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ON THE SCHRÖDINGER EQUATIONS WITH ISOTROPIC AND ANISOTROPIC FOURTH-ORDER DISPERSION

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Abstract. This article concerns the Cauchy problem associated with the nonlinear fourth-order Schrödinger equation with isotropic and anisotropic mixed dispersion. This model is given by the equation

$$i\partial_t u + \epsilon\Delta u + \delta A u + \lambda |u|^\alpha u = 0, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R},$$

where A is either the operator Δ^2 (isotropic dispersion) or $\sum_{i=1}^d \partial_{x_i x_i x_i x_i}$, $1 \le d < n$ (anisotropic dispersion), and $\alpha, \epsilon, \lambda$ are real parameters. We obtain local and global well-posedness results in spaces of initial data with low regularity, based on weak-L^p spaces. Our analysis also includes the biharmonic and anisotropic biharmonic equation ($\epsilon = 0$); in this case, we obtain the existence of self-similar solutions because of their scaling invariance property. In a second part, we analyze the convergence of solutions for the nonlinear fourth-order Schrödinger equation

$$i\partial_t u + \epsilon\Delta u + \delta\Delta^2 u + \lambda |u|^\alpha u = 0, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R},$$

as ϵ approaches zero, in the H^2-norm, to the solutions of the corresponding biharmonic equation $i\partial_t u + \delta\Delta^2 u + \lambda |u|^\alpha u = 0$.

1. Introduction

This article is devoted to the study of the Cauchy problem associated with the fourth-order Schrödinger equation in $\mathbb{R}^n \times \mathbb{R}$,

$$\begin{gathered} i\partial_t u + \epsilon\Delta u + \delta A u + f(|u|)u = 0, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{1.1}$$

where the unknown $u(x,t)$ is a complex-valued function in space-time $\mathbb{R}^n \times \mathbb{R}$, $n \ge 1$, u_0 denotes the initial data and ϵ, δ, are real parameters. The operator A is defined by

$$Au = \begin{cases} \Delta^2 u = \Delta\Delta u, & \text{(isotropic dispersion)}, \\ \sum_{i=1}^d u_{x_i x_i x_i x_i},\ 1 \le d < n, & \text{(anisotropic dispersion)}. \end{cases} \tag{1.2}$$

The nonlinear term is given by $f(|u|)u$, where $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$|f(x) - f(y)| \le C_f |x - y|(|x|^{\alpha-1} + |y|^{\alpha-1}), \tag{1.3}$$

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for some $1 \leq \alpha < \infty$, $f(0) = 0$, and the constant $C_f > 0$ is independent of $x, y \in \mathbb{R}$. A typical case of a function f is $f(x) = |x|^\alpha$.

The class of fourth-order Schrödinger equations has been widely used in many branches of applied science such as nonlinear optics, deep water wave dynamics, plasma physics, superconductivity, quantum mechanics and so on [1, 9, 23, 24, 26, 27, 37]. If we consider $\epsilon = 0$ in (1.1), the resulting equation is the fourth-order nonlinear Schrödinger equation

$$i\partial_t u + \delta A u + f(|u|)u = 0. \tag{1.4}$$

In particular, if we take $A = \Delta^2$ in (1.4) we obtain the well-known biharmonic equation

$$i\partial_t u + \delta \Delta^2 u + f(|u|)u = 0, \tag{1.5}$$

introduced by Karpman [26], and Karpman and Shagalov [27] to take into account the role played by the higher fourth-order dispersion terms in formation and propagation of intense laser beams in a bulk medium with Kerr nonlinearity [24]. Historically, (1.5) has been extensively studied in Sobolev spaces, see for instance [22, 28, 29, 30, 31, 32, 33, 36, 39] and references therein. Fibich et al [22] established sufficient conditions for the existence of global solutions to (1.5), for $\delta < 0$ and $\delta > 0$, with initial data in $H^2(\Omega)$ being Ω a smooth bounded domain of $\mathbb{R}^n$. Global existence and scattering theory for the defocusing biharmonic equation, in $H^2(\mathbb{R}^n)$, was established in Pausander [30, 31]. Wang in [36] showed the global existence of solutions and a scattering result for biharmonic equation (with a nonlinearity of the form $|u|^p u$) with small initial radial data in the homogeneous Sobolev space $\dot{H}^{s_c}(\mathbb{R}^n)$ and dimensions $n \geq 2$. Here $s_c = \frac{n}{2} - \frac{4}{p}$ and $s_c > -\frac{3n-2}{2n+1}$. The main ingredient of [36] is the improvement of the Strichartz estimatives associated with (1.5) for radial initial data; see also Zhu, Yang and Zhang [39], where some results on blow-up solitons for biharmonic equation are established. More recently, Guo in [17] analyzed the existence of global solutions in Sobolev spaces and the asymptotic behavior for the Cauchy problem associated with (1.5) with combined power-type nonlinearities. Finally, we recall a recent result of Miao et al [28] about the defocusing energy-critical nonlinear biharmonic equation $iu_t + \Delta^2 u = -|u|^{\frac{8}{d-4}} u$, which establishes that any finite energy solution is global and scatters both forward and backward in time for dimensions $d \geq 9$.

When $\epsilon \neq 0$ and A is the biharmonic operator, equation (1.1) corresponds to the following nonlinear Schrödinger equation with isotropic mixed-dispersion:

$$i\partial_t u + \epsilon \Delta u + \delta \Delta^2 u + f(|u|)u = 0. \tag{1.6}$$

This equation was also introduced by Karpman [26], and Karpman and Shagalov [27], and it has been used as a model to investigate the role played by the higher-order dispersion terms, in formation and propagation of solitary waves in magnetic materials where the effective quasi-particle mass becomes infinite. From the mathematical point of view, equation (1.6) has been studied extensively in Sobolev and Besov spaces, see for instance [18, 19, 16, 21, 22] and some references therein. Fibich et al [22] investigated the existence of global solutions to (1.6) in the class $C(\mathbb{R}; H^2(\mathbb{R}^n))$ by using the conservation laws. Moreover, the dynamic of the solutions and numerical simulations were also analyzed. These results were improved by Guo and Cui in [18]. Local well-posedness of the Cauchy problem associated with (1.6) in Sobolev spaces $H^s(\mathbb{R}^n)$, with $f(u) = |u|^\alpha$, $\frac{\alpha}{2} \geq \frac{4}{n}$, $s > s_0 := \frac{n}{2} - \frac{4}{\alpha}$, was obtained by Cui and Guo in [21]. Additionally, by using the local existence and the conservation laws, a global well-posedness results in $H^2(\mathbb{R}^n)$ was also established. In [19] the authors proved some results of local and global well-posedness on Besov spaces for dimensions $1 \le n \le 4$; more exactly, the authors proved that the Cauchy problem associated with (1.6), with $f(u) = |u|^{\alpha}$, is local well possed in $C([-T,T]; \dot{B}^{s_\alpha}_{2,q}(\mathbb{R}^n))$ and $C([-T,T]; B^{s}_{2,q}(\mathbb{R}^n))$ for some $T > 0$, where $s_\alpha = \frac{n}{2} - \frac{4}{\alpha}$, $s > s_\alpha$, $1 \le q \le \infty$. With respect to the global well-posedness in Sobolev space, Guo in [16], considering $f(u) = |u|^{2m}$, and using the I-method, proved the existence of global solutions in $H^s(\mathbb{R}^n)$ for $s > 1 + \frac{mn-9+\sqrt{(4m-mn+7)^2+16}}{4m}$, $4 < mn < 4m+2$.

Another important model considered in (1.1) is given by the case of anisotropic dispersion, that is,

$$i\partial_t u + \epsilon \Delta u + \delta \sum_{i=1}^{d} u_{x_i x_i x_i x_i} + f(|u|)u = 0. \tag{1.7}$$

This model appears in the propagation of ultrashort laser pulses in a planar waveguide medium with anomalous time-dispersion, and the propagation of solitons in fiber arrays (see Wen and Fan [37] and Acevedes *et al* [1]). Results of local and global well-posedness for initial data in H^s-spaces were given in [21] and [38]

In this article we are interested in the local and global well-posedness of the general fourth-order Schrödinger equation outside the framework of finite energy H^s-spaces. More exactly, we analyze the existence of local and global solutions for the Cauchy problem (1.1) in a new class of initial data based on weak-L^p spaces. Weak-L^p spaces, also denoted by $L^{(p,\infty)}$, are natural extensions of Lebesgue spaces L^p, in view of the Chebyshev inequality [4]. They contain singular functions with infinite L^2-mass such as homogeneous functions of degree $-\frac{n}{p}$. However, $L^{(p,\infty)} \subset L^2_{\rm loc}$ for $p > 2$. Making a comparison between weak-L^p spaces and $H^{s,l}$-spaces, it is known that the continuous inclusion $H^{s,l}(\mathbb{R}^n) \subset L^{(p,\infty)}(\mathbb{R}^n)$ holds for $s \ge 0$ and $\frac{1}{p} \ge \frac{1}{l} - \frac{s}{n}$, and $H^{s,l}$-spaces do not contain any weak-L^p spaces if $s \in \mathbb{R}$, $1 \le l \le 2$ and $l \le p$. In particular, $L^{(p,\infty)}(\mathbb{R}^n) \not\subset H^{s,2}(\mathbb{R}^n) = H^s(\mathbb{R}^n)$ for all $s \in \mathbb{R}$, when $p \ge 2$. On the other hand, comparing equations (1.4) with (1.6) and (1.7), we observe that equation (1.4), with $f(|u|) = |u|^{\alpha}$, unlike equations (1.6) and (1.7), is invariant under the group of transformations $u(x,t) \to u_\lambda(x,t)$, where $u_\lambda(x,t) = \lambda^{\frac{4}{\alpha}} u(\lambda x, \lambda^4 t)$, $\lambda > 0$. Solutions which are invariant under the transformation $u \to u_\lambda$ are called self-similar solutions. As pointed out in Dudley et al [10] (see also [13]), self-similarity type properties appear in a wide range of physical situations and they reproduce the structure of a phenomena in different spatio-temporal scales. A universal law governing self-similar scale invariance reveals the existence of internal symmetry and structure in a system. Thus, self-similar solutions naturally provide such a law for system (1.4). In ultrafast nonlinear optics, self-similar dynamics have attracted a lot of interest and constitute an increasing field of research (see [10] and references therein). For instance, in Fermann *et al.* [11] was showed that a type of self-similar parabolic pulse is an asymptotic solution to a nonlinear Schrödinger equation with gain. In order to obtain self-similar solutions we need to consider a norm $\|\cdot\|$ defined on a space of initial data u_0, which is invariant with respect to the group of transformations $u \to u_\lambda$, that is, $\|u_{0\lambda}\| = \|u_0\|$ for all $\lambda > 0$; therefore u_0 must be a homogeneous function of degree $-\frac{4}{\alpha}$. However, H^s-spaces are not well adapted for studying this kind of solutions. This fact represents an additional motivation to study the existence of global solutions of the Cauchy problem associated with (1.4) with initial data outside H^s-spaces, by using norms based on $L^{(p,\infty)}$. As consequence, the existence of forward self-similar solutions for (1.4) is obtained by assuming u_0 a sufficiently small homogeneous function of degree $-\frac{4}{\lambda}$. Because equation appearing in (1.1) does not verify any scaling symmetries (in particular equations (1.6) and (1.7)), it is not likely to possess self-similar solutions. However, by using time decay estimates for the respective fourth-order Schrödinger group in weak-L^p spaces, we are able to obtain a result of existence of global solutions for the Cauchy problem (1.1) in a class of function spaces generated by the scaling of the biharmonic equation (1.5) with $f(|u|) = |u|^\alpha$. In relation to the existence of local in time solutions for (1.1) and in particular, the Cauchy problem associated with the equation (1.4), we will prove a result of existence and uniqueness for a large class of singular initial data, which includes homogeneous functions of degree $-\frac{n}{p}$ for adequate values of p. The solutions obtained here can be physically interesting because, as was said, elements of $L^{(p,\infty)}$ have local finite L^2-mass (that is, they belong to $L^2_{\rm loc}$), for $p > 2$. In addition, for initial data in $H^s(\mathbb{R}^n)$, the corresponding solution belongs to $H^s(\mathbb{R}^n)$, which shows that the constructed data-solution map in $L^{(p,\infty)}$ recovers the H^s-regularity and it is compatible with the H^s-theory.

It is worthwhile to remark that the existence of local and global solutions for dispersive equations with initial data outside the context of finite L^2-mass, such as weak-L^r spaces, has been analyzed for the classical Schrödinger equation, coupled Schrödinger equations, Davey-Stewartson system, which are models characterized by having scaling relation (cf. [5, 13, 15, 35]). Existence of solutions in the framework of weak-L^r spaces for models which have no scaling relation, have been explored in the case of Boussinesq and Schorödinger-Boussinesq system in [2, 12] and more recently, in the context of Klein-Gordon-Schrödinger system [3].

To state our results, we establish the definition of mild solution for the Cauchy problem (1.1). A mild solution for (1.1) is a function u satisfying the integral equation

$$u(x,t) = G_{\epsilon,\delta}(t)u_0(x) + i\int_0^t G_{\epsilon,\delta}(t-\tau)f(|u(x,\tau)|)u(x,\tau)d\tau, \tag{1.8}$$

where $G_{\epsilon,\delta}(t)$ is the free group associated with the linear Fourth-order Schrödinger equation, that is,

$$G_{\epsilon,\delta}(t)\varphi = \begin{cases} J_{\epsilon,\delta}(\cdot,t) * \varphi, & \text{if } A = \Delta^2, \\ I_{\epsilon,\delta}(\cdot,t) * \varphi, & \text{if } A = \sum_{i=1}^d \partial_{x_ix_ix_ix_i}, \end{cases} \tag{1.9}$$

for all $\varphi \in \mathcal{S}'(\mathbb{R}^n)$, where

$$\begin{aligned} J_{\epsilon,\delta}(x,t) &= (2\pi)^{-n}\int_{\mathbb{R}^n} e^{ix\xi - it\left(\epsilon|\xi|^2-\delta|\xi|^4\right)}d\xi \\ I_{\epsilon,\delta}(x,t) &= \Big((2\pi)^{-d}\prod_{j=1}^d\int_{\mathbb{R}} e^{ix_j\xi_j - it(\epsilon\xi_j^2-\delta\xi_j^4)}d\xi_j\Big) \\ &\quad\times\Big((2\pi)^{-(n-d)}\prod_{j=d+1}^n\int_{\mathbb{R}} e^{ix_j\xi_j - it\epsilon\xi_j^2}d\xi_j\Big) \\ &\equiv I^1_{\epsilon,\delta}(x,t)I^2_{\epsilon,\delta}(x,t). \end{aligned}$$

Before to precise our results, briefly we recall some notation and facts about Lorentz spaces, see Bergh and Löfström [4], which will be our scenario to establish existence results. Lorentz spaces $L^{(p,d)}$ are defined as the set of measurable function g on $\mathbb{R}^n$ such that the quantity

$$\|g\|_{(p,d)} = \begin{cases} \left(\frac{p}{d}\int_0^\infty [t^{1/p}g^{**}(t)]^d \frac{dt}{t}\right)^{1/d}, & \text{if } 1 < p < \infty,\ 1 \le d < \infty, \\ \sup_{t>0} t^{1/p}g^{**}(t), & \text{if } 1 < p \le \infty,\ d = \infty, \end{cases}$$

is finite. Here $g^{**}(t) = \frac{1}{t}\int_0^t g^*(s)\, ds$ and

$$g^*(t) = \inf\{s > 0 : \mu(\{x \in \Omega : |g(x)| > s\}) \le t\}, \quad t > 0,$$

with μ denoting the Lebesgue measure. In particular, $L^p(\Omega) = L^{(p,p)}(\Omega)$ and, when $d = \infty$, $L^{(p,\infty)}(\Omega)$ are called weak-L^p spaces. Furthermore, $L^{(p,d_1)} \subset L^p \subset L^{(p,d_2)} \subset L^{(p,\infty)}$ for $1 \le d_1 \le p \le d_2 \le \infty$. In particular, weak-$L^p$ spaces contain singular functions with infinite L^2-mass such as homogeneous functions of degree $-\frac{n}{p}$. Finally, a helpful fact about Lorentz spaces is the validity of the Hölder inequality, which reads

$$\|gh\|_{(r,s)} \le C(r)\|g\|_{(p_1,d_1)}\|h\|_{(p_2,d_2)},$$

for $1 < p_1 \le \infty$, $1 < p_2$, $r < \infty$, $\frac{1}{p_1} + \frac{1}{p_2} < 1$, $\frac{1}{r} = \frac{1}{p_1} + \frac{1}{p_2}$, and $s \ge 1$ satisfies $\frac{1}{d_1} + \frac{1}{d_2} \ge \frac{1}{s}$.

In this paper we obtain new results for the existence of local and global solutions to Schrödinger equations with isotropic and anisotropic fourth-order dispersion. First, we prove the existence of local-in-time solutions to the integral equation (1.8) (see Theorem 3.1). For the existence of local solutions, fixed $0 < T < \infty$, we consider the space $\mathcal{G}_\beta^T$ of Bochner measurable functions $u : (-T, T) \to L^{(p(\alpha+1),\infty)}$ such that

$$\|u\|_{\mathcal{G}_\beta^T} = \sup_{-T<t<T} |t|^\beta \|u(t)\|_{(p(\alpha+1),\infty)},$$

where

$$\beta = \begin{cases} \frac{n\alpha}{4p(\alpha+1)}, & \text{if } A = \Delta^2, \\ \frac{(2n-d)\alpha}{4p(\alpha+1)}, & \text{if } A = \sum_{i=1}^d \partial_{x_ix_ix_ix_i}, \end{cases} \tag{1.10}$$

and p is such that the pair $(\frac{1}{p}, \frac{1}{p(\alpha+1)})$ belongs to the set $\Xi_0 \setminus \partial\Xi_0$ where Ξ_0 is the quadrilateral $R_0P_0BQ_0$, with $B = (1,0)$, $P_0 = (2/3, 0)$, $Q_0 = (1, 1/3)$ and $R_0 = (1/2, 1/2)$. The exponent β in (1.10), and the restriction of p, correspond to the time decay of the group $G_{\epsilon,\delta}(t)$ on Lorentz spaces (see Proposition 2.3 below). The initial data is such that $\|G_{\epsilon,\delta}(t)u_0\|_{\mathcal{G}_\beta^T}$ is finite. As a consequence, some results of local existence in Sobolev spaces can be recovered (see Remark 3.2).

We also analyze the existence of global-in-time solutions (see Theorem 3.4). For that we define the space $\mathcal{G}_\sigma^\infty$ as the set of Bochner measurable functions $u : (-\infty, \infty) \to L^{(\alpha+2,\infty)}$ such that

$$\|u\|_{\mathcal{G}_\sigma^\infty} = \sup_{-\infty<t<\infty} |t|^\sigma \|u(t)\|_{(\alpha+2,\infty)} < \infty,$$

where σ is given by

$$\sigma = \begin{cases} \frac{1}{\alpha} - \frac{n}{4(\alpha+2)}, & \text{if } A = \Delta^2, \\ \frac{1}{\alpha} - \frac{2n-d}{4(\alpha+2)}, & \text{if } A = \sum_{i=1}^d \partial_{x_ix_ix_ix_i}. \end{cases} \tag{1.11}$$

Observe that the value $\sigma = \frac{1}{\alpha} - \frac{n}{4(\alpha+2)}$ in (1.11) is the unique one such that the norm $\|u\|_{\mathcal{G}_\sigma^\infty}$ becomes invariant by the scaling of biharmonic equation with $f(u) = |u|^\alpha$. In order to obtain existence of global solutions, we consider the following class of initial data

$$\mathcal{D}_\sigma \equiv \{\varphi \in \mathcal{S}'(\mathbb{R}^n) : \sup_{-\infty<t<\infty} t^\sigma \|G_{\epsilon,\delta}(t)\varphi\|_{(\alpha+2,\infty)} < \infty\}. \tag{1.12}$$

Consequently, if we consider the biharmonic or anisotropic biharmonic equation, that is, $\epsilon = 0$ in (1.1), we obtain the existence of self-similar solutions by assuming u_0 a sufficiently small homogeneous function of degree $-\frac{4}{\alpha}$ (see Corollary 3.6).

As it was said, formally, when we drop the second order dispersion term in $i\partial_t u + \epsilon\Delta u + \delta\Delta^2 u + f(|u|)u = 0$, that is, taking $\epsilon = 0$, we obtain the biharmonic equation $i\partial_t u + \delta\Delta^2 u + f(|u|)u = 0$. However, to the best of our knowledge, the vanishing second order dispersion limit has not been addressed. We observe that the analysis of vanishing dispersion limits can be seen as an interesting issue in dispersive PDE theory, because it permits to describe qualitative properties between different models. We recall, for instance, that in fluid mechanics, the vanishing viscosity limit of the incompressible Navier-Stokes equations is a classical issue [14, 25]. This is the motivation of the second aim of this paper. We study the convergence as ϵ goes to zero, in the H^2-norm, of the solution of Cauchy problem (1.1), with $A = \Delta^2$, to the corresponding Cauchy problem associated with the biharmonic equation (1.5). In the anisotropic case, that is, $A = \sum_{i=1}^d \partial_{x_i x_i x_i x_i}$, the vanishing second order dispersion limit is not clear, because we are not able to bound $\|\nabla u_\epsilon\|_{L^2}$ or $\|u_\epsilon\|_{H^1}^2 + \sum_{i=1}^d \|u_{\epsilon_{x_i x_i}}\|_{L^2}^2$ in terms of the conserved quantities associated to (1.1) and independently of ϵ (see Remark 4.3). This is an interesting question to be considered as future research.

The rest of this article is organized as follows. In Section 2 we establish some linear and nonlinear estimates which are fundamental for obtaining our results of local and global mild solutions. In Section 3 we state and prove our results of local and global solutions. Finally, in Section 4, we give a result about vanishing second order dispersion limit.

2. Linear and nonlinear estimates

In this section we establish some linear and nonlinear estimates which are fundamental for obtain our results of local and global mild solutions. We start by rewriting Theorem 2, Section 3, of Cui [6] for the case $n = 1$ and Theorem 2, Section 3, of Cui [7] for the case $n \geq 2$ (see also Lemma 2.1 in Guo and Cui [20, 21]). For this purpose we denote Ξ_0 the quadrilateral $R_0P_0BQ_0$ in the $(1/p, 1/q)$ plane, where

$$B = (1,0), \quad P_0 = (2/3, 0), \quad Q_0 = (1, 1/3), \quad R_0 = (1/2, 1/2).$$

Ξ_0 comprises the apices B, R_0 and all the edges BP_0, BQ_0, P_0R_0 and Q_0R_0, but does not comprise the apices P_0 and Q_0.

Proposition 2.1. *Given $T > 0$ and a pair of positive numbers (p, q) satisfying $(1/p, 1/q) \in \Xi_0$, there exists a constant $C = C(T, p, q) > 0$ such that for any $\varphi \in L^p(\mathbb{R}^n)$ and $-T \leq t \leq T$ it holds*

$$\|G_{\epsilon,\delta}(t)\varphi\|_{L^q} \leq C|t|^{-b_l}\|\varphi\|_{L^p},$$

where

$$b_l = \begin{cases} \frac{n}{4}\big(\frac{1}{p} - \frac{1}{q}\big), & \text{if } A = \Delta^2, \\ \frac{2n-d}{4}\big(\frac{1}{p} - \frac{1}{q}\big), & \text{if } A = \sum_{i=1}^{d} \partial_{x_i x_i x_i x_i}. \end{cases} \tag{2.1}$$

Moreover, if $\epsilon = 0$ the above estimate holds for all $t \neq 0$.

The above inequality is not convenient to obtain a result of global well-posedness because the constant C depends on T. To overcome this problem we establish a different result which follows from a standard scaling argument.

Lemma 2.2. *If $\frac{1}{p}+\frac{1}{p'} = 1$ with $p \in [1, 2]$, then there exists a constant C independent of ϵ, δ and t such that*

$$\|G_{\epsilon,\delta}(t)\varphi\|_{L^{p'}} \leq C|t|^{-b_g}\|\varphi\|_{L^p}, \quad \varphi \in L^p(\mathbb{R}^n),$$

for all $t \neq 0$, where

$$b_g = \begin{cases} \frac{n}{4}\big(\frac{2}{p} - 1\big), & \text{if } A = \Delta^2, \\ \frac{2n-d}{4}\big(\frac{2}{p} - 1\big), & \text{if } A = \sum_{i=1}^{d} \partial_{x_i x_i x_i x_i}. \end{cases} \tag{2.2}$$

Proof. It is clear that $\|G_{\epsilon,\delta}(t)\varphi\|_{L^2} = \|\varphi\|_{L^2}$, in both cases, the isotropic and anisotropic dispersion. Now, for the isotropic case, we define $h(\xi) := \frac{z\xi}{t} - (\epsilon\xi^2 - \delta\xi^4)$ and since $|h^{(4)}(\xi)| = 24$, we can use [34, Proposition VIII. 2] to obtain

$$\Big|\int_{-\infty}^{\infty} e^{ith(\xi)} d\xi\Big| \leq C|t|^{-1/4}.$$

Note that the constant C given above does not depend on ϵ and δ. From Young inequality we have

$$\|G_{\epsilon,\delta}(t)\varphi\|_{L^\infty} \leq C|t|^{-1/4}\|\varphi\|_{L^1}.$$

Then the result follows by real interpolation.

The anisotropic case is obtained in a similar way. Indeed, we only need to note that

$$|I^1_{\epsilon,\delta}(x,t)| \leq C_1|t|^{-d/4} \quad \text{and} \quad |I^1_{\epsilon,\delta}(x,t)| \leq C_2|t|^{-(n-d)/2},$$

where C_1 and C_2 are independent of t, ϵ and δ. Consequently

$$|I_{\epsilon,\delta}(x,t)| = |I^1_{\epsilon,\delta}(x,t) I^2_{\epsilon,\delta}(x,t)| \leq C|t|^{-\frac{2n-d}{4}}.$$

The proof is finished. □

Lemma 2.3. *Let $T > 0$, $1 \leq d \leq \infty$ and $1 \leq p, q \leq \infty$ satisfying $(1/p, 1/q) \in \Xi_0 \setminus \partial\Xi_0$. Then, there exists a positive constant $C = C(T, p, q) > 0$ such that*

$$\|G_{\epsilon,\delta}(t)\varphi\|_{(q,d)} \leq C|t|^{-b_l}\|\varphi\|_{(p,d)}, \tag{2.3}$$

for all $-T \leq t \leq T$ and $\varphi \in L^{(p,d)}$. Here b_l is defined in (2.1). Moreover, if $\epsilon = 0$ the above estimate holds for all $t \neq 0$.

Proof. We prove only the isotropic case; the anisotropic case can be proved in an analogous way. Since Ξ_0 is convex we can chose $(1/p_0, 1/q_0)$, $(1/p_1, 1/q_1) \in \Xi_0$ such that $\frac{1}{p} = \frac{\theta}{p_0} + \frac{1-\theta}{p_1}$ and $\frac{1}{q} = \frac{\theta}{q_0} + \frac{1-\theta}{q_1}$, with $0 < \theta < 1$. From Proposition 2.1 we have $G_{\epsilon,\delta}(t) : L^{p_0} \to L^{q_0}$ and $G_{\epsilon,\delta}(t) : L^{p_1} \to L^{q_1}$, with norms bounded by

$$\|G_{\epsilon,\delta}(t)\|_{p_0\to q_0} \leq C|t|^{-n/4(1/p_0-1/q_0)},$$
$$\|G_{\epsilon,\delta}(t)\|_{p_1\to q_1} \leq C|t|^{-n/4(1/p_1-1/q_1)}.$$

Since $L^p = L^{(p,p)}$, using real interpolation we obtain

$$
\begin{aligned}
\|G_{\epsilon,\delta}(t)\|_{(p,d)\to(q,d)} &\le C|t|^{-n/4(1/p_0-1/q_0)\theta}|t|^{-n/4(1/p_1-1/q_1)(1-\theta)}\\
&= C|t|^{-n/4(1/p-1/q)},
\end{aligned}
$$

which completes the proof. □

In the same spirit of Lemma 2.3 one can obtain the next result, which gives a linear estimate in Lorentz spaces. The proof follows from Lemma 2.2 and real interpolation. We omit it.

Lemma 2.4. *Let $1 \le d \le \infty$, $1 < p < 2$ and p' such that $\frac{1}{p} + \frac{1}{p'} = 1$. Then, there exists a positive constant C such that*

$$
\|G_{\epsilon,\delta}(t)\varphi\|_{(p',d)} \le C|t|^{-b_g}\|\varphi\|_{(p,d)}, \tag{2.4}
$$

for all $t \neq 0$ and $\varphi \in L^{(p,d)}$. Here b_g is defined in (2.2).

For the rest of this article, we denote the nonlinear part of the integral equation (1.8) by

$$
\mathcal{F}(u) = i\int_0^t G_{\epsilon,\delta}(t-\tau)f(|u(x,\tau)|)u(x,\tau)d\tau.
$$

In the next lemma we estimate the nonlinear term $\mathcal{F}(u)$ in the norm $\|\cdot\|_{\mathcal{G}_\sigma^\infty}$, which is crucial in order to obtain existence of global mild solutions.

Lemma 2.5. *Let $1 \le \alpha < \infty$ and assume that $(\alpha+1)\sigma < 1$. Then*

(1) If $\frac{n\alpha}{4(\alpha+2)} < 1$ and $A = \Delta^2$, then there exists a constant $C_1 > 0$ such that

$$
\begin{aligned}
&\|\mathcal{F}(u)-\mathcal{F}(v)\|_{\mathcal{G}_\sigma^\infty}\\
&\le C_1 \sup_{-\infty<t<\infty}|t|^{\sigma}\|u-v\|_{(\alpha+2,\infty)} \sup_{-\infty<t<\infty}|t|^{\alpha\sigma}\big[\|u\|^{\alpha}_{(\alpha+2,\infty)} + \|v\|^{\alpha}_{(\alpha+2,\infty)}\big],
\end{aligned} \tag{2.5}
$$

for all u, v such that the right hand side of (2.5) is finite.

(2) If $\frac{(2n-d)\alpha}{4(\alpha+2)} < 1$ and $A = \sum_{i=1}^d \partial_{x_ix_ix_ix_i}$, then there exists a constant $C_2 > 0$ such that

$$
\begin{aligned}
&\|\mathcal{F}(u)-\mathcal{F}(v)\|_{\mathcal{G}_\sigma^\infty}\\
&\le C_2 \sup_{-\infty<t<\infty}|t|^{\sigma}\|u-v\|_{(\alpha+2,\infty)} \sup_{-\infty<t<\infty}|t|^{\alpha\sigma}\big[\|u\|^{\alpha}_{(\alpha+2,\infty)} + \|v\|^{\alpha}_{(\alpha+2,\infty)}\big],
\end{aligned} \tag{2.6}
$$

for all u, v such that the right hand side of (2.6) is finite.

Proof. Without loss of generality we consider only the case $t > 0$. Using Lemma 2.4, the property of f established in (1.3), and the Hölder inequality, we have

$$
\begin{aligned}
\|\mathcal{F}(u)-\mathcal{F}(v)\|_{(p',\infty)} &\le C\int_0^t (t-\tau)^{-\frac{n(2-p)}{4p}}\|f(|u|)u - f(|v|)v\|_{(p,\infty)}d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{n(2-p)}{4p}}\||u-v|(|u|^{\alpha}+|v|^{\alpha})\|_{(p,\infty)}d\tau\\
&\le C\int_0^t (t-\tau)^{-\frac{n(2-p)}{4p}}\|u-v\|_{(p',\infty)}\big[\|u\|^{\alpha}_{(p',\infty)} + \|v\|^{\alpha}_{(p',\infty)}\big]d\tau.
\end{aligned}
$$

Since $\frac{1}{p} + \frac{1}{p'} = 1$ and we used the Hölder inequality, we obtain the restriction $p' = \alpha + 2$. Hence

$$
\|\mathcal{F}(u)-\mathcal{F}(v)\|_{(\alpha+2,\infty)}
$$

$$
\begin{aligned}
&\le C\int_0^t (t-\tau)^{-\frac{n\alpha}{4(\alpha+2)}}\|u-v\|_{(\alpha+2,\infty)}\big[\|u\|^{\alpha}_{(\alpha+2,\infty)}+\|v\|^{\alpha}_{(\alpha+2,\infty)}\big]d\tau\\
&\le C\sup_{t>0} t^{\sigma}\|u-v\|_{(\alpha+2,\infty)}\sup_{t>0} t^{\alpha\sigma}\big[\|u\|^{\alpha}_{(\alpha+2,\infty)}+\|v\|^{\alpha}_{(\alpha+2,\infty)}\big]t^{-\sigma}t^{1-\frac{n\alpha}{4(\alpha+2)}-\sigma\alpha}.
\end{aligned}
$$

From $1-\frac{n\alpha}{4(\alpha+2)}-\sigma\alpha=0$, we conclude that

$$
\begin{aligned}
&t^{\sigma}\|\mathcal{F}(u)-\mathcal{F}(v)\|_{(\alpha+2,\infty)}\\
&\le C\sup_{t>0} t^{\sigma}\|u-v\|_{(\alpha+2,\infty)}\sup_{t>0} t^{\alpha\sigma}\big[\|u\|^{\alpha}_{(\alpha+2,\infty)}+\|v\|^{\alpha}_{(\alpha+2,\infty)}\big].
\end{aligned}
\tag{2.7}
$$

Taking the supremum in (2.7) we conclude the proof of the estimate (2.5). The proof of (2.6) follows in a similar way. □

In the next lemma we estimate the nonlinear term $\mathcal{F}(u)$ in the norm $\|\cdot\|_{\mathcal{G}^T_\beta}$, which is crucial in order to obtain existence of local-in-time mild solutions. Here we use the notation $A\lesssim B$ which means that there exists a constant $c>0$ such that $A\le cB$.

Lemma 2.6. *Let $1\le\alpha<\infty$, and $(1/p,1/(\alpha+1)p)\in\Xi_0\setminus\partial\Xi_0$.*

(1) If $\frac{n\alpha}{4p}<1$ and $A=\Delta^2$, then there exists a constant $C_3>0$ such that

$$
\begin{aligned}
&\|\mathcal{F}(u)-\mathcal{F}(v)\|_{\mathcal{G}^T_\beta}\\
&\le C_3\sup_{-T<t<T}|t|^{\beta}\|u-v\|_{((\alpha+1)p,\infty)}\sup_{-T<t<T}|t|^{\beta\alpha}\Big[\|u\|^{\alpha}_{((\alpha+1)p,\infty)}\\
&\quad+\|v\|^{\alpha}_{((\alpha+1)p,\infty)}\Big]T^{1-\beta(\alpha+1)},
\end{aligned}
\tag{2.8}
$$

for all u,v such that the right hand side of (2.8) is finite.

(2) If $\frac{(2n-d)\alpha}{4p}<1$ and $A=\sum_{i=1}^d\partial_{x_ix_ix_ix_i}$, then there exists a constant $C_4>0$ such that

$$
\begin{aligned}
&\|\mathcal{F}(u)-\mathcal{F}(v)\|_{\mathcal{G}^T_\beta}\\
&\le C_4\sup_{-T<t<T}|t|^{\beta}\|u-v\|_{((\alpha+1)p,\infty)}\sup_{-T<t<T}|t|^{\beta\alpha}\Big[\|u\|^{\alpha}_{((\alpha+1)p,\infty)}\\
&\quad+\|v\|^{\alpha}_{((\alpha+1)p,\infty)}\Big]T^{1-\beta(\alpha+1)},
\end{aligned}
\tag{2.9}
$$

for all u,v such that the right hand side of (2.9) is finite.

Proof. We only prove the first inequality; the proof of the second one is analogous. Without loss of generality suppose that $t>0$. Then, from Lemma 2.3, the property of f established in (1.3) and the Hölder inequality, we obtain

$$
\begin{aligned}
\|\mathcal{F}(u)-\mathcal{F}(v)\|_{(q,\infty)}&\le\int_0^t (t-\tau)^{-b_l}\|f(|u|)u-f(|v|)v\|_{(p,\infty)}d\tau\\
&\le C\int_0^t (t-\tau)^{-b_l}\||u-v|(|u|^{\alpha}+|v|^{\alpha})\|_{(p,\infty)}d\tau\\
&\le C\int_0^t (t-\tau)^{-b_l}\|u-v\|_{(q,\infty)}\big(\|u\|^{\alpha}_{(q,\infty)}+\|v\|^{\alpha}_{(q,\infty)}\big)d\tau.
\end{aligned}
$$

Since we used the Hölder inequality the next restriction appears $q=(\alpha+1)p$. Therefore,

$$
\|\mathcal{F}(u)-\mathcal{F}(v)\|_{((\alpha+1)p,\infty)}
$$

$$
\begin{aligned}
&\lesssim \int_0^t (t-\tau)^{-\frac{n\alpha}{4p(\alpha+1)}} \|u-v\|_{((\alpha+1)p,\infty)} \big(\|u\|^{\alpha}_{((\alpha+1)p,\infty)} + \|v\|^{\alpha}_{((\alpha+1)p,\infty)}\big) d\tau \\
&\lesssim \sup_{0<t<T} t^{\beta} \|u-v\|_{((\alpha+1)p,\infty)} \sup_{0<t<T} t^{\alpha\beta} \big[\|u\|^{\alpha}_{((\alpha+1)p,\infty)} + \|v\|^{\alpha}_{((\alpha+1)p,\infty)}\big] t^{1-\beta(\alpha+2)}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
& t^{\beta}\|\mathcal{F}(u)-\mathcal{F}(v)\|_{((\alpha+1)p,\infty)} \\
& \leq C \sup_{0<t<T} t^{\beta}\|u-v\|_{((\alpha+1)p,\infty)} \\
& \quad \times \sup_{0<t<T} t^{\beta\alpha}\big[\|u\|^{\alpha}_{((\alpha+1)p,\infty)} + \|v\|^{\alpha}_{((\alpha+1)p,\infty)}\big] T^{1-\beta(\alpha+1)}.
\end{aligned}
$$

Taking supremum on t in the last inequality, we obtain the desired result. $\square$

3. Local and global solutions

In this section we prove some results of local and global well-posedness for the Schrödinger equations with isotropic and anisotropic fourth-order dispersion in the setting of Lorentz spaces.

3.1. Local-in-time solutions.

Theorem 3.1 (Local-in-time solutions). *Let $1 \leq \alpha < \infty$, and $(1/p, 1/(\alpha+1)p) \in \Xi_0 \setminus \partial\Xi_0$. Consider $\frac{n\alpha}{4p} < 1$ if $A = \Delta^2$, or $\frac{(2n-d)\alpha}{4p} < 1$ if $A = \sum_{i=1}^{d} \partial_{x_i x_i x_i x_i}$. If $u_0 \in \mathcal{S}'(\mathbb{R}^n)$ such that $\|G_{\epsilon,\delta}(t)u_0\|_{\mathcal{G}^T_\beta}$ is finite, then there exists $0 < T^* \leq T < \infty$ such that the initial value problem (1.1) has a mild solution $u \in \mathcal{G}^{T^*}_\beta$, satisfying $u(t) \rightharpoonup u_0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $t \to 0^+$. The solution u is unique in a given ball of $\mathcal{G}^{T^*}_\beta$, and the data-solution map $u_0 \mapsto u$ into $\mathcal{G}^{T^*}_\beta$ is Lipschitz.*

Remark 3.2. (i) (Large class of initial data) From the definition of the norm $\|\cdot\|_{\mathcal{G}^T_\beta}$ and Lemma 2.3, if we take $u_0 \in L^{(p,\infty)}$, the quantity $\|G_{\epsilon,\delta}(t)u_0\|_{\mathcal{G}^T_\beta}$ is finite.

(ii) (Regularity) If the initial data is such that

$$
\sup_{-T<t<T} |t|^{\beta} \|G_{\epsilon,\delta}(t)u_0\|_{(p(\alpha+1),d)} < \infty,
$$

for $1 \leq d < \infty$, then the local mild solution satisifes

$$
\sup_{-T^*<t<T^*} |t|^{\beta} \|u\|_{(p(\alpha+1),d)} < \infty,
$$

(possibly reducing the time of existence T^*).

(iii) (Finite energy solutions) From Theorem 3.1 some results of local existence in Sobolev spaces can be recovered. For that, notice that $H^s(\mathbb{R}^n) \hookrightarrow L^{(p(\alpha+1),\infty)}$, for $s > 0$ such that $2 < p(\alpha+1) \leq \frac{2n}{n-2s}$ if $n > 2s$ ($2 < p(\alpha+1) \leq \infty$ if $n < 2s$). Therefore, if $u_0 \in H^s$, then

$$
\begin{aligned}
\|G_{\epsilon,\delta}(t)u_0\|_{\mathcal{G}^T_\beta} &\leq C \sup_{-T<t<T} |t|^{\beta} \|G_{\epsilon,\delta}(t)u_0\|_{H^s} \\
&\leq C \sup_{-T<t<T} |t|^{\beta} \|u_0\|_{H^s} < \infty.
\end{aligned}
$$

Consequently, by Theorem 3.1 there exists a mild solution $u : (-T^*, T^*) \to L^{(p(\alpha+1))}(\mathbb{R}^n)$ in $\mathcal{G}^{T^*}_\beta$. On the other hand, for the same initial data $u_0 \in H^s(\mathbb{R}^n)$, suppose $v \in C([-T_0, T_0]; H^s(\mathbb{R}^n))$ the unique energy finite solution for some T_0 small enough. By the embedding $H^s \hookrightarrow L^{(p(\alpha+1),\infty)}$, we obtain that $v \in \mathcal{G}^{T_0}_\beta$.

Thus, taking T_0 small enough, the uniqueness of solution given in Theorem 3.1, implies that $u=v$ on $[-T_0,T_0]$ and consequently, $u\in C([-T_0,T_0];H^s)$.

Before proving Theorem 3.1, we enunciate a result related to the existence of radial solutions. First of all, we recall that a solution u in $\mathcal{G}_\beta^T$ is said to be radially symmetric, or simply radial, for a.e. $0<|t|<T$, if $u(Rx,t)=u(x,t)$ a.e. $x\in\mathbb{R}^n$ for all $n\times n$-orthogonal matrix R. Then, we have the following corollary.

Corollary 3.3. *Under the hypotheses of Theorem 3.1, if the initial data u_0 is radially symmetric, then the corresponding solution u is radially symmetric for a.e. $0<|t|<T$.*

Proof of Theorem 3.1. This proof will be obtained as an application of the Banach fixed point theorem. First, notice that by hypothesis on the initial data, we have

$$\|G_{\epsilon,\delta}(t)u_0\|_{\mathcal{G}_\beta^T} := \sup_{-T<t<T} |t|^\beta \|G_{\epsilon,\delta}(t)u_0\|_{(p(\alpha+1),\infty)} \equiv \frac{K}{2} < \infty.$$

We consider the mapping Υ defined by

$$\Upsilon(u(t)) = G_{\epsilon,\delta}(t)u_0 + i\int_0^t G_{\epsilon,\delta}(t-\tau)f(|u(x,\tau)|)u(x,s)d\tau. \tag{3.1}$$

Then, we prove that Υ defines a contraction on (B_K,d) where B_K denotes the closed ball $\{u\in\mathcal{G}_\beta^{T^*} : \|u\|_{\mathcal{G}_\beta^{T^*}} \le K\}$ endowed with the complete metric $d(u,v)=\|u-v\|_{\mathcal{G}_\beta^{T^*}}$ for some $0<T^*\le T$. In fact, let us consider $0<T^*\le T$ such that $\tilde{C}K^\alpha(T^*)^{1-\beta(\alpha+1)} < \frac{1}{2}$ where $\tilde{C}$ denotes the constant C_3 or C_4 in Lemma 2.6. Then, from Lemma 2.6 with $v=0$ we obtain

$$\begin{aligned}
\|\Upsilon(u)\|_{\mathcal{G}_\beta^{T^*}} &\le \|G_{\epsilon,\delta}(t)u_0\|_{\mathcal{G}_\beta^{T^*}} + \|\mathcal{F}(u)\|_{\mathcal{G}_\beta^{T^*}} \\
&\le \frac{K}{2} + \tilde{C}K^{\alpha+1}(T^*)^{1-\beta(\alpha+1)} \le \frac{K}{2}+\frac{K}{2} = K,
\end{aligned}$$

for all $u\in B_K$. Consequently, $\Upsilon(B_K)\subset B_K$. Now, assuming that $u,v\in B_K$, from Lemma 2.6 we obtain

$$\begin{aligned}
\|\Upsilon(u(t))-\Upsilon(v(t))\|_{\mathcal{G}_\beta^{T^*}} &= \|\mathcal{F}(u)-\mathcal{F}(v)\|_{\mathcal{G}_\beta^{T^*}} \\
&\le 2\tilde{C}K^\alpha(T^*)^{1-\beta(\alpha+1)}\|u-v\|_{\mathcal{G}_\beta^{T^*}}.
\end{aligned} \tag{3.2}$$

Thus, as $\tilde{C}K^\alpha(T^*)^{1-\beta(\alpha+1)} < 1/2$, the map Υ is a contraction on (B_K,d). Consequently, the Banach fixed point theorem implies the existence of a unique solution $u\in\mathcal{G}_\beta^{T^*}$. Through standard argument one can prove that $u(t)\to u_0$ as $t\to 0$, in the sense of distributions [15]. On the other hand, in order to prove the local Lipschitz continuity of the data-solution map, we consider u,v two local mild solutions with initial data u_0,v_0, respectively. Then, as in estimate (3.2) we obtain

$$\begin{aligned}
\|u-v\|_{\mathcal{G}_\beta^{T^*}} &= \|G_{\epsilon,\delta}(t)(u_0-v_0)\|_{\mathcal{G}_\beta^{T^*}} + \|\mathcal{F}(u)-\mathcal{F}(v)\|_{\mathcal{G}_\beta^{T^*}} \\
&\le \|G_{\epsilon,\delta}(t)(u_0-v_0)\|_{\mathcal{G}_\beta^{T^*}} + 2\tilde{C}K^\alpha(T^*)^{1-\beta(\alpha+1)}\|u-v\|_{\mathcal{G}_\beta^{T^*}}.
\end{aligned} \tag{3.3}$$

Since $2\tilde{C}K^\alpha(T^*)^{1-\beta(\alpha+1)} < 1$, from the above inequality, the local Lipschitz continuity of the data-solution map holds. □

Proof of Corollary 3.3. From the fixed point argument used in the proof of Theorem 3.1, we can see the local solution u as the limit in $\mathcal{G}_\beta^T$ of the Picard sequence

$$u_1 = G_{\epsilon,\delta}(t)(u_0), \quad u_{k+1} = u_1 + \mathcal{F}(u_k), \quad k \in \mathbb{N}. \tag{3.4}$$

Since the symbol of the group $G_{\epsilon,\delta}(t)$ is radially symmetric for each fixed $0 < t < T$, it follows that $G_{\epsilon,\delta}(t)u_0$ is radial, provided that u_0 is radial. Furthermore, since the nonlinear term $\mathcal{F}(u)$ is radial when u is radial, an induction argument gives that the sequence $\{u_k\}_{k\in\mathbb{N}}$ given in (3.4) is radial. Since pointwise convergence preserves radial symmetry, and $\mathcal{G}_\beta^T$ implies (up to a subsequence) almost everywhere pointwise convergence in the variable x, for a.e. fixed $t \neq 0$, it follows that $u(x,t)$ is radially symmetric. $\square$

3.2. **Global-in-time solutions.**

Theorem 3.4. *Let $1 \le \alpha < \infty$ and assume that $(\alpha+1)\sigma < 1$. Consider either $\frac{n\alpha}{4(\alpha+2)} < 1$ if $A = \Delta^2$, or $\frac{(2n-d)\alpha}{4(\alpha+2)} < 1$ if $A = \sum_{i=1}^d \partial_{x_i x_i x_i x_i}$. Suppose further that $\xi > 0$ and $M > 0$ satisfy the inequality $\xi + \widetilde{C}M^{\alpha+1} \le M$ where $\widetilde{C} = \widetilde{C}(\alpha, n)$ is the constant C_1 or C_2 in Lemma 2.5. If $u_0 \in \mathcal{D}_\sigma$, with $\sup_{t>0} t^\sigma \|G_{\epsilon,\delta}(t)u_0\|_{(\alpha+2,\infty)} < \xi$, then the initial value problem (1.1) has a unique global-in-time mild solution $u \in \mathcal{G}_\sigma^\infty$ with $\|u\|_{\mathcal{G}_\sigma^\infty} \le M$, such that $\lim_{t\to 0} u(t) = u_0$ in distribution sense. Moreover, if u, v are two global mild solutions with respective initial data u_0, v_0, then*

$$\|u - v\|_{\mathcal{G}_\sigma^\infty} \le C \|G_{\epsilon,\delta}(t)(u_0 - v_0)\|_{\mathcal{G}_\sigma^\infty}. \tag{3.5}$$

Additionally, if $G_{\epsilon,\delta}(t)(u_0 - v_0)$ satisifes the stronger decay

$$\sup_{t>0} |t|^\sigma (1+|t|)^\varsigma \|G_\epsilon(t)(u_0 - v_0)\|_{(\alpha+2,\infty)} < \infty,$$

for some $\varsigma > 0$ such that $\sigma(\alpha+1) + \varsigma < 1$, then

$$\begin{aligned} &\sup_{t>0} |t|^\sigma (1+|t|)^\varsigma \|u(t) - v(t)\|_{(\alpha+2,\infty)} \\ &\le C \sup_{t>0} |t|^\sigma (1+|t|)^\varsigma \|G_\epsilon(t)(u_0 - v_0)\|_{(\alpha+2,\infty)}. \end{aligned} \tag{3.6}$$

Remark 3.5. (i) (Regularity) In addition to the assumptions of Theorem 3.4, if we consider that the initial data satisifes

$$\sup_{-\infty<t<\infty} t^\sigma \|G_{0,\delta}(t)u_0\|_{(\alpha+2,d)} < \infty$$

for some $1 \le d < \infty$, then there exists ξ_0 such that if

$$\sup_{-\infty<t<\infty} t^\sigma \|G_{0,\delta}(t)u_0\|_{(\alpha+2,d)} \le \xi_0,$$

then global solution provided in Theorem 3.4 satisfies that

$$\sup_{-\infty<t<\infty} t^\sigma \|u(t)\|_{(\alpha+2,d)} < \infty.$$

(ii) (Radial solutions) As in Corollary 3.3, if the initial data u_0 is radially symmetric, then the global-in-time solution u is radially symmetric for a.e. $t \neq 0$.

(iii) (Asymptotic stability) Following the proof of (3.6) we can obtain that if u, v are global mild solutions of the Cauchy problem (1.1) given by Theorem 3.4, with initial data $u_0, v_0 \in \mathcal{D}_\sigma$ respectively, satisfying

$$\lim_{t\to\infty} t^\sigma (1+t)^\varsigma \|G_\epsilon(t)(u_0 - v_0)\|_{(\alpha+2,\infty)} = 0,$$

then $\lim_{t\to\infty} t^\sigma (1+t)^\varsigma \|u(t) - v(t)\|_{(\alpha+2,\infty)} = 0$.

(iv) (biharmonic and anisotropic biharmonic global solutions) Theorem 3.4 gives existence of global mild solution for Cauchy problem associated with equation (1.4) in the class $\mathcal{G}_\sigma^\infty$. The proof was based on the time-decay estimate of the group $G_{0,\delta}(t)$ given in Lemma 2.4. However, taking into account that if $\epsilon = 0$ the time-decay estimate in Lemma 2.3 holds true for all $t \neq 0$, we are able to prove the existence of global-in-time mild solutions for the Cauchy problem associated with equation (1.4) in the class $G_{\sigma(p)}^\infty$ defined as the set of Bochner measurable functions $u : (-\infty, \infty) \to L^{(p(\alpha+1),\infty)}$ such that

$$\|u\|_{\mathcal{G}_{\sigma(p)}^\infty} = \sup_{-\infty<t<\infty} |t|^{\sigma(p)} \|u(t)\|_{(p(\alpha+1),\infty)} < \infty,$$

where

$$\sigma(p) = \begin{cases} \frac{1}{\alpha} - \frac{n}{4p(\alpha+1)}, & \text{if } A = \Delta^2, \\ \frac{1}{\alpha} - \frac{2n-d}{4p(\alpha+1)}, & \text{if } A = \sum_{i=1}^d \partial_{x_i x_i x_i x_i}. \end{cases} \tag{3.7}$$

Here p, α must verify $1 \leq \alpha < \infty$, $(1/p, 1/(\alpha+1)p) \in \Xi_0 \setminus \partial\Xi_0$ and $\frac{4p}{n\alpha} < 1 < \frac{4p(\alpha+1)}{n\alpha}$ if $A = \Delta^2$ or, $\frac{4p}{(2n-d)\alpha} < 1 < \frac{4p(\alpha+1)}{(2n-d)\alpha}$ if $A = \sum_{i=1}^d \partial_{x_i x_i x_i x_i}$.

Corollary 3.6 (Biharmonic and anisotropic biharmonic self-similar solutions)**.** *Let $\epsilon = 0$, $1 \leq \alpha < \infty$ and assume that $(\alpha+1)\sigma < 1$. Consider either $\frac{n\alpha}{4(\alpha+2)} < 1$ if $A = \Delta^2$, or $\frac{(2n-d)\alpha}{4(\alpha+2)} < 1$ if $A = \sum_{i=1}^d \partial_{x_i x_i x_i x_i}$. Assume that the initial data u_0 is a homogeneous function of degree $\frac{-4}{\alpha}$. Then the solution $u(t,x)$ provided by Theorem 3.4 is self-similar, that is, $u(t,x) = \lambda^{\frac{4}{\alpha}} u(\lambda^4 t, \lambda x)$ for all $\lambda > 0$, almost everywhere for $x \in \mathbb{R}^n$ and $t > 0$.*

Remark 3.7. An admissible class of initial data for the existence of self-similar solutions in Corollary 3.6 is given by the set of functions $u_0(x) = P_m(x)|x|^{-m-\frac{4}{\alpha}}$ where $P_m(x)$ is a homogeneous polynomial of degree m.

Proof of Theorem 3.4. It will be also obtained as an application of the Banach fixed point Theorem. We denote by B_M the set of $u \in \mathcal{G}_\sigma^\infty$ such that

$$\|u\|_{\mathcal{G}_\sigma^\infty} \equiv \sup_{-\infty<t<\infty} |t|^\sigma \|u(t)\|_{(\alpha+2,\infty)} \leq M,$$

endowed with the complete metric $d(u,v) = \sup_{-\infty<t<\infty} |t|^\sigma \|u(t) - v(t)\|_{(\alpha+2,\infty)}$. We will show that the mapping

$$\Upsilon(u(t)) = G_{\epsilon,\delta}(t)u_0 + i\int_0^t G_{\epsilon,\delta}(t-\tau) f(|u(x,\tau)|) u(x,s)\, d\tau, \tag{3.8}$$

is a contraction on (B_M, d). From the assumptions on the initial data and Lemma 2.5 (with $v = 0$), we have (for all $u \in B_M$)

$$\begin{aligned} \|\Upsilon(u)\|_{\mathcal{G}_\sigma^\infty} &\leq \|G_{\epsilon,\delta}(t)u_0\|_{\mathcal{G}_\sigma^\infty} + \|\mathcal{F}(u)\|_{\mathcal{G}_\sigma^\infty} \\ &\leq \xi + \tilde{C}\|u\|_{\mathcal{G}_\sigma^\infty}^{\alpha+1} \\ &\leq \xi + \tilde{C}M^{\alpha+1} \leq M, \end{aligned}$$

because M and ξ satisfy $\xi + \tilde{C}M^{\alpha+1} \leq M$. Thus, Υ maps B_M itself. On the other hand, Lemma 2.5, we obtain

$$\|\Upsilon(u) - \Upsilon(v)\|_{\mathcal{G}_\sigma^\infty} \leq \|\mathcal{F}(u) - \mathcal{F}(v)\|_{\mathcal{G}_\sigma^\infty} \leq 2\tilde{C}M^\alpha \|u - v\|_{\mathcal{G}_\sigma^\infty}. \tag{3.9}$$

Since $\tilde{C}M^\alpha < 1$, it follows that Υ is a contraction on (B_M, d) and consequently, the Banach fixed point theorem implies the existence of a unique global solution $u \in \mathcal{G}_\sigma^\infty$. To prove the continuous dependence of the mild solutions with respect to the initial data, it suffices to observe that (3.9) implies that

$$\|u - v\|_{\mathcal{G}_\sigma^\infty} \le \|G_{\epsilon,\delta}(t)u_0 - G_{\epsilon,\delta}(t)v_0\|_{\mathcal{G}_\sigma} + CM^\alpha\|u - v\|_{\mathcal{G}_\sigma^\infty}.$$

Thus, as $\tilde{C}M^\alpha < 1$, then $\|u - v\|_{\mathcal{G}_\sigma^\infty} \le C\|G_{\epsilon,\delta}(t)u_0 - G_{\epsilon,\delta}(t)v_0\|_{\mathcal{G}_\sigma^\infty}$. Finally, to prove the stronger decay, notice that

$$\begin{aligned}
&t^\sigma(1+t)^\varsigma\|u(t) - v(t)\|_{(\alpha+2,\infty)} \\
&\le C\sup_{t>0} t^\sigma(1+t)^\varsigma\|G_{\epsilon,\delta}(t)(u_0 - v_0)\|_{(\alpha+2,\infty)} + t^\sigma(1+t)^\varsigma\|\mathcal{F}(u) - \mathcal{F}(v)\|_{(\alpha+2,\infty)}.
\end{aligned} \tag{3.10}$$

Since $\|u\|_{\mathcal{G}_\sigma^\infty}, \|v\|_{\mathcal{G}_\sigma^\infty} \le M$, using the change of variable $\tau \mapsto \tau t$ and noting that $(1+t)^\varsigma(1+t\tau)^{-\varsigma} \le t^\varsigma(t\tau)^{-\varsigma}$ for $\tau \in [0,1]$, we obtain

$$\begin{aligned}
&t^\sigma(1+t)^\varsigma\|\mathcal{F}(u) - \mathcal{F}(v)\|_{(\alpha+2,\infty)} \\
&\le t^\sigma(1+t)^\varsigma \int_0^t (t-\tau)^{-\frac{n\alpha}{4(\alpha+2)}}\tau^{-\sigma(\alpha+1)}(1+\tau)^\varsigma \\
&\times (\tau^\sigma(1+\tau)^\varsigma\|u(\tau) - v(\tau)\|_{(\alpha+2,\infty)})\big[\tau^\sigma\|u(\tau)\|^\alpha_{(\alpha+2,\infty)} + \tau^\sigma\|v(\tau)\|^\alpha_{(\alpha+2,\infty)}\big]ds \\
&\le 2M^\alpha \int_0^1 (1-\tau)^{-\frac{n\alpha}{4(\alpha+2)}}\tau^{-\sigma(\alpha+1)}(1+t)^\varsigma(1+t\tau)^{-\varsigma}((t\tau)^\sigma(1+(t\tau))^\varsigma \\
&\quad \times \|u(t\tau) - v(t\tau)\|_{(\alpha+2,\infty)})ds \\
&\le 2M^\alpha \int_0^1 (1-\tau)^{-\frac{n\alpha}{4(\alpha+2)}}\tau^{-\sigma(\alpha+1)}\tau^{-\varsigma}((t\tau)^\sigma(1+(t\tau))^\varsigma \\
&\quad \times \|u(t\tau) - v(t\tau)\|_{(\alpha+2,\infty)})d\tau.
\end{aligned} \tag{3.11}$$

Therefore, by denoting $A = \sup_{t>0} t^\sigma(1+t)^\varsigma\|u(t) - v(t)\|_{(\alpha+2,\infty)}$, from (3.10) and (3.11) we obtain

$$\begin{aligned}
A &\le C\sup_{t>0} t^\sigma(1+t)^\varsigma\|G_{\epsilon,\delta}(t)(u_0 - v_0)\|_{(\alpha+2,\infty)} \\
&\quad + \Big(2M^\alpha \int_0^1 (1-\tau)^{-\frac{n\alpha}{4(\alpha+2)}}\tau^{-\sigma(\alpha+1)}\tau^{-\varsigma}d\tau\Big)A.
\end{aligned}$$

Choosing M small enough such that $2M^\alpha \int_0^1 (1-\tau)^{-\frac{n\alpha}{4(\alpha+2)}}\tau^{-\sigma(\alpha+1)}\tau^{-\varsigma}d\tau < 1$, we conclude the proof. $\square$

Proof of Corollary 3.6. We recall that by the fixed point argument used in the proof of Theorem 3.4, the solution u is the limit in $\mathcal{G}_\sigma^\infty$ of the Picard sequence

$$u_1 = G_{0,\delta}(t)u_0, \quad u_{k+1} = u_1 + \mathcal{F}(u_k), \quad k \in \mathbb{N}. \tag{3.12}$$

Notice that the initial data u_0 satisfying $u_0(\lambda x) = \lambda^{-\frac{4}{\alpha}}u_0(x)$ belongs to the class $\mathcal{D}_\sigma$ (see [15, Corollary 2.6]). Since $\epsilon = 0$, we obtain

$$u_1(\lambda x, \lambda^4 t) = \lambda^{-\frac{4}{\alpha}}u_1(x,t) \tag{3.13}$$

and then u_1 is invariant by the scaling

$$u(x,t) \to u_\lambda(x,t) := \lambda^{\frac{4}{\lambda}}u(\lambda x, \lambda^4 t), \quad \lambda > 0. \tag{3.14}$$

Moreover, the nonlinear term $\mathcal{F}(u)$ is invariant by scaling (3.14) when u is also. Therefore, we can employ an induction argument in order to obtain that all elements u_k have the scaling invariance property (3.14). Because the norm of $\mathcal{G}_\alpha^\infty$ is scaling invariant, we obtain that the limit u also is invariant by the scaling transformation $u \to u_\lambda$, as required. $\square$

4. Vanishing dispersion limit

This section is devoted to the analysis of the solutions of (1.1) as the second order dispersion vanishes. More exactly, we study the convergence, $\epsilon \to 0$, of the solutions of the Cauchy problem

$$\begin{gathered} i\partial_t u + \epsilon\Delta u + \delta A u + \lambda |u|^\alpha u = 0, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n, \end{gathered} \tag{4.1}$$

to the solutions of

$$\begin{gathered} i\partial_t u + \delta A u + \lambda |u|^\alpha u = 0, \quad x \in \mathbb{R}^n,\ t \in \mathbb{R}, \\ u(x,0) = u_0(x), \quad x \in \mathbb{R}^n. \end{gathered} \tag{4.2}$$

in the framework of the $H^2(\mathbb{R}^n)$ space. Throughout this section we consider α as a positive even integer. Before to establish our main results, we give some preliminary facts. First, we recall the following conserved quantities of (4.1):

$$M(u) = \|u\|_{L^2(\mathbb{R}^n)}^2; \tag{4.3}$$

$$E_{\epsilon,\delta,\lambda}(u) = \delta\|\Delta u\|_{L^2}^2 - \epsilon\|\nabla u\|_{L^2}^2 + \frac{2\lambda}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2}, \quad \text{if } A = \Delta^2; \tag{4.4}$$

$$E_{\epsilon,\delta,\lambda}(u) = \delta\sum_{i=1}^d \|u_{x_i x_i}\|_{L^2}^2 - \epsilon\|\nabla u\|_{L^2}^2 + \tfrac{2\lambda}{\alpha+2}\|u\|_{L^{\alpha+2}}^{\alpha+2}, \quad \text{if } A = \sum_{i=1}^d \partial_{x_i x_i x_i x_i}. \tag{4.5}$$

According to the signs of the pair (δ,λ), we have two cases: Case 1: $\delta\lambda > 0$ and $\epsilon \in \mathbb{R}$. Case 2: $\delta\lambda < 0$ and $\epsilon \in \mathbb{R}$. Thus we have the next result.

Proposition 4.1. *Fix $\delta = \pm 1$, $\lambda = \pm 1$ and let $u_\epsilon \in C([-T,T]; H^2(\mathbb{R}^n))$ be the local solution of (4.1) with initial data $u_0 \in H^2(\mathbb{R}^n)$ and $A = \Delta^2$. Assume that*

- *$(\epsilon,\delta,\lambda)$ is as in Case 1 or*
- *$(\epsilon,\delta,\lambda)$ is as in Case 2, $n\alpha < 8$, $\frac{n\alpha}{4(\alpha+2)} \le 1$, if $n \ne 2,4$, and $0 \le \frac{n\alpha}{4(\alpha+2)} < 1$ if $n = 2,4$.*

Then the following estimate holds

$$\|u_\epsilon(t)\|_{H^2(\mathbb{R}^n)} \le C(\|u_0\|_{H^2}, \|u_0\|_{L^{\alpha+2}}). \tag{4.6}$$

Proof. First we consider Case 1. Using the conserved quantities of (4.1) given in (4.3)-(4.4), we obtain

$$\begin{aligned} &\|u_\epsilon(t)\|_{L^2}^2 + \|\Delta u_\epsilon(t)\|_{L^2}^2 \\ &= M(u_0) + \delta^{-1}E_{\epsilon,\delta,\lambda}(u_0) + \delta^{-1}\epsilon\|\nabla u_\epsilon\|_{L^2}^2 - \frac{2\delta^{-1}\lambda}{\alpha+2}\|u_\epsilon\|_{L^{\alpha+2}}^{\alpha+2} \\ &\le M(u_0) + \delta^{-1}E_{\epsilon,\delta,\lambda}(u_0) + \delta^{-1}\epsilon\|\nabla u_\epsilon(t)\|_{L^2}^2. \end{aligned} \tag{4.7}$$

At this point we have to consider two subcases. If $\delta^{-1}\epsilon < 0$, taking $0 < |\epsilon| < \frac{1}{2}$, we arrived at

$$\|u_\epsilon(t)\|_{L^2}^2 + \|\Delta u_\epsilon(t)\|_{L^2}^2 \leq M(u_0) + \delta^{-1} E_{\epsilon,\delta,\lambda}(u_0) \leq M(u_0) + E_{-\frac{1}{2},1,\delta^{-1}\lambda}(u_0).$$

On the other hand, if $\delta^{-1}\epsilon > 0$, from (4.7) we have

$$\begin{aligned}\|u_\epsilon(t)\|_{H^2}^2 &\leq C(\|u_\epsilon(t)\|_{L^2}^2 + \|\Delta u_\epsilon(t)\|_{L^2}^2)\\ &\leq CM(u_0) + CE_{0,1,\delta^{-1}\lambda}(u_0) + \delta^{-1}\epsilon C\|u_\epsilon(t)\|_{H^2}^2.\end{aligned}$$

Again, consider $0 < |\epsilon| < \frac{1}{2C}$ to arrive at

$$\|u_\epsilon(t)\|_{H^2}^2 \lesssim M(u_0) + E_{0,1,\delta^{-1}\lambda}(u_0).$$

In both subcases we obtain the desired result.

Now, we consider the Case 2. Consider the restrictions $n\alpha < 8$, $0 \leq \frac{n\alpha}{4(\alpha+2)} \leq 1$ if $n \neq 2, 4$, and $0 \leq \frac{n\alpha}{4(\alpha+2)} < 1$ if $n = 2, 4$. Thus, by applying the Douglas-Niremberg and Young inequalities we obtain

$$\begin{aligned}&\|u_\epsilon(t)\|_{L^2}^2 + \|\Delta u_\epsilon(t)\|_{L^2}^2\\ &= M(u_0) + \delta^{-1}E_{\epsilon,\delta,\lambda}(u_0) + \delta^{-1}\epsilon\|\nabla u_\epsilon(t)\|_{L^2}^2 - \frac{2\delta^{-1}\lambda}{\alpha+2}\|u_\epsilon(t)\|_{L^{\alpha+2}}^{\alpha+2}\\ &\leq M(u_0) + \delta^{-1}E_{\epsilon,\delta,\lambda}(u_0) + \delta^{-1}\epsilon\|\nabla u_\epsilon(t)\|_{L^2}^2 + C_1\|u_\epsilon(t)\|_{H^2}^{\frac{n\alpha}{4}}\|u_\epsilon(t)\|_{L^2}^{\alpha+2-\frac{n\alpha}{4}}\\ &= M(u_0) + \delta^{-1}E_{\epsilon,\delta,\lambda}(u_0) + \delta^{-1}\epsilon\|\nabla u_\epsilon(t)\|_{L^2}^2 + C_1\|u_\epsilon(t)\|_{H^2}^{\frac{n\alpha}{4}}\|u_0\|_{L^2}^{\alpha+2-\frac{n\alpha}{4}}\\ &\leq M(u_0) + \delta^{-1}E_{\epsilon,\delta,\lambda}(u_0) + \delta^{-1}\epsilon\|\nabla u_\epsilon(t)\|_{L^2}^2 + C_1\mu_0\|u_\epsilon(t)\|_{H^2}^2\\ &\quad + C(\mu_0)\|u_0\|_{L^2}^{\kappa},\end{aligned} \tag{4.8}$$

with $\kappa = \frac{8(\alpha+2)-8n\alpha}{8-n\alpha}$. Taking $0 < \mu_0 < \frac{1}{2C_1}$, from (4.8) we obtain

$$\|u_\epsilon(t)\|_{H^2}^2 \lesssim M(u_0) + \delta^{-1}E_{\epsilon,\delta,\lambda}(u_0) + \delta^{-1}\epsilon\|\nabla u_\epsilon(t)\|_{L^2}^2 + C(\|u_0\|_{L^2}). \tag{4.9}$$

Again, we have two subcases. If $\delta^{-1}\epsilon < 0$, it is easy to see that for $0 < |\epsilon| < \frac{1}{2}$,

$$\|u_\epsilon(t)\|_{H^2}^2 \lesssim M(u_0) + E_{-\frac{1}{2},1,\delta^{-1}\lambda}(u_0) + C(\mu_0, \|u_0\|_{L^2}). \tag{4.10}$$

Finally, if $\delta^{-1}\epsilon > 0$, we use that $\delta^{-1}\epsilon\|\nabla u_\epsilon(t)\|_{L^2}^2 \leq \frac{1}{2}\|u_\epsilon(t)\|_{H^2}^2$ for $0 < |\epsilon| < \frac{1}{2}$ in (4.9) to obtain again inequality (4.10). $\square$

Now we are in a position to establish our main result of this section.

Theorem 4.2. *Consider u_ϵ and u in $C([-T,T]; H^2(\mathbb{R}^n))$, the solutions of (4.1) and (4.2) respectively, with common initial data $u_0 \in H^2(\mathbb{R}^n)$ and $A = \Delta^2$. Here $[-T,T]$ is the common interval of local existence for u_ϵ and u. Suppose $n < 4$, if $\delta\lambda < 0$ assume that $n\alpha < 8$, $\frac{n\alpha}{4(\alpha+2)} \leq 1$, if $n \neq 2$, and $0 \leq \frac{n\alpha}{4(\alpha+2)} < 1$ if $n = 2$. Then*

$$\lim_{\epsilon\to 0}\|u_\epsilon(t) - u(t)\|_{H^2} = 0,$$

for all $t \in [-T,T]$.

Remark 4.3. A version of Theorem 4.2 for the anisotropic dispersion case, $A = \sum_{i=1}^{d}\partial_{x_ix_ix_ix_i}$, by replacing the norm convergence in H^2 by the natural norm $H^2(\mathbb{R}^d)H^1(\mathbb{R}^{n-d})$, is not clear. In fact, we are not able to bound $\|\nabla u_\epsilon\|_{L^2}$ or $\|u_\epsilon\|_{H^1}^2 + \sum_{i=1}^{d}\|u_{\epsilon_{x_ix_i}}\|_{L^2}^2$ in terms of the conserved quantities associated to (4.1) and independently of ϵ.

Proof of Theorem 4.2. As usual, the mild solutions associated with (4.2) satisfy the integral equation

$$u(x,t) = G_{0,\delta}(t)u_0(x) + i\int_0^t G_{0,\delta}(t-\tau)f(|u(x,\tau)|)u(x,\tau)d\tau, \tag{4.11}$$

where $G_{0,\delta}$ is defined in (1.9) with $\epsilon = 0$. Computing the difference between the integral equations (1.8) and (4.11) we obtain

$$\begin{aligned}
&\|u_\epsilon(t) - u(t)\|_{H^2} \\
&\le \|[G_{\epsilon,\delta}(t) - G_{0,\delta}(t)]u_0\|_{H^2} \\
&\quad + \|\int_0^t G_{\epsilon,\delta}(t-\tau)|u_\epsilon(\tau)|^\alpha u_\epsilon(\tau)d\tau - \int_0^t G_{0,\delta}(t-\tau)|u(\tau)|^\alpha u(\tau)d\tau\|_{H^2} \\
&\le \int_0^t \|G_{\epsilon,\delta}(t-\tau)[|u_\epsilon(\tau)|^\alpha u_\epsilon(\tau) - |u(\tau)|^\alpha u(\tau)]\|_{H^2}d\tau + \|[G_{\epsilon,\delta}(t) - G_{0,\delta}(t)]u_0\|_{H^2} \\
&\quad + \int_0^t \|[G_{\epsilon,\delta}(t-\tau) - G_{0,\delta}(t-\tau)]|u(\tau)|^\alpha u(\tau)\|_{H^2}d\tau
\end{aligned}$$

Since $G_{\epsilon,\delta}(t)$ is a unitary group on H^2, from last inequality we obtain

$$\begin{aligned}
&\|u_\epsilon(t) - u(t)\|_{H^2} \\
&\le \int_0^t \|[|u_\epsilon(\tau)|^\alpha u_\epsilon(\tau) - |u(\tau)|^\alpha u(\tau)]\|_{H^2}d\tau + \|[G_{\epsilon,\delta}(t) - G_{0,\delta}(t)]u_0\|_{H^2} \\
&\quad + \int_0^t \|[G_{\epsilon,\delta}(t-\tau) - G_{0,\delta}(t-\tau)]|u(\tau)|^\alpha u(\tau)\|_{H^2}d\tau \\
&\quad \le \int_0^t \||u_\epsilon(\tau) - u(\tau)|(|u_\epsilon(\tau)|^\alpha + |u(\tau)|^\alpha)\|_{H^2}d\tau + \|[G_{\epsilon,\delta}(t) - G_{0,\delta}(t)]u_0\|_{H^2} \\
&\quad + \int_0^t \|[G_{\epsilon,\delta}(t-\tau) - G_{0,\delta}(t-\tau)]|u(\tau)|^\alpha u(\tau)\|_{H^2}d\tau.
\end{aligned} \tag{4.12}$$

From (4.12) and Proposition 4.1 we have

$$\begin{aligned}
\|u_\epsilon(t) - u(t)\|_{H^2} &\le C\int_0^t \|u_\epsilon(\tau) - u(\tau)\|_{H^2}d\tau + \|[G_{\epsilon,\delta}(t) - G_{0,\delta}(t)]u_0\|_{H^2} \\
&\quad + \int_0^t \|[G_{\epsilon,\delta}(t-\tau) - G_{0,\delta}(t-\tau)]|u(\tau)|^\alpha u(\tau)\|_{H^2}d\tau.
\end{aligned} \tag{4.13}$$

From the Gronwall inequality we arrived at

$$\|u_\epsilon(t) - u(t)\|_{H^2} \le \Psi_{\epsilon,\delta}(t) + C\int_0^t \Psi_{\epsilon,\delta}(\tau)e^{C(t-\tau)}d\tau,$$

where

$$\begin{aligned}
\Psi_{\epsilon,\delta}(t) &= \|[G_{\epsilon,\delta}(t) - G_{0,\delta}(t)]u_0\|_{H^2} \\
&\quad + \int_0^t \|[G_{\epsilon,\delta}(t-\tau) - G_{0,\delta}(t-\tau)]|u(\tau)|^\alpha u(\tau)\|_{H^2}d\tau.
\end{aligned}$$

Note that because α is a positive integer, we have

$$\Psi_{\epsilon,\delta}(t) \le \|u_0\|_{H^2} + \int_0^t \||u(\tau)|^\alpha u(\tau)\|_{H^2}d\tau$$

$$
\begin{aligned}
&\le \|u_0\|_{H^2} + \int_0^t \|u(\tau)\|_{H^2}^{\alpha+1} d\tau \\
&\le \|u_0\|_{H^2} + t\|u_0\|_{H^2}^{\alpha+1}.
\end{aligned}
$$

Thus $|\Psi_{\epsilon,\delta}(\tau)e^{C(t-\tau)}| \lesssim e^{C(t-\tau)}$. Since $e^{C(t-\tau)} \in L^1(0,T)$, to obtain our result we just have to show that $\Psi_{\epsilon,\delta}(t) \to 0$ as $\epsilon \to 0$, for any $t \in [0,T]$. First, observe that

$$
\|[G_{\epsilon,\delta}(t) - G_{0,\delta}(t)]u_0\|_{H^2}^2 = \int_{\mathbb{R}^n} \langle\xi\rangle^4 |e^{-it\epsilon|\xi|^2} - 1|^2 |\widehat{u_0}(\xi)|^2 d\xi.
$$

Since

$$
\langle\xi\rangle^4 |e^{-it\epsilon|\xi|^2} - 1|^2 |\widehat{u_0}(\xi)|^2 \lesssim \langle\xi\rangle^4 |\widehat{u_0}(\xi)|^2 \quad \text{in } L^1(\mathbb{R}^n)
$$

and $\langle\xi\rangle^4 |e^{-it\epsilon|\xi|^2} - 1|^2 |\widehat{u_0}(\xi)|^2 \to 0$, as $\epsilon \to 0$, a.e. on $\mathbb{R}^n$, by the Lebesgue dominated convergence theorem we have

$$
\lim_{\epsilon\to 0} \|[G_{\epsilon,\delta}(t) - G_{0,\delta}(t)]u_0\|_{H^2} = 0\,.
$$

From Proposition 4.1 we obtain

$$
\begin{aligned}
\|[G_{\epsilon,\delta}(t-\tau) - G_{0,\delta}(t-\tau)]|u(\tau)|^\alpha u(\tau)\|_{H^2} &\le \||u(\tau)|^\alpha u(\tau)\|_{H^2} \le \|u(\tau)\|_{H^2}^{\alpha+1} \\
&\lesssim [C(\|u_0\|_{H^2}, \|u_0\|_{L^{\alpha+2}})]^{\alpha+1}.
\end{aligned}
$$

Moreover, $\|[G_{\epsilon,\delta}(t-\tau) - G_{0,\delta}(t-\tau)]|u(\tau)|^\alpha u(\tau)\|_{H^2} \to 0$, as $\epsilon \to 0$; then we arrived at

$$
\lim_{\epsilon\to 0} \int_0^t \|[G_{\epsilon,\delta}(t-\tau) - G_{0,\delta}(t-\tau)]|u(\tau)|^\alpha u(\tau)\|_{H^2} d\tau = 0,
$$

which completes the proof. □

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References

[1] A. Aceves, C. De Angelis, S. Turitsyn; *Multidimensional solitons in fiber arrays,* Optim. Lett. 19 (1995), 329-331.

[2] C. Banquet, L. C. F. Ferreira, E.J. Villamizar-Roa; *On the Schrödinger-Boussinesq system with singular initial data,* J. Math. Anal. Appl. 400 (2013), 487-496.

[3] C. Banquet, L. C. F. Ferreira, E. J. Villamizar-Roa; *On the existence and scattering theory for the Klein-Gordon-Schrödinger system in an infinite L^2-norm setting,* Annali di Matematica Pura ed Applicata, 194 (2015), 781-804.

[4] J. Bergh, J. Löfström; *Interpolation Spaces*, Springer-Verlag, Berlin-New York, 1976.

[5] T. Cazenave, F. Weissler; *Asymptotically self-similar global solutions of the nonlinear Schrödinger and heat equations,* Math. Z. 228 (1998), 83-120.

[6] S. Cui; *Pointwise estimates for a class of oscillatory integrals and related $L^p - L^q$ estimates,* J. Fourier Anal. Appl. 11 (2005) 441-457.

[7] S. Cui; *Pointwise estimates for a class of oscillatory integrals and related $L^p - L^q$ estimates II: Multi-dimension case,* J. Fourier Anal. Appl. 6 (2006) 605-626.

[8] S. Cui, A. Guo; *Well-posedness of higher-order nonlinear Schrödinger equations in Sobolev spaces $H^s(\mathbb{R}^n)$ and applications,* Nonlinear Anal. 67 (2007) 687-707.

[9] K. Dysthe; *Note on a modification to the nonlinear Schrödinger equation for application to deep water waves,* Proc. R. Soc. Lond. Ser. A 369 (1979) 105-114.

[10] J. M. Dudley, C. Finot, D. J. Richardson, G. Millot; *Self-similarity in ultrafast nonlinear optics* Nature Phys. 3 (9) (2007), 597-603.

[11] M. E. Fermann, V. I. Kruglov, B. C. Thomsen, J. M. Dudley, J. D. Harvey, *Self-similar propagation and amplification of parabolic pulses in optical fibers*, Phys. Rev. Lett. 84 (2000), 6010-6013.

[12] L. C. F. Ferreira; *Existence and scattering theory for Boussinesq type equations with singular data,* J. Differential Equations, 250 (2011), 2372-2388.

[13] L. C. F. Ferreira, E. J. Villamizar-Roa; *Self-similarity and asymptotic stability for coupled nonlinear Schrödinger equations in high dimensions*, Phys. D 241 (2012), 534-542.

[14] L. C. F. Ferreira, G. Planas, E. J. Villamizar-Roa; *On the nonhomogeneous Navier-Stokes system with Navier friction boundary conditions,* SIAM J. Math. Anal. 45 (2013), no. 4, 2576-2595.

[15] L. C. F. Ferreira, E. J. Villamizar-Roa, P. Braz e Silva; *On the existence of infinite energy solutions for nonlinear Schrödinger equations,* Proc. Amer. Math. Soc. 137 (2009), 1977-1987.

[16] C. Guo; *Global existence of solutions for a fourth-order nonlinear Schrödinger equation in $n+1$ dimensions,* Nonlinear Anal. 73 (2010), 555-563.

[17] C. Guo; *Global existence and asymptotic behavior of the Cauchy problem for fourth-order Schrödinger equations with combined power-type nonlinearities*, J. Math. Anal. Appl., 392 (2012), 111-122.

[18] A. Guo, S. Cui; *Global existence of solutions for a fourth-order nonlinear Schrödinger equation,* Appl. Math. Lett. 19 (2006), 706-711.

[19] A. Guo, S. Cui; *On the Cauchy problem of fourth-order nonlinear Schrödinger equations,* Nonlinear Anal. 66 (2007), 2911-2930.

[20] A. Guo, S. Cui; *On the Cauchy problem of fourth-order nonlinear Schrödinger equations,* Nonlinear Anal. 66 (2007), 2911-2930.

[21] C. Guo, S. Cui; *Well-posedness of the Cauchy problem of high dimension non-isotropic fourth-order Schrödinger equations in Sobolev spaces,* Nonlinear Anal. 70 (2009), 3761-3772.

[22] G. Fibich, B. Ilan, G. Papanicolaou; *Self-focusing with fourth-order dispersion,* SIAM J. Appl. Math., 62 (2002), 1437-1462.

[23] R. Hirota; *Direct Methods in Soliton Theory,* Springer, Berlin, 1980.

[24] B. Ivano, A. Kosevich; *Stable three-dimensional small-amplitude soliton in magnetic materials,* Sov. J. Low Temp. Phys. 9 (1983), 439-442.

[25] D. Iftimie, F. Sueur; *Viscous boundary layers for the Navier-Stokes equations with the Navier slip conditions,* Arch. Ration. Mech. Anal. 199 (2011), no. 1, 145175.

[26] V. Karpman; *Stabilization of soliton instabilities by higher-order dispersion: fourth order nonlinear Schrödinger-type equations,* Phys. Rev. E, 53 (1996), 1336-1339.

[27] V. Karpman, A. Shagalov; *Stability of soliton described by nonlinear Schrödinger type equations with higher-order dispersion,* Phys. D 144 (2000), 194-210.

[28] C. Miao, G. Xua, L. Zhao; *Global well-posedness and scattering for the defocusing energy-critical nonlinear Schrödinger equations of fourth order in dimensions $d \geq 9$,* J. Differential Equations 251 (2011), 3381-3402.

[29] C. Miao, G. Xu, L. Zhao; *Global well-posedness and scattering for the focusing energy-critical nonlinear Schrödinger equations of fourth order in the radial case,* J. Differential Equations 246 (2009), 3715-3749.

[30] B. Pausader; *Global well-posedness for energy critical fourth-order Schrödinger equations in the radial case,* Dyn. Partial Differ. Equ. 4 (2007), 197-225.

[31] B. Pausader; *The cubic fourth-order Schrödinger equation,* J. Funct. Anal., 256 (2009), 2473-2515.

[32] B. Pausader; *The focusing energy-critical fourth-order Schrödinger equation with radial data, Discrete Contin. Dyn. Syst. Ser. A*, 24 (2009), 1275-1292.

[33] J. Segata; *Modified wave operators for the fourth-order nonlinear Schrödinger-type equation with cubic nonlinearity*, Math. Methods Appl. Sci., 26 (2006), 1785-1800.

[34] E. Stein; *Harmonic analysis: Real-variable methods orthogonality and oscillatory integrals,* Princeton University Press, New Yersey, 1993.

[35] E. J. Villamizar-Roa, J. E. Pérez-López; *On the Davey-Stewartson system with singular initial data,* Comptes Rendus Mathematique, 350 (2012), 959-964.

[36] Y. Wang; *Nonlinear fourth-order Schrödinger equations with radial data,* Nonlinear Anal. 75 (2012), 2534-2541.

[37] S. Wen, D. Fan; *Spatiotemporal instabilities in nonlinear Kerr media in the presence of arbitrary higher order dispersions,* J. Opt. Soc. Amer. B 19 (2002), 1653-1659.

[38] X. Zhao, C. Guo, W. Sheng, X. Wei; *Well-posedness of the fourth-order perturbed Schrödinger type equation in non-isotropic Sobolev spaces,* J. Math. Anal. Appl. 382 (2011), 97-109.

[39] S. Zhu, H. Yang, J. Zhang; *Blow-up of rough solutions Nonlinear to the fourth-order nonlinear Schrödinger equation,* Nonlinear Anal. 74 (2011), 6186-6201.

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