)\\ \dots\\ \phi_{n+1}(x,\lambda), & x\in(\xi_n,\pi],\end{cases}\quad \chi(x,\lambda)=\begin{cases}\chi_1(x,\lambda), & x\in[0,\xi_1)\\ \chi_2(x,\lambda), & x\in(\xi_1,\xi_2)\\ \chi_3(x,\lambda), & x\in(\xi_2,\xi_3)\\ \dots\\ \chi_{n+1}(x,\lambda), & x\in(\xi_n,\pi],\end{cases}$$

by a special process. Let $\phi_1(x,\lambda)$ and $\chi_{n+1}(x,\lambda)$ be solutions of (1.4) on $(0,\xi_1)$ and (ξ_n,π) satisfy the initial conditions

$$u(0,\lambda)=\sin\alpha,\quad u'(0,\lambda)=-\cos\alpha, \tag{2.3}$$
$$u(\pi,\lambda)=-\sin\beta,\quad u'(\pi,\lambda)=\cos\beta \tag{2.4}$$

respectively. In terms of these solution we shall define the other solutions $\phi_i(x,\lambda)$ and $\chi_i(x,\lambda)$ by the initial conditions

$$\phi_{i+1}(\xi_i+0,\lambda)=\frac{1}{\Delta_{i12}}(\Delta_{i23}\phi_i(\xi_i-0,\lambda)+\Delta_{i24}\frac{\partial\phi_i(\xi_i-0,\lambda)}{\partial x}) \tag{2.5}$$
$$\frac{\partial\phi_{i+1}(\xi_i+0,\lambda)}{\partial x}=\frac{-1}{\Delta_{i12}}(\Delta_{i13}\phi_i(\xi_i-0,\lambda)+\Delta_{i14}\frac{\partial\phi_i(\xi_i-0,\lambda)}{\partial x}) \tag{2.6}$$

and

$$\chi_i(\xi_i-0,\lambda)=\frac{-1}{\Delta_{i34}}(\Delta_{i14}\chi_{i+1}(\xi_i+0,\lambda)+\Delta_{i24}\frac{\partial\chi_{i+1}(\xi_i+0,\lambda)}{\partial x}) \tag{2.7}$$
$$\frac{\partial\chi_i(\xi_i-,\lambda)}{\partial x})=\frac{1}{\Delta_{i34}}(\Delta_{i13}\chi_{i+1}(\xi_i+0,\lambda)+\Delta_{i23}\frac{\partial\chi_{i+1}(\xi_i+0,\lambda)}{\partial x}) \tag{2.8}$$

respectively, where Δ_{ijk} $(1\le j<k\le 4)$ denotes the determinant of the j-th and k-th columns of the matrix

$$\begin{bmatrix}\delta'_{2i-1} & \delta_{2i-1} & \gamma'_{2i-1} & \gamma_{2i-1}\\ \delta'_{2i} & \delta_{2i} & \gamma'_{2i} & \gamma_{2i}\end{bmatrix}$$

where $i=1,2,\dots n$. Everywhere in below we shall assume that $\Delta_{ijk}>0$ for all i,j,k. The existence and uniqueness of these solutions are follows from the by Lemma (2.1). Moreover by applying the method of [15] we can prove that each of these solutions are entire functions of parameter $\lambda\in\mathbb{C}$ for each fixed x. Taking into account (2.5)–(2.8) and the fact that the Wronskians $\omega_i(\lambda):=W[\phi_i(x,\lambda),\chi_i(x,\lambda)]$ $(i=1,2,\dots n+1)$ are independent of the variable x, we have

$$\begin{aligned}\omega_{i+1}(\lambda)&=\phi_i(\xi_i+,\lambda)\frac{\partial\chi_i(\xi_i+,\lambda)}{\partial x}-\frac{\partial\phi_i(\xi_i+,\lambda)}{\partial x}\chi_i(\xi_i+,\lambda)\\ &=\frac{\Delta_{i34}}{\Delta_{i12}}(\phi_i(\xi_i-,\lambda)\frac{\partial\chi_i(\xi_i-,\lambda)}{\partial x}-\frac{\partial\phi_i(\xi_i-,\lambda)}{\partial x}\chi_i(\xi_i-,\lambda))\\ &=\frac{\Delta_{i34}}{\Delta_{i12}}\omega_i(\lambda)=\prod_{j=1}^{i}\frac{\Delta_{j34}}{\Delta_{j12}}\omega_1(\lambda)\quad(i=1,2,\dots n).\end{aligned}$$

It is convenient to define the characteristic function $\omega(\lambda)$ for problem (1.4)–(1.8) as

$$\omega(\lambda):=\omega_1(\lambda)=\prod_{j=1}^{i}\frac{\Delta_{j12}}{\Delta_{i34}}\omega_{i+1}(\lambda)\quad(i=1,2,\dots n).$$

Obviously, $\omega(\lambda)$ is an entire function. By applying the technique of [9] we can prove that there are infinitely many eigenvalues $\lambda_n,\ n = 1, 2, \dots$ of the problem (1.4)–(1.8) which coincide with the zeros of characteristic function $\omega(\lambda)$.

Theorem 2.2. *Let λ_0 be zero of $w(\lambda)$. Then the solutions $\phi(x, \lambda_0)$ and $\chi(x, \lambda_0)$ are linearly dependent.*

Proof. From $w_i(\lambda_0) = 0$ it follows that

$$\chi_i(x, \lambda_0) = k_i \phi_i(x, \lambda_0) \tag{2.9}$$

for some $k_i \neq 0$ $(i = 1, 2, \dots n + 1)$. Next we show that $k_i = k_{i+1}$ $(i = 1, 2, \dots n)$. Suppose that $k_i \neq k_{i+1}$. Using (1.8) and (2.9) we have

$$\begin{aligned} \chi_i(\xi_i - 0, \lambda_0) &= \frac{-1}{\Delta_{i34}}(\Delta_{i14}\chi_{i+1}(\xi_i + 0, \lambda_0) + \Delta_{i24}\frac{\partial \chi_{i+1}(\xi_i + 0, \lambda_0)}{\partial x}) \\ &= \frac{-k_{i+1}}{\Delta_{i34}}(\Delta_{i14}\phi_{i+1}(\xi_i + 0, \lambda_0) + \Delta_{i24}\frac{\partial \phi_{i+1}(\xi_i + 0, \lambda_0)}{\partial x}) \\ &= \frac{k_{i+1}(\Delta_{i14}\Delta_{i23} - \Delta_{i24}\Delta_{i13})}{k_i \Delta_{i12}\Delta_{i34}}\chi_i(\xi_i - 0, \lambda_0) \end{aligned} \tag{2.10}$$

The direct calculations gives $\Delta_{i14}\Delta_{i23} - \Delta_{i24}\Delta_{i13} = \Delta_{i12}\Delta_{i34}$. Consequently

$$\chi_i(\xi_i - 0, \lambda) = \frac{k_{i+1}}{k_i}\chi_i(\xi_i - 0, \lambda) \tag{2.11}$$

Hence $\chi_i(\xi_i - 0, \lambda_0) = 0$. Similarly from (2.8) and (2.9) we derive that

$$\chi_i'(\xi_i - 0, \lambda_0) = 0. \tag{2.12}$$

Thus we have $\chi_i(x, \lambda_0) = 0$ for any $x \in \Omega_i = (\xi_i, \xi_{i+1})$, $i = 1, 2, \dots n - 1$ (2.11) and (2.12). But this is contradict with (2.7)-(2.8). Consequently, $k_i = k_{i+1}$ $(i = 1, 2, \dots n)$, so $\phi(x, \lambda_0)$ and $\chi(x, \lambda_0)$ are linearly dependent. ☐

Corollary 2.3. *If $w(\lambda_0) = 0$, then λ_0 is eigenvalue and $\phi(x, \lambda_0)$, $\chi(x, \lambda_0)$ are corresponding eigenfunctions.*

Lemma 2.4. *The set of eigenvalues of problem* (1.4)–(1.8) *coincide with the set of zeros of the function $w(\lambda)$.*

Proof. By Corollary 2.3 each zero $w(\lambda)$ is eigenvalue. Now let λ_0 be eigenvalue. We must show that $w(\lambda_0) = 0$ Suppose , it possible, that $w(\lambda_0) \neq 0$. Let u_0 be eigenfunction corresponding to this eigenvalue. Since $w_i(\lambda_0) \neq 0$ $(i = 1, 2, \dots n+1)$ the eigenfunction $u_0(x)$ can be represent in the form

$$u_0(x) = \begin{cases} \hbar_{11}\phi_1(x, \lambda_0) + \hbar_{12}\chi_1(x, \lambda_0), & x \in [0, \xi_1) \\ \hbar_{21}\phi_2(x, \lambda_0) + \hbar_{22}\chi_2(x, \lambda_0), & x \in (\xi_1, \xi_2) \\ \hbar_{31}\phi_3(x, \lambda_0) + \hbar_{32}\chi_3(x, \lambda_0), & x \in (\xi_2, \xi_3) \\ \dots \\ \hbar_{(n+1)1}\phi_{n+1}(x, \lambda) + \hbar_{(n+1)2}\chi_{n+1}(x, \lambda), & x \in (\xi_n, \pi] \end{cases}$$

where at least one of the constants $\hbar_{ij}$ $(i = 1, 2 \dots n+1;\ j = 1, 2)$ is not zero. Putting in conditions (1.5)–(1.8) we obtain homogenous linear simultaneous equation of the variables $\hbar_{ij}$ $(i = 1, 2 \dots n + 1;\ j = 1, 2)$. By direct calculations it is easy to show that the determinant of this system has the form $cw(\lambda_0$, where $c \neq 0$ is an constant. Consequently this linear simultaneous equation has the only trivial solution $\hbar_{ij}$ $(i = 1, 2 \dots n+1;\ j = 1, 2)$, and so we reach a contraction, which completes the proof. $\square$

Now by modifying the standard method we prove that all eigenvalues of problem (1.4)–(1.8) are real.

Theorem 2.5. *All the eigenvalues of the boundary value transmission problem* (1.4)–(1.8) *are real.*

Proof. Let λ_0 be eigenvalue and u_0 be eigenfunction corresponding to this eigenvalue. Then integrating by parts we obtain

$$
\begin{aligned}
&\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \Delta_{i12} \prod_{i=k+1}^{n+1} \Delta_{i34} \int_{\xi_k+}^{\xi_{k+1}-} \tau u_0(x)\overline{u_0(x)}dx \\
&- \sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \Delta_{i12} \prod_{i=k+1}^{n+1} \Delta_{i34} \int_{\xi_k+}^{\xi_{k+1}-} u_0(x)\overline{\lambda u_0(x)}dx \\
&= \Delta_{134}\Delta_{234}\dots\Delta_{n34}(W(u_0, \overline{u_0}; \xi_1-) - W(u_0, \overline{u_0}; 0)) \\
&\quad + \Delta_{112}\Delta_{234}\dots\Delta_{n34}(W(u_0, \overline{u_0}; \xi_2-) - W(u_0, \overline{u_0}; \xi_1+)) \\
&\quad + \dots + \Delta_{112}\Delta_{212}\dots\Delta_{n12}(W(u_0, \overline{u_0}; \pi) - W(u_0, \overline{u_0}; \xi_n+))
\end{aligned}
\tag{2.13}
$$

where, as usual, $W(u_0, \overline{u_0}; x)$ denotes the Wronskians of the functions u_0 and $\overline{u_0}$. From the definitions of transmission functionals we have

$$
\Delta_{i34}W(u_0, \overline{u_0}; \xi_i-) = \Delta_{i12}W(u_0, \overline{u_0}; \xi_i+)\ i = 1, 2 \dots n
\tag{2.14}
$$

Finally, substituting (2.14) in (2.13) we have

$$
\begin{aligned}
&(\lambda_0 - \overline{\lambda_0})[\sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \Delta_{i12} \prod_{i=k+1}^{n+1} \Delta_{i34} \int_{\xi_k+}^{\xi_{k+1}-} (u_0(x))^2 dx \\
&- \sum_{k=0}^{n} \frac{1}{a_{k+1}^2} \prod_{i=0}^{k} \Delta_{i12} \prod_{i=k+1}^{n+1} \Delta_{i34} \int_{\xi_k+}^{\xi_{k+1}-} (u_0(x))^2 dx] = 0
\end{aligned}
$$

Since $\Delta_{ijk} > 0$ we obtain $\lambda_0 = \overline{\lambda_0}$. Consequently all eigenvalues of the problem (1.4)–(1.8) are real. The proof is complete. $\square$

Theorem 2.6. *For each eigenvalue λ_0 of problem* (1.4)–(1.8) *there is at least one real valued eigenfunction $u_0(x)$ corresponding to this eigenvalue.*

Proof. Let λ_0 be any eigenvalue of the problem and let $\phi_0(x) = \varphi_0(x) + i\psi_0(x)$ be an eigenfunction corresponding to this eigenvalue. This implies $\varphi_0(x)$, $\psi_0(x)$ are not both zero. It is easy to see that $\overline{\phi_0}(x) = \varphi_0(x) - i\psi_0(x)$ is also an eigenfunction, corresponding to same eigenvalue λ_0. Since $\phi_0(x)$ and $\overline{\phi_0}(x)$ are eigenfunctions of the problem (1.4)–(1.8), at least one of the real valued functions $\frac{\phi_0(x)+\overline{\phi_0}(x)}{2} = \varphi_0(x)$ and $\frac{\phi_0(x)-\overline{\phi_0}(x)}{2i} = \psi_0(x)$ is not zero and therefore also is eigenfunction of the problem (1.4)–(1.8). Thus, there is at least one real valued eigenfunction, corresponding to the eigenvalue λ_0. $\square$

3. Asymptotic behavior of the functions $\phi(x,\lambda)$ and $\chi(x,\lambda)$

To abbreviate notation we use the notation $\phi_{i\lambda}(x) := \phi_i(x,\lambda)$, $\chi_{i\lambda}(x) := \chi_i(x,\lambda)$ $(i = 1, 2, \dots n+1)$. We can prove that the next integral and integro-differential equations hold for $k = 0$ and $k = 1$.

$$\begin{aligned}\frac{d^k}{dx^k}\phi_{1\lambda}(x) &= \sin\alpha \frac{d^k}{dx^k}\cos\big(\frac{sx}{a_1}\big) + \frac{a_1\cos\alpha}{s}\frac{d^k}{dx^k}\sin\big(\frac{sx}{a_1}\big)\\ &\quad + \frac{1}{a_1 s}\int_0^x \frac{d^k}{dx^k}\sin\big(\frac{s(x-y)}{a_1}\big)q(y)\phi_{1\lambda}(y)dy,\end{aligned} \tag{3.1}$$

$$\begin{aligned}\frac{d^k}{dx^k}\phi_{(i+1)\lambda}(x) &= \frac{1}{\Delta_{i12}}(\Delta_{i23}\phi_{i\lambda}(\xi_i) + \Delta_{i24}\phi'_{i\lambda}(\xi_i))\frac{d^k}{dx^k}\cos\big(\frac{s(x-\xi_i)}{a_{i+1}}\big)\\ &\quad - \frac{a_{i+1}}{s\Delta_{i12}}(\Delta_{i13}\phi_{i\lambda}(\xi_i) + \Delta_{i14}\phi'_{i\lambda}(\xi_i))\frac{d^k}{dx^k}\sin\big(\frac{s(x-\xi_i)}{a_{i+1}}\big)\\ &\quad + \frac{1}{a_{i+1}s}\int_{\xi_i}^x \frac{d^k}{dx^k}\sin\big(\frac{s(x-y)}{a_{i+1}}\big)q(y)\phi_{(i+1)\lambda}(y)dy\end{aligned} \tag{3.2}$$

for $x \in (a,\xi_1)$ and for $x \in (\xi_i,\xi_{i+1})$, $i = 1, 2, \dots n$ respectively. Also

$$\begin{gathered}\frac{d^k}{dx^k}\chi_{(n+1)\lambda}(x) = -\sin\beta\frac{d^k}{dx^k}\cos\big(\frac{s(\pi-x)}{a_{n+1}}\big) + \frac{a_{n+1}}{s}\cos\beta\frac{d^k}{dx^k}\sin\big(\frac{s(\pi-x)}{a_{n+1}}\big)\\ + \frac{1}{a_{n+1}s}\int_x^\pi \frac{d^k}{dx^k}\sin\big(\frac{s(x-y)}{a_{n+1}}\big)q(y)\chi_{(n+1)\lambda}(y)dy\end{gathered}$$

$$\begin{aligned}&\frac{d^k}{dx^k}\chi_{i\lambda}(x)\\ &= -\frac{1}{\Delta_{i34}}(\Delta_{i14}\chi_{(i+1)\lambda}(\xi_i) + \Delta_{i24}\chi'_{(i+1)\lambda}(\xi_i))\frac{d^k}{dx^k}\cos\big(\frac{s(x-\xi_i)}{a_i}\big)\\ &\quad + \frac{a_i}{s\Delta_{i34}}(\Delta_{i13}\chi_{(i+1)\lambda}(\xi_i) + \Delta_{i23}\chi'_{(i+1)\lambda}(\xi_i))\frac{d^k}{dx^k}\sin\big(\frac{s(x-\xi_i)}{a_i}\big)\\ &\quad + \frac{1}{a_i s}\int_x^{\xi_i}\frac{d^k}{dx^k}\sin\big(\frac{s(x-y)}{a_i}\big)q(y)\chi_{i\lambda}(y)dy\end{aligned}$$

for $x \in (\xi_n, b)$ and (ξ_i,ξ_{i+1}) $(i = 1, 2, \dots n)$, respectively.

Theorem 3.1. *Let $Im\mu = t$. If $\sin\alpha \neq 0$, then*

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = \sin\alpha\frac{d^k}{dx^k}\cos[\frac{sx}{a_1}] + O\Big(|s|^{k-1}e^{|t|\frac{x}{a_1}}\Big) \tag{3.3}$$

$$\begin{aligned}&\frac{d^k}{dx^k}\phi_{(i+1)\lambda}(x)\\ &= (-1)^i\sin\alpha s^i\Big(\prod_{j=1}^i \frac{\Delta_{j24}}{\Delta_{j12}a_j}\sin\big[\frac{s(\xi_j-\xi_{j-1})}{a_j}\big]\Big)\frac{d^k}{dx^k}\cos\big[\frac{s(x-\xi_i)}{a_{i+1}}\big]\\ &\quad + O\Big(|s|^{i-1}e^{|t|((\sum_{j=1}^i\frac{(\xi_j-\xi_{j-1})}{a_j}) + \frac{(x-\xi_i)}{a_{i+1}})}\Big), \quad i = 1, 2, \dots n\end{aligned} \tag{3.4}$$

as $|s| \to \infty$, while if $\sin\alpha = 0$,

$$\frac{d^k}{dx^k}\phi_{1\lambda}(x) = \frac{a_1\cos\alpha}{s}\frac{d^k}{dx^k}\sin[\frac{sx}{a_1}] + O\Big(|s|^{k-2}e^{|t|\frac{x}{a_1}}\Big) \tag{3.5}$$

$$\begin{aligned} \frac{d^k}{dx^k}\phi_{2\lambda}(x) &= -\cos\alpha\frac{\Delta_{124}}{\Delta_{112}}\frac{d^k}{dx^k}\cos\big[\frac{s\xi_1}{a_1}\big]\cos\big[\frac{s(x-\xi_1)}{a_2}\big] \\ &\quad + O\Big(|s|^{k-1}e^{|t|(\frac{\xi_1}{a_1}+\frac{(x-\xi_1)}{a_2})}\Big) \end{aligned} \tag{3.6}$$

$$\begin{aligned} &\frac{d^k}{dx^k}\phi_{(i+1)\lambda}(x) \\ &= (-1)^i\cos\alpha s^{i-1}\frac{\Delta_{124}}{\Delta_{112}}\cos\big[\frac{s\xi_1}{a_1}\big]\Big(\prod_{j=2}^{i}\frac{\Delta_{j24}}{\Delta_{j12}a_j}\sin\big[\frac{s(\xi_j-\xi_{j-1})}{a_j}\big]\Big) \\ &\quad\times\frac{d^k}{dx^k}\cos\big[\frac{s(x-\xi_i)}{a_{i+1}}\big] + O\Big(|s|^{i-2}e^{|t|((\sum_{j=1}^{i}\frac{(\xi_j-\xi_{j-1})}{a_j})+\frac{(x-\xi_i)}{a_{i+1}})}\Big) \end{aligned} \tag{3.7}$$

$i=2,\dots n$ as $|s|\to\infty$, $(k=0,1)$. Each of this asymptotic equalities hold uniformly for x.

Theorem 3.2. *Let $Ims=t$. If $\sin\beta\neq 0$, then*

$$\frac{d^k}{dx^k}\chi_{(n+1)\lambda}(x) = -\sin\beta\frac{d^k}{dx^k}\cos\big[\frac{s(\pi-x)}{a_{n+1}}\big] + O\Big(|s|^{k-1}e^{|t|\frac{(\pi-x)}{a_{n+1}}}\Big) \tag{3.8}$$

$$\begin{aligned} &\frac{d^k}{dx^k}\chi_{(n-i)\lambda}(x) \\ &= (-1)^{i+1}\sin\beta s^{i+1}\Big(\prod_{j=0}^{i}\frac{\Delta_{(n-j)24}}{\Delta_{(n-j)34}a_{n+1-j}}\sin\big[\frac{s(\xi_{n+1-j}-\xi_{n-j})}{\rho_{n+1-j}}\big]\Big) \\ &\quad\times\frac{d^k}{dx^k}\cos\big[\frac{s(x-\xi_{n-i})}{a_{n-i}}\big] + O\Big(|s|^{i}e^{|t|((\sum_{j=0}^{i}\frac{(\xi_{n+1-j}-\xi_{n-j})}{a_{n+1-j}})+\frac{(x-\xi_{n-i})}{a_{n-i}})}\Big), \end{aligned} \tag{3.9}$$

for $i=0,1,2,\dots n-1$ as $|s|\to\infty$, while if $\sin\beta=0$,

$$\frac{d^k}{dx^k}\chi_{(n+1)\lambda}(x) = -\frac{a_{n+1}\cos\beta}{s}\frac{d^k}{dx^k}\sin\big[\frac{s(\pi-x)}{a_{n+1}}\big] + O\Big(|s|^{k-2}e^{|t|\frac{(\pi-x)}{a_{n+1}}}\Big) \tag{3.10}$$

$$\begin{aligned} \frac{d^k}{dx^k}\chi_{n\lambda}(x) &= -cos\beta\frac{\Delta_{n24}s}{a_{n+1}\Delta_{n34}}\cos\big[\frac{s(\pi-\xi_n)}{a_{n+1}}\big]\cos\big[\frac{s(x-\xi_n)}{a_n}\big] \\ &\quad + O\Big(|s|^{k-1}e^{|t|(\frac{(\pi-\xi_n)}{a_{n+1}}+\frac{(x-\xi_n)}{a_n})}\Big) \end{aligned} \tag{3.11}$$

$$\begin{aligned} &\frac{d^k}{dx^k}\chi_{(n-i)\lambda}(x) \\ &= (-1)^{i+1}\sin\beta s^{i+1}\frac{\Delta_{n24}}{a_{n+1}\Delta_{n34}}\cos\big[\frac{s(\pi-\xi_n)}{a_{n+1}}\big]\frac{d^k}{dx^k}\cos\big[\frac{s(x-\xi_{n-i})}{a_{n-i}}\big] \\ &\quad\times\Big(\prod_{j=1}^{i}\frac{\Delta_{(n-j)24}}{\Delta_{(n-j)34}a_{n+1-j}}\sin\big[\frac{s(\xi_{n-1-j}-\xi_{n-j})}{a_{n-j}}\big]\Big) \\ &\quad + O\Big(|s|^{i-1}e^{|t|((\sum_{j=0}^{i}\frac{(\xi_{n+1-j}-\xi_{n-j})}{a_{n+1-j}})+\frac{(x-\xi_{n-i})}{a_{n-i}})}\Big) \end{aligned} \tag{3.12}$$

as $|s|\to\infty$ $i=1,\dots n-1$ $(k=0,1)$. Each of this asymptotic equalities hold uniformly for x.

4. Asymptotic behaviour of eigenvalues and eigenfunctions

Since the Wronskians of $\phi_\lambda(x)$ and $\chi_\lambda(x)$ are independent of x in each Ω_i ($i = 0, 1, \dots, n+1$), in particular, by putting $x = \pi$ we have

$$\begin{aligned}
\omega(\lambda) &= \prod_{j=1}^{n} \frac{\Delta_{j12}}{\Delta_{j34}} \omega_{n+1}(\lambda)|_{x=\pi} = \prod_{j=1}^{n} \frac{\Delta_{j12}}{\Delta_{j34}} \omega(\phi_{n+1}(\pi,\lambda), \chi_{n+1}(\pi,\lambda)) \\
&= \prod_{j=1}^{n} \frac{\Delta_{j12}}{\Delta_{j34}} \{\cos\beta\phi_{(n+1)}(\pi,\lambda) - \sin\beta\phi'_{(n+1)}(\pi,\lambda)\}.
\end{aligned} \tag{4.1}$$

Let $Im\mu = t$. By substituting (3.8) and (3.12) in (4.1) we obtain the following asymptotic representations:

(i) If $\sin\alpha \neq 0$ and $\sin\beta \neq 0$, then

$$\begin{aligned}
w(\lambda) &= (-1)^{n+1} \sin\alpha \sin\beta \frac{s^{n+1}}{a_{n+1}} \Big(\prod_{j=1}^{i} \frac{\Delta_{j24}}{\Delta_{j34} a_j} \sin\big[\frac{s(\xi_j - \xi_{j-1})}{a_j}\big]\Big) \\
&\quad \times \sin\big[\frac{s(\pi - \xi_n)}{a_{n+1}}\big] + O\Big(|s|^n e^{|t|(\sum_{j=1}^{n+1} \frac{(\xi_j - \xi_{j-1})}{a_j})}\Big),
\end{aligned} \tag{4.2}$$

(ii) If $\sin\alpha \neq 0$ and $\sin\beta = 0$, then

$$\begin{aligned}
w(\lambda) &= (-1)^{n} \sin\alpha \cos\beta s^{n} \Big(\prod_{j=1}^{i} \frac{\Delta_{j24}}{\Delta_{j34} a_j} \sin\big[\frac{s(\xi_j - \xi_{j-1})}{a_j}\big]\Big) \\
&\quad \times \cos\big[\frac{s(x - \xi_n)}{a_{n+1}}\big] + O\Big(|s|^{n-1} e^{|t|\left(\sum_{j=1}^{n+1} \frac{(\xi_j - \xi_{j-1})}{a_j}\right)}\Big)
\end{aligned} \tag{4.3}$$

(iii) If $\sin\alpha = 0$ and $\sin\beta \neq 0$, then

$$\begin{aligned}
&w(\lambda) \\
&= (-1)^{n+2} \sin\beta \cos\alpha \frac{s^n}{a_{n+1}} \frac{\Delta_{124}}{\Delta_{134}} \cos\big[\frac{s\xi_1}{a_1}\big] \Big(\prod_{j=2}^{n} \frac{\Delta_{j24}}{\Delta_{j34} a_j} \sin\big[\frac{s(\xi_j - \xi_{j-1})}{a_j}\big]\Big) \\
&\quad \times \sin\big[\frac{s(\pi - \xi_n)}{a_{n+1}}\big] + O\Big(|s|^{n-1} e^{|t|(\sum_{j=1}^{n+1} \frac{(\xi_j - \xi_{j-1})}{a_j})}\Big),
\end{aligned} \tag{4.4}$$

(iv) If $\sin\alpha = 0$ and $\sin\beta = 0$, then

$$\begin{aligned}
&w(\lambda) \\
&= (-1)^{n} \cos\beta \cos\alpha s^{n-1} \frac{\Delta_{124}}{\Delta_{134}} \cos\big[\frac{s\xi_1}{a_1}\big] \Big(\prod_{j=2}^{n} \frac{\Delta_{j24}}{\Delta_{j34} a_j} \sin\big[\frac{s(\xi_j - \xi_{j-1})}{a_j}\big]\Big) \\
&\quad \times \cos\big[\frac{s(\pi - \xi_n)}{a_{n+1}}\big] + O\Big(|s|^{n-2} e^{|t|\left(\sum_{j=1}^{n+1} \frac{(\xi_j - \xi_{j-1})}{a_j}\right)}\Big),
\end{aligned} \tag{4.5}$$

Corollary 4.1. *The eigenvalues of problem* (1.4)–(1.8) *are bounded below, and they are countably infinite and can cluster only at* ∞.

Proof. Indeed, putting $s = it(t > 0)$ in (4.2)–(4.5) it follows that $\omega(\lambda) \to \infty$ as $t \to \infty$, so $\omega(\lambda) \neq 0$ for λ negative and sufficiency large. □

Now we are ready to derived the needed asymptotic formulas for eigenvalues and eigenfunctions.

Theorem 4.2. *The boundary-value-transmission problem* (1.4)–(1.8) *has an precisely numerable many real eigenvalues, whose behavior may be expressed by the sequence* $\{s_k^{(t)}\}$ $(t = 1, 2, \dots, n+1)$ *with the following asymptotic behavior as* $n \to \infty$*:*

(i) If $\sin\alpha \neq 0$ *and* $\sin\beta \neq 0$*, then*

$$s_k^{(t)} = \frac{a_t \pi k}{(\xi_t - \xi_{t-1})} + O\big(\frac{1}{k}\big), \quad (t = 1, 2, \dots n+1), \tag{4.6}$$

(ii) If $\sin\alpha \neq 0$ *and* $\sin\beta = 0$*, then*

$$s_k^{(n+1)} = (k + \frac{1}{2})\frac{a_{n+1}\pi}{(\pi - \xi_n)} + O\big(\frac{1}{k}\big), \quad s_k^{(t)} = \frac{(k+1)a_t\pi}{2(\xi_t - \xi_{t-1})} + O\big(\frac{1}{k}\big), \tag{4.7}$$

for $t = 1, \dots n$,

(iii) If $\sin\alpha = 0$ *and* $\sin\beta \neq 0$*, then*

$$s_k^{(1)} = (k + \frac{1}{2})\frac{a_1\pi}{\xi_1} + O\big(\frac{1}{k}\big), \quad s_k^{(t)} = \frac{(k+1)a_j\pi}{2(\xi_t - \xi_{t-1})} + O\big(\frac{1}{k}\big), \tag{4.8}$$

for $t = 2, \dots n+1$,

(iv) If $\sin\alpha = 0$ *and* $\sin\beta = 0$*, then*

$$\begin{gathered} s_k^{(1)} = (k + \frac{1}{2})\frac{a_1\pi}{\xi_1} + O\big(\frac{1}{k}\big), \quad s_k^{(n+1)} = (k + \frac{1}{2})\frac{a_{n+1}\pi}{(\pi - \xi_n)} + O\big(\frac{1}{k}\big), \\ s_k^{(t)} = \frac{(k+2)a_t\pi}{2(\xi_t - \xi_{t-1})} + O\big(\frac{1}{k}\big), \quad (t = 2, \dots n), \end{gathered} \tag{4.9}$$

Proof. Let $\sin\alpha \neq 0$ and $\sin\beta \neq 0$. By applying the well-known Rouche Theorem which asserts that if $f(z)$ and $g(z)$ are analytic inside and on a closed contour Γ, and $|g(z)| < |f(z)|$ on Γ then $f(z)$ and $f(z) + g(z)$ have the same number zeros inside Γ provided that the zeros are counted with multiplicity on a sufficiently large contour, it follows that $w(\lambda)$ has the same number of zeros inside the suitable contour as the leading term

$$w_0(\lambda) = -\sin\alpha \sin\beta s^{r+5}\Big(\prod_{j=1}^{r+1} \frac{1}{\rho_j} \sin\big[\frac{s(\xi_{j-1} - \xi_j)}{\rho_j}\big]\Big)$$

in (4.2). Hence, if $\lambda_0 < \lambda_1 < \lambda_2 \dots$ are the zeros of $w(\lambda)$, we have the needed asymptotic formulas (4.6). Other cases can be proved similarly. $\square$

Using this asymptotic expressions of eigenvalues we can obtain the corresponding asymptotic expressions for eigenfunctions of the problem (1.4)–(1.7). Recalling that $\phi_{\lambda_n}(x)$ is an eigenfunction according to the eigenvalue λ_n, and by putting (4.6) in the (3.3)-(3.4) for $k = 0, 1$ and denoting the corresponding eigenfunction as $\phi_k^{(t)}(x)$ $(t = 1, 2, \dots n+1)$ we get the following asymptotic representation for the eigenfunctions if $\sin\alpha \neq 0$ and $\sin\beta \neq 0$, then

$$\phi_k^{(t)}(x) = \begin{cases} \sin\alpha \cos\big[\frac{a_t\pi k x}{a_1(\xi_t - \xi_{t-1})}\big] + O(\frac{1}{k}), \quad \text{for } x \in (a, \xi_1) \\ (-1)^i \sin\alpha\big[\frac{a_t\pi k}{(\xi_t - \xi_{t-1})}\big]^i \Big(\prod_{j=1}^{i} \frac{\Delta_{j24}}{\Delta_{j12}a_j} \sin\big[\frac{a_t\pi(\xi_j - \xi_{j-1})k}{a_j(\xi_t - \xi_{t-1})}\big]\Big) \\ \times \cos\big[\frac{a_t\pi k(x - \xi_i)}{a_{i+1}(\xi_t - \xi_{t-1})}\big] + O(k^{i-1}), \\ \quad \text{for } x \in (\xi_i, \xi_{i+1}),\ i = 1, 2, \dots n, \end{cases}$$

where $t = 1, 2, \dots n+1$. If $\sin\alpha \neq 0$ and $\sin\beta = 0$, then

$$\phi_k^{(n+1)}(x) = \begin{cases} \sin\alpha\cos\left[(k+\frac{1}{2})\frac{a_{n+1}\pi x}{(\pi-\xi_n)}\right] + O(1), \quad \text{for } x \in (a, \xi_1) \\ (-1)^i \sin\alpha\left[(k+\frac{1}{2})\frac{a_{n+1}\pi x}{(\pi-\xi_n)}\right]^i \\ \times\Big(\prod_{j=1}^i \frac{\Delta_{j24}}{\Delta_{j12}a_j}\sin\left[(k+\frac{1}{2})\frac{a_{n+1}(\xi_j-\xi_{j-1})\pi}{a_j(\pi-\xi_n)}\right]\Big) \\ \times\cos\left[(k+\frac{1}{2})\frac{a_{n+1}\pi(x-\xi_i)}{a_{i+1}(\pi-\xi_n)}\right] + O(k^{i-1}), \\ \quad \text{for } x \in (\xi_i, \xi_{i+1}) \quad i = 1, 2, \dots n\,; \end{cases}$$

and

$$\phi_k^{(t)}(x) = \begin{cases} \sin\alpha\cos\left[\frac{a_t\pi(k+1)x}{2(\xi_t-\xi_{t-1})}\right] + O(\frac{1}{k}), \quad \text{for } x \in (a, \xi_1), \\ (-1)^i \sin\alpha\left[\frac{a_t\pi(k+1)}{2(\xi_t-\xi_{t-1})}\right]^i \\ \times\Big(\prod_{j=1}^i \frac{\Delta_{j24}}{\Delta_{j12}a_j}\sin\left[\frac{a_t\pi(\xi_j-\xi_{j-1})(k+1)}{2a_j(\xi_t-\xi_{t-1})}\right]\Big) \\ \times\cos\left[\frac{a_t\pi(k+1)(x-\xi_i)}{2a_{i+1}(\xi_t-\xi_{t-1})}\right] + O(k^{i-1}) \\ \quad \text{for } x \in (\xi_i, \xi_{i+1}) \quad i = 1, 2, \dots n\,, \end{cases}$$

where t=1,2,... n. If $\sin\alpha = 0$ and $\sin\beta \neq 0$, then

$$\phi_k^{(1)}(x) = \begin{cases} -a_1\cos\alpha\left[(k+\frac{1}{2})\frac{a_1\pi}{\xi_1}\right]^{-1}\sin[(k+\frac{1}{2})\frac{\pi x}{\xi_1}] + O(\frac{1}{k^2}), \quad \text{for } x \in (a, \xi_1) \\ -\cos\alpha\frac{\Delta_{124}}{\Delta_{112}}\cos\left[(k+\frac{1}{2})\frac{\pi\xi_1}{\xi_1}\right]\cos\left[(k+\frac{1}{2})\frac{a_1\pi(x-\xi_1)}{a_2\xi_1}\right] + O(\frac{1}{k}) \\ \quad \text{for } x \in (\xi_1, \xi_2) \\ (-1)^i\cos\alpha\left[(k+\frac{1}{2})\frac{a_1\pi}{\xi_1}\right]^{i-1}\cos\left[(k+\frac{1}{2})\frac{a_1\pi\xi_1}{a_1\xi_1}\right]\cos\left[(k+\frac{1}{2})\frac{a_1\pi(x-\xi_i)}{a_{i+1}\xi_1}\right] \\ \times\Big(\prod_{j=2}^i \frac{\Delta_{(j-1)24}}{\Delta_{(j-1)12}a_j}\sin\left[(k+\frac{1}{2})\frac{a_1\pi(\xi_j-\xi_{j-1})}{a_j\xi_1}\right]\Big) + O(k^{i-2}) \\ \quad \text{for } x \in (\xi_i, \xi_{i+1}) \quad i = 2, \dots n, \end{cases}$$

and

$$\phi_k^{(t)}(x) = \begin{cases} -a_1\cos\alpha\left[(k+1)\frac{a_t\pi}{2(\xi_t-\xi_{t-1})}\right]^{-1}\sin\left[(k+1)\frac{a_t\pi x}{2a_1(\xi_t-\xi_{t-1})}\right] + O(\frac{1}{k^2}) \\ \quad \text{for } x \in (a, \xi_1), \\ -\cos\alpha\frac{\Delta_{124}}{\Delta_{112}}\cos\left[(k+1)\frac{a_t\pi\xi_1}{2a_1(\xi_t-\xi_{t-1})}\right]\cos\left[(k+1)\frac{a_t\pi(x-\xi_1)}{2a_2(\xi_t-\xi_{t-1})}\right] + O(\frac{1}{k}) \\ \quad \text{for } x \in (\xi_1, \xi_2), \\ (-1)^i\cos\alpha\left[(k+1)\frac{a_t\pi}{2(\xi_t-\xi_{t-1})}\right]^{i-1}\cos\left[(k+1)\frac{a_t\pi\xi_1}{a_1(\xi_t-\xi_{t-1})}\right] \\ \times\cos\left[(k+1)\frac{a_t\pi(x-\xi_i)}{2a_{i+1}(\xi_t-\xi_{t-1})}\right] \\ \Big(\prod_{j=2}^i \frac{\Delta_{(j-1)24}}{\Delta_{(j-1)12}a_j}\sin\left[(k+1)\frac{a_t\pi(\xi_j-\xi_{j-1})}{2a_j(\xi_t-\xi_{t-1})}\right]\Big) \\ +O(k^{i-2}) \\ \quad \text{for } x \in (\xi_i, \xi_{i+1}),\ i = 2, \dots n \end{cases}$$

where $t=2,\dots n+1$. If $\sin\alpha=0$ and $\sin\beta=0$, then

$$\phi_k^{(1)}(x)=\begin{cases} -a_1\cos\alpha\big[(k+\frac{1}{2})\frac{a_1\pi}{\xi_1}\big]^{-1}\sin[(k+\frac{1}{2})\frac{\pi x}{\xi_1}]+O(\frac{1}{k^2}) \\ \quad \text{for } x\in(a,\xi_1), \\ -\cos\alpha\frac{\Delta_{124}}{\Delta_{112}}\cos\big[(k+\frac{1}{2})\frac{\pi\xi_1}{\xi_1}\big]\cos\big[(k+\frac{1}{2})\frac{a_1\pi(x-\xi_1)}{a_2\xi_1}\big]+O(\frac{1}{k}) \\ \quad \text{for } x\in(\xi_1,\xi_2), \\ (-1)^i\cos\alpha\big[(k+\frac{1}{2})\frac{a_1\pi}{\xi_1}\big]^{i-1}\cos\big[(k+\frac{1}{2})\frac{a_1\pi\xi_1}{a_1\xi_1}\big]\cos\big[(k+\frac{1}{2})\frac{a_1\pi(x-\xi_i)}{a_{i+1}\xi_1}\big] \\ \times\Big(\prod_{j=2}^{i}\frac{\Delta_{(j-1)24}}{\Delta_{(j-1)12}a_j}\sin\big[(k+\frac{1}{2})\frac{a_1\pi(\xi_j-\xi_{j-1})}{a_j\xi_1}\big]\Big)+O(k^{i-2}) \\ \quad \text{for } x\in(\xi_i,\xi_{i+1})\ i=2,\dots n\,; \end{cases}$$

$$\phi_k^{(n+1)}(x)=\begin{cases} -a_1\cos\alpha\big[(k+\frac{1}{2})\frac{a_{n+1}\pi}{(\pi-\xi_n)}\big]^{-1}\sin\big[(k+\frac{1}{2})\frac{a_{n+1}\pi x}{a_1(\pi-\xi_n)}\big]+O(\frac{1}{k^2}) \\ \quad \text{for } x\in(a,\xi_1), \\ -\cos\alpha\frac{\Delta_{124}}{\Delta_{112}}\cos\big[(k+\frac{1}{2})\frac{a_{n+1}\pi\xi_1}{a_1(\pi-\xi_n)}\big]\cos\big[(k+\frac{1}{2})\frac{a_{n+1}\pi(x-\xi_1)}{a_2(\pi-\xi_n)}\big]+O(\frac{1}{k}) \\ \quad \text{for } x\in(\xi_1,\xi_2), \\ (-1)^i\cos\alpha\big[(k+\frac{1}{2})\frac{a_{n+1}\pi}{(\pi-\xi_n)}\big]^{i-1}\cos\big[(k+\frac{1}{2})\frac{a_{n+1}\pi\xi_1}{a_1(\pi-\xi_n)}\big] \\ \times\cos\big[(k+\frac{1}{2})\frac{a_{n+1}\pi(x-\xi_i)}{a_{i+1}(\pi-\xi_n)}\big] \\ \times\Big(\prod_{j=2}^{i}\frac{\Delta_{(j-1)24}}{\Delta_{(j-1)12}a_j}\sin\big[(k+\frac{1}{2})\frac{a_{n+1}\pi(\xi_j-\xi_{j-1})}{a_j(\pi-\xi_n)}\big]\Big) \\ +O(k^{i-2}) \\ \quad \text{for } x\in(\xi_i,\xi_{i+1}), i=2,\dots n\,; \end{cases}$$

and

$$\phi_k^{(t)}(x)=\begin{cases} -a_1\cos\alpha\big[(k+2)\frac{a_t\pi}{2(\xi_t-\xi_{t-1})}\big]^{-1}\sin\big[(k+2)\frac{a_t\pi x}{2a_1(\xi_t-\xi_{t-1})}\big]+O(\frac{1}{k^2}), \\ \quad \text{for } x\in(a,\xi_1) \\ -\cos\alpha\frac{\Delta_{124}}{\Delta_{112}}\cos\big[(k+2)\frac{a_t\pi\xi_1}{2a_1(\xi_t-\xi_{t-1})}\big]\cos\big[(k+2)\frac{a_t\pi(x-\xi_1)}{2a_2(\xi_t-\xi_{t-1})}\big]+O(\frac{1}{k}) \\ \quad \text{for } x\in(\xi_1,\xi_2), \\ (-1)^i\cos\alpha\big[(k+2)\frac{a_t\pi}{2(\xi_t-\xi_{t-1})}\big]^{i-1}\cos\big[(k+2)\frac{a_t\pi\xi_1}{2a_1(\xi_t-\xi_{t-1})}\big] \\ \times\cos\big[(k+2)\frac{2a_t\pi(x-\xi_i)}{2a_{i+1}(\xi_t-\xi_{t-1})}\big] \\ \Big(\prod_{j=2}^{i}\frac{\Delta_{(j-1)24}}{\Delta_{(j-1)12}a_j}\sin\big[(k+2)\frac{a_t\pi(\xi_j-\xi_{j-1})}{2a_j(\xi_t-\xi_{t-1})}\big]\Big) \\ +O(k^{i-2}) \\ \quad \text{for } x\in(\xi_i,\xi_{i+1},\ i=2,\dots n\,. \end{cases}$$

Conclusion. The main results of this study are derived under the simple condition $\Delta_{ijk}>0$. We can show that this condition cannot be omitted. Indeed, let us consider the next simple special case of the problem (1.4)–(1.8),

$$-y''(x)=\lambda y(x)\quad x\in[-1,0)\cup(0,1], \tag{4.10}$$

$$y(-1)=0,\quad (\lambda-1)y'(-1)+\lambda y(1)=0, \tag{4.11}$$

$$y(0-)=y(0+),\quad y'(0-)=-y'(0+). \tag{4.12}$$

for which the condition $\Delta_{112}>0$ is not valid ($\Delta_{112}<0$). It is well known that the standard Sturm-liouville problems has infinitely many real eigenvalues. But it can be shown by direct calculation that the problem (4.10)–(4.12) has only one eigenvalue $\lambda=1$.

References

[1] Z. S. Aliyev, N. B. Kerimov; *Spectral Properties of the Differential Operators of the Fourth-Order with Eigenvalue Parameter Dependent Boundary Condition*, International Journal of Mathematics and Mathematical Sciences, Volume 2012 (2012), Article ID 456517.

[2] B. P. Allahverdiev, E. Uğurlu; *On dilation, scattering and spectraltheory for two-interval singular dierential operators*, Bulletin Mathmatiquede la Socit des Sciences Mathmatiques de Roumanie, 58(106), No. 4 (2015), 383-392..

[3] B. P. Allahverdiev, E. Bairamov, E. Uğurlu; *Eigenparameter depen-dent Sturm-Liouville problems in boundary conditions with transmission con-ditions*, J. Math. Anal. Appl. 401(1) (2013), 388-396.

[4] K. Aydemir, O. Sh. Mukhtarov; *Second-order differential operators with interior singularity*, Advances in Difference Equations (2015) 2015:26 DOI 10.1186/s13662-015-0360-7.

[5] J. Ao, J. Sun, M. Zhang; *Matrix representations of Sturm-Liouville problems with transmission conditions*, Comput. Math. Appl., 63 (2012), 1335-1348.

[6] C. T. Fulton; *Two-point boundary value problems with eigenvalue parameter contained in the boundary conditions,* Proc. Roy. Soc. Edin. 77A (1977), 293-308.

[7] F. Hıra, N. Altınışık; *Sampling theorems for Sturm-Liouville problem with moving discontinuity points*, Boundary Value Problems (2015) 2015:237.

[8] M. Kandemir; *Irregular boundary value problems for elliptic differential-operator equations with discontinuous coefficients and transmission conditions*, Kuwait J. Sci. Eng. 39 (1A) (2010),71-97.

[9] M. Kadakal, O. Sh. Mukhtarov; *Sturm-Liouville problems with discontinuities at two points*, Computers and Mathematics with Applications 54(2007) 1367-1379

[10] L. Kun, Z. Zhaowen; *Spectral Properties for Sturm-Liouville Equations with Transmission Conditions*, Acta Mathematica Scientia, 2015,35A(5):910926

[11] B. M. Levitan, I. S. Sargsyan; *Sturm-Liouville and Dirac Operators*, Springer-Verlag New York, 1991.

[12] A. V. Likov, Yu. A. Mikhailov; *The theory of Heat and Mass Transfer*, Qosenergaizdat, 1963 (Russian).

[13] Kh. R. Mamedov, F.A. Cetinkaya; *Inverse problem for a class of Sturm-Liouville operator with spectral parameter in boundary condition*, Boundary Value Problems 2013, 2013:183.

[14] O. Sh. Mukhtarov, H. Olğar, K. Aydemir; *Resolvent Operator and Spectrum of New Type Boundary Value Problems*, Filomat, 29:7 , 1671-1680 (2015).

[15] O. Sh. Mukhtarov, M. Kadakal; *Some spectral properties of one Sturm-Liouville type problem with discontinuous weight*, Sib. Math. J., Vol. 46 (2005), 681-694.

[16] O. Sh. Mukhtarov, K. Aydemir; *New type Sturm-Liouville problems in associated Hilbert spaces*, Journal of Function Spaces and Applications, vol. 2014, Article ID 606815.

[17] O. Sh. Mukhtarov, K. Aydemir, H. Olğar; *Spectral theory of one perturbed boundary value problem with interior singularities* IP Conference Proceedings; AIP Conference Proceedings; Vol. 1611(2014), p283.

[18] O. Sh. Mukhtarov, H. Olğar, K. Aydemir; *Transmission problems for the Sturm-Liouville equation involving an abstract linear operator* IP Conference Proceedings; Vol. 1611 (2014), 325-332.

[19] M. L. Rasulov; *Methods of Contour Integration*, North-Holland Publishing Company, Amsterdam, 1967.

[20] M. Shahriari, A. j. Akbarfam, G. Teschl; *Uniqueness for inverse Sturm-Liouville problems with a finite number of transmission conditions*, J. Math. Anal. Appl. Vol. 395(2012), 19-29.

[21] A. N. Tikhonov, A. A. Samarskii; *Equations of Mathematical Physics*, Oxford and New York, Pergamon, 1963.

[22] E. C. Titchmarsh; *Eigenfunctions Expansion Associated with Second Order Differential Equations I*, second edn. Oxford Univ. Press, London, 1962.

[23] I. Titeux, Ya. Yakubov; *Completeness of root functions for thermal conduction in a strip with piecewise continuous coefficients*, Math. Models Methods Appl. Sc., 7(7)(1997), 1035-1050.

[24] E. Uğurlu, B. P. Allahverdiev; *On selfadjoint dilation of the dissipative extension of a direct sum differential operator*, Banach Journal of Mathematical Analysis, 7(2)(2013), 194-207 .

[25] N. N. Voitovich , B. Z. Katsenelbaum, A. N. Sivov; *Generalized Method of Eigen-vibration in the theory of Diffraction*, Nakua, Mockow, 1997 (Russian).

[26] A. Wang, A. Zettl; *Green's Function for Two-Interval SturmLiouville problems* , Electronic Journal of Differential Equations, Vol. 2013 (2013), No. 76, pp. 1-13.
[27] A. Wang, J. Sun, A. Zettl; *Two-interval SturmLiouville operators in modified Hilbert spaces*, J. Math. Anal. Appl. 328(2007), 390-399.
[28] A. Zettl; *Sturm-Liouville Theory*, American Mathematical Society, Mathematical Surveys and Monographs, vol. 121, 2005.
[29] X. Zhanga, J.Suna; *The determinants of fourth order dissipative operators with transmission conditions*, J. Math. Anal. Appl., 410(2014), 55-69.

Kadriye Aydemir
Faculty of Education, Giresun University, 28100 Giresun, Turkey
E-mail address: kadriyeaydemr@gmail.com

Oktay Sh. Mukhtarov
Department of Mathematics, Faculty of Arts and Science, Gaziosmanpaşa University, 60250 Tokat, Turkey.
Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, Baku, Azerbaijan
E-mail address: omukhtarov@yahoo.com