Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 128, pp. 1–8.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

BLOW-UP CRITERIA OF SMOOTH SOLUTIONS TO A 3D MODEL OF ELECTRO-KINETIC FLUIDS IN A BOUNDED DOMAIN

MIAOCHAO CHEN, QILIN LIU

Abstract. We prove that a smooth solution of a 3D model for electro-kinetic fluids in a bounded domain breaks down blows up at the same time as certain norm of vorticity. This norm is weaker than bmo-norm.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded, simply connected domain with smooth boundary $\partial\Omega$, and ν is the unit outward normal vector to $\partial\Omega$. We consider the following model of electro-hydrodynamics in $\Omega \times (0, \infty)$ [1, 2]:

$$\partial_t u + (u \cdot \nabla)u + \nabla\pi = \Delta\phi\nabla\phi, \tag{1.1}$$

$$\operatorname{div} u = 0, \tag{1.2}$$

$$\partial_t n + u \cdot \nabla n = \nabla \cdot (\nabla n - n\nabla\phi), \tag{1.3}$$

$$\partial_t p + u \cdot \nabla p = \nabla \cdot (\nabla p + p\nabla\phi), \tag{1.4}$$

$$-\Delta\phi = p - n, \quad \int_\Omega \phi dx = 0, \tag{1.5}$$

$$u \cdot \nu = 0, \quad \frac{\partial n}{\partial \nu} = \frac{\partial p}{\partial \nu} = \frac{\partial \phi}{\partial \nu} = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{1.6}$$

$$(u, n, p)(x, 0) = (u_0, n_0, p_0)(x), \quad x \in \Omega \subset \mathbb{R}^3. \tag{1.7}$$

The unknowns u, π, ϕ, n and p denote the velocity, pressure, electric potential, anion concentration and cation concentration, respectively.

Equations (1.3)–(1.5) are known as the electro-chemical equations [3] or semiconductor equations [4, 5, 6], and electro-rheological systems [2, 7] when formally setting $u = 0$.

Equations (1.1) and (1.2) are the Euler equations with the Lorentz force $(n - p)\nabla\phi = \Delta\phi\nabla\phi$. Ogawa-Taniuchi [8] proved that a smooth solution breaks down if a certain norm of vorticity blows up at the same time. Here this norm is weaker than bmo-norm. Zhang and Yin [9] proved the global well-posedness of problem (1.1)–(1.7) when $\Omega := \mathbb{R}^2$.

Before presenting our results, we introduce some function spaces, and notation.

2010 *Mathematics Subject Classification.* 35Q30, 76D03, 76D05, 76D07.
Key words and phrases. Euler system; regularity criterion; bounded domain; bmo.

Submitted August 8, 2015. Published May 19, 2016.

Let η, ϕ_j, $j = 0, \pm 1, \pm 2, \pm 3, \dots$ be the Littlewood-Paley dyadic decomposition of unity that satisfies

$$\eta \in C_0^\infty(B(0,1)), \quad \phi \in C_0^\infty(B(0,2) \setminus B(0,\frac{1}{2})),$$
$$\phi_j(\xi) = \phi(2^{-j}\xi), \quad \eta(\xi) + \sum_{j=0}^{\infty} \phi_j(\xi) = 1$$

for all $\xi \in \mathbb{R}^3$, where $B(x,r)$ denotes the ball centered at x of radius r. We first recall the space of Besov type introduced by Vishik [10].

Definition 1.1 ([10])**.** Let $\Theta(\alpha)(\geq 1)$ be a nondecreasing function on $[1,\infty)$. $V_\Theta := \{f \in \mathscr{S}' : \|f\|_{V_\Theta} < \infty\}$ with the norm

$$\|f\|_{V_\Theta} := \sup_{N=1,2,\dots} \frac{\|(n\hat{f})^\vee\|_{L^\infty} + \sum_{j=0}^{N} \|(\phi_j \hat{f})^\vee\|_{L^\infty}}{\Theta(N)},$$

where $\hat{f}$ and $\check{f}$ denote the Fourier and inverse Fourier transforms.

We note that

$$\|f\|_{V_\Theta} \leq C\|f\|_{B^o_{\infty,\infty}} \leq C\|f\|_{bmo} \leq C\|f\|_{L^\infty}, \quad \text{if } \Theta(N) \geq N.$$

Now let us introduce the space of bmo type used in [8].

Definition 1.2. Let $\beta(r)$ be a positive function on $(0,1]$ and $\Omega \subset \mathbb{R}^3$ be a domain with $\partial\Omega \in C^\infty$.

(1) $bmo_\beta(\mathbb{R}^3)$ is defined as the set of functions f in $L^1_{\rm loc}(\mathbb{R}^3)$ such that

$$\|f\|_{bmo_\beta(\mathbb{R}^3)} := \sup_{0<r<1, x\in\mathbb{R}^3} \frac{1}{|B(x,r)|\beta(r)} \int_{B(x,r)} |f(y) - \bar{f}_{B(x,r)}| dy + \sup_{x\in\mathbb{R}^3} \frac{1}{|B(x,1)|} \int_{B(x,1)} |f(y)| dy \leq \infty,$$

where $\bar{f}_B := \frac{1}{|B|}\int_B f(y)dy$.

(2) On $\Omega \subset \mathbb{R}^3$ we define bmo_β as restrictions of the above space $bmo_\beta(\mathbb{R}^3)$:

$$bmo_\beta(\Omega) := \{f|_\Omega;\ f \in bmo_\beta(\mathbb{R}^3)\},$$

where $f|_\Omega$ is the restriction of f on Ω. The norm of this space is defined by

$$\|f\|_{bmo_\beta(\Omega)} := \inf\big\{\|\tilde{f}\|_{bmo_\beta(\mathbb{R}^3)}; \tilde{f} \in bmo_\beta(\mathbb{R}^3) \text{ with } \tilde{f} = f \text{ in } \Omega\big\}.$$

In particular if $\beta(r) = 1$, we write $bmo_\beta(\mathbb{R}^3) = bmo(\mathbb{R}^3)$ and $bmo_\beta(\Omega) = bmo(\Omega)$. Obviously, $bmo \subset bmo_\beta$ if $\beta \geq 1$.

Definition 1.3. Let $\Theta(\alpha)(\geq 1)$ be a nondecreasing function on $[1,\infty)$.

$$Y_\Theta(\Omega) := \{f \in L^1(\Omega) : \|f\|_{Y_\Theta(\Omega)} < \infty\},$$

where

$$\|f\|_{Y_\Theta(\Omega)} := \sup_{p\geq 1} \frac{\|f\|_{L^p}}{\Theta(p)}.$$

$$M_\Theta(\Omega) := \{f \in L^1(\Omega) : \|f\|_{M_\Theta(\Omega)} < \infty\},$$

where

$$\|f\|_{M_\Theta(\Omega)} := \sup_{p\geq 1} \frac{1}{\Theta(p)} \sup_{0<r<1, x\in\mathbb{R}^3} \Big(r^{-3+\frac{3}{p}} \int_{B(x,r)\cap\Omega} |f(y)| dy\Big).$$

We note that these spaces have the following relations.

$$\|f\|_{M_\Theta(\Omega)} \le C\|f\|_{Y_\Theta(\Omega)} \le C\|f\|_{bmo(\Omega)}. \tag{1.8}$$

Let

$$\beta(r) := \frac{\Theta(\log(e+\frac{1}{r}))}{\log(e+\frac{1}{r})}.$$

In this article we use the following assumptions:

(H1) $\Theta(\alpha)$ is a positive and nondecreasing function on $[1,\infty)$ satisfying

$$\int^{+\infty} \frac{d\alpha}{\Theta(\alpha)} = \infty, \quad \Theta(\alpha) \ge \alpha. \tag{1.9}$$

(H2) For all $s \ge 1$ there exists $C(s)$ such that

$$\Theta(s\alpha) \le C(s)\Theta(\alpha) \quad \text{for all } \alpha \ge 1.$$

(H3) $\beta(r)$ is a non-increasing function on $(0,1]$.

Ogawa-Taniuchi [8] proved the following blowup criterion

$$\int_0^T \|\omega(t)\|_{bmo_\beta(\Omega)} + \|\omega(t)\|_{M_\Theta(\Omega_\epsilon)} dt = \infty, \tag{1.10}$$

where $\omega := \operatorname{curl} u$ and for all $\epsilon > 0$ and $\Omega_\epsilon := \{x \in \Omega; \operatorname{dist}(x,\partial\Omega) < \epsilon\}$ or

$$\int_0^T \|\omega(t)\|_{bmo_\beta(\Omega_{3\epsilon})} + \|\omega(t)\|_{M_\Theta(\Omega_{3\epsilon})} + \|\rho\omega(t)\|_{V_\Theta} dt = \infty, \tag{1.11}$$

for all $0 < \epsilon < \epsilon_0$ and all $\rho \in C^\infty(\mathbb{R}^3)$ with $\rho \equiv 1$ in $\Omega \setminus \Omega_\epsilon$ and $\rho \equiv 0$ in $\mathbb{R}^3 \setminus \Omega$. ϵ_0 is a small positive constant depending only on Ω.

Since $\beta(r) \ge 1$, we have

$$\|f\|_{bmo_\beta(\Omega)} \le \|f\|_{bmo(\Omega)}.$$

By this inequality and (1.8), (1.10) implies

$$\int_0^T \|\omega(t)\|_{bmo(\Omega)} dt = \infty. \tag{1.12}$$

The aim of this article is to prove a similar result for problem (1.1)–(1.7). It is easy to show that (1.1)–(1.7) has a unique local smooth solution with $u_0 \in H^3$ and $(n_0, p_0) \in H^2$. Thus we omit the details here. However, the global regularity is still open, which this paper aims to study. We will prove the following result.

Theorem 1.4. *Let $u_0 \in H^3, (n_0, p_0) \in H^2$, $n_0, p_0 \ge 0, \operatorname{div} u_0 = 0$ in Ω, $u_0 \cdot \nu = \frac{\partial n_0}{\partial \nu} = \frac{\partial p_0}{\partial \nu}$ on $\partial\Omega$ and $\int_\Omega n_0 dx = \int_\Omega p_0 dx$. Suppose that (u, n, p) is a local smooth solution to (1.1)–(1.7) on $[0, T)$. If T is maximal, then (1.10) and (1.11) hold.*

In Section 2, we will give some preliminaries. Section 3 is devoted to the proof of Theorem 1.4.

2. Preliminaries

Lemma 2.1 ([11]). *For any $u \in W^{s,p}$ with $\operatorname{div} u = 0$ in Ω and $u \cdot \nu = 0$ on $\partial\Omega$, there holds*

$$\|u\|_{W^{s,p}} \le C(\|u\|_{L^p} + \|\operatorname{curl} u\|_{W^{s-1,p}})$$

for any $s \ge 1$ and $p \in (1, \infty)$.

Lemma 2.2 ([12]). *Let $s \ge 1$.*

(1) If $f, g \in H^s(\Omega) \cap C(\Omega)$, then

$$\|fg\|_{H^s(\Omega)} \le C\left(\|f\|_{H^s(\Omega)}\|g\|_{L^\infty(\Omega)} + \|f\|_{L^\infty(\Omega)}\|g\|_{H^s(\Omega)}\right).$$

(2) If $f \in H^s(\Omega) \cap C^1(\Omega)$ and $g \in H^{s-1}(\Omega) \cap C(\Omega)$, then for $|\alpha| \le s$,

$$\|D^\alpha(fg) - fD^\alpha g\|_{L^2(\Omega)} \le C\left(\|f\|_{H^s(\Omega)}\|g\|_{L^\infty(\Omega)} + \|f\|_{W^{1,\infty}(\Omega)}\|g\|_{H^{s-1}(\Omega)}\right).$$

Lemma 2.3 ([8]). *For all $\epsilon > 0$, we have*

$$\begin{aligned}&\|\nabla u\|_{L^\infty(\Omega)}\\ &\le C\left(1 + \|u\|_{L^2(\Omega)} + \|\operatorname{curl} u\|_{bmo_\beta(\Omega)} + \|\operatorname{curl} u\|_{M_\Theta(\Omega_\epsilon)}\right)\end{aligned}$$
$$\Theta(\log(e + \|u\|_{H^3(\Omega)}))$$

for all $u \in H^3(\Omega)$ with $\operatorname{div} u = 0$ in Ω and $u \cdot \nu = 0$ on $\partial\Omega$.

Lemma 2.4 ([8]). *There exists a constant ϵ_0 depending only on Ω such that: For all $0 < \epsilon < \epsilon_0$ and for all $\rho \in C^\infty(\mathbb{R}^3)$ with $\rho \equiv 1$ in $\Omega \setminus \Omega_\epsilon$ and $\rho \equiv 0$ in $\mathbb{R}^3 \setminus \Omega$ there exists constant C depending only on ϵ, ρ, Ω and Θ such that*

$$\begin{aligned}\|\nabla u\|_{L^\infty(\Omega)} \le C\Big(&1 + \|u\|_{L^2(\Omega)} + \|\operatorname{curl} u\|_{bmo_\beta(\Omega_{3\epsilon})} + \|\operatorname{curl} u\|_{M_\Theta(\Omega_{3\epsilon})}\\ &+ \|\rho \operatorname{curl} u\|_{V_\Theta}\Big)\Theta(\log(e + \|u\|_{H^3(\Omega)}))\end{aligned}$$

for all $u \in H^3(\Omega)$ with $\operatorname{div} u = 0$ in Ω and $u \cdot \nu = 0$ on $\partial\Omega$.

Lemma 2.5 ([13]). *Let ψ be nonnegative function on $(0, T)$ with $\int_0^T \psi(t)dt < \infty$, let $\Theta(\alpha)$ be a positive and nondecreasing for $\alpha \ge 1$ and $\int^{+\infty} \frac{d\alpha}{\Theta(\alpha)} = \infty$. Assume that $v \in C([0, T))$ and*

$$0 \le v(t) \le v(0) + \int_0^t \psi(s)\Theta(v(s))ds \quad \text{for all } 0 \le t < T.$$

Then $\sup_{0 \le t \le T} v(t) < \infty$.

3. Proof of Theorem 1.4

Since the proof of (1.11) is similar to that of (1.10), we only need to prove (1.10). By the standard argument of continuation of local solutions, it suffices to prove that if

$$\int_0^T \|\omega(t)\|_{bmo_\beta(\Omega)} + \|\omega(t)\|_{M_\Theta(\Omega_\epsilon)} dt < \infty \quad \text{for some } \epsilon > 0, \tag{3.1}$$

then

$$u \in L^\infty(0, T; H^3), \quad (n, p) \in L^\infty(0, T; H^2) \cap L^2(0, T; H^3). \tag{3.2}$$

First, by the maximum principle, it is easy to prove that $n, p \ge 0$ in $\Omega \times (0, \infty)$.

Testing (1.3) by n and testing (1.4) by p, using (1.5), (1.2) and summing up the resulting inequality, we easily get

$$\frac{1}{2}\int n^2+p^2dx+\int_0^T\int|\nabla n|^2+|\nabla p|^2+\frac{1}{2}(p-n)^2(n+p)dxdt\le\frac{1}{2}\int u_0^2+p_0^2dx,$$

whence

$$\|(n,p)\|_{L^\infty(0,T;L^2)}+\|(n,p)\|_{L^2(0,T;H^1)}\le C. \tag{3.3}$$

Testing (1.3) by n^{k-1} and testing (1.4) by p^{k-1}, using (1.2), (1.5) and $n,p\ge 0$, we find that

$$\int n^k+p^kdx\le\int n_0^k+p_0^k\le\int(n_0+p_0)^kdx,$$

which gives

$$\|n\|_{L^k}\le\|n_0+p_0\|_{L^k},\quad \|p\|_{L^k}\le\|n_0+p_0\|_{L^k}.$$

Taking $k\to\infty$, we obtain

$$\|(n,p)\|_{L^\infty(0,T;L^\infty)}\le C. \tag{3.4}$$

Testing (1.1) by u, using (1.2)-(1.5), we infer that

$$\frac{1}{2}\frac{d}{dt}\int u^2+|\nabla\phi|^2dx+\int|\Delta\phi|^2+(n+p)|\nabla\phi|^2dx=0, \tag{3.5}$$

which leads to

$$\|u\|_{L^\infty(0,T;L^2)}\le C. \tag{3.6}$$

It follows from (3.5), (3.4), (3.3) and (1.5) that

$$\nabla\phi\in L^\infty(0,T;H^1\cap L^\infty)\cap L^2(0,T;H^2). \tag{3.7}$$

Testing (1.3) by $-\Delta n$, using (1.2), (1.5), (1.6), (3.4) and (3.7), we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int|\nabla n|^2dx+\int|\Delta n|^2dx\\
&=\int(u\cdot\nabla)n\cdot\Delta ndx+\int(n\Delta\phi+\nabla n\cdot\nabla\phi)\Delta ndx\\
&=\sum_{i,j}\int u_i\partial_in\partial_j^2ndx+\int(n\Delta\phi+\nabla n\cdot\nabla\phi)\Delta ndx\\
&=-\sum_{i,j}\int\partial_ju_i\partial_in\partial_jndx+\int(n(n-p)+\nabla n\cdot\nabla\phi)\Delta ndx\\
&\le C\|\nabla u\|_{L^\infty}\|\nabla n\|_{L^2}^2+C\|\Delta n\|_{L^2}+C\|\nabla n\|_{L^2}\|\nabla\phi\|_{L^\infty}\|\Delta n\|_{L^2}\\
&\le\frac{1}{2}\|\Delta n\|_{L^2}^2+C\|\nabla u\|_{L^\infty}\|\nabla n\|_{L^2}^2+C+C\|\nabla n\|_{L^2}^2,
\end{aligned}$$

which implies

$$\frac{d}{dt}\int|\nabla n|^2dx+\int|\Delta n|^2dx\le C+C\|\nabla n\|_{L^2}^2+C\|\nabla u\|_{L^\infty}\|\nabla n\|_{L^2}^2. \tag{3.8}$$

Similarly for the p-equation, we have

$$\frac{d}{dt}\int|\nabla p|^2dx+\int|\Delta p|^2dx\le C+C\|\nabla p\|_{L^2}^2+C\|\nabla u\|_{L^\infty}\|\nabla p\|_{L^2}^2. \tag{3.9}$$

Equations (1.3) and (1.6) can be rewritten as

$$\Delta n=f:=\partial_tn+u\cdot\nabla n+\nabla\cdot(n\nabla\phi),\quad\text{in }\Omega\times(0,\infty)$$

$$\frac{\partial n}{\partial \nu} = 0, \quad \text{on } \partial\Omega \times (0, \infty).$$

By the classical regularity theory of elliptic equation, using (3.6), (3.4) and (3.7), we deduce that

$$\begin{aligned}
\|n\|_{H^3} &\le C\|f\|_{H^1} \\
&\le C\|\partial_t n\|_{H^1} + C\|u \cdot \nabla n\|_{H^1} + C\|\nabla \cdot (n\nabla\phi)\|_{H^1} \\
&\le C\|\partial_t n\|_{H^1} + C\|u\|_{L^2}\|\nabla n\|_{L^\infty} + C\|u\|_{L^6}\|\Delta n\|_{L^3} \\
&\quad + C\|\nabla u\|_{L^\infty}\|\nabla n\|_{L^2} + C\|n\Delta\phi\|_{L^2} + C\|\nabla n \cdot \nabla\phi\|_{L^2} \\
&\quad + C\|n\|_{L^\infty}\|\nabla\Delta\phi\|_{L^2} + C\|\nabla n\|_{L^\infty}\|\nabla^2\phi\|_{L^2} + C\|\nabla\phi\|_{L^6}\|\Delta n\|_{L^3} \\
&\le C\|\partial_t n\|_{H^1} + C\|\nabla n\|_{L^\infty} + C\|u\|_{L^6}\|\Delta n\|_{L^3} \\
&\quad + C\|\nabla u\|_{L^\infty}\|\nabla n\|_{L^2} + C + C\|\nabla n\|_{L^2} \\
&\quad + C\|\nabla(n-p)\|_{L^2} + C\|\Delta n\|_{L^3}.
\end{aligned} \tag{3.10}$$

Now we use the following Gagliardo-Nirenberg inequalities:

$$\|\nabla n\|_{L^\infty} \le C\|n\|_{L^\infty}^{1/3}\|n\|_{H^3}^{2/3}, \tag{3.11}$$

$$\|\nabla n\|_{L^3} \le C\|n\|_{L^\infty}^{1/3}\|n\|_{H^3}^{2/3}, \tag{3.12}$$

$$\|u\|_{L^6}^3 \le C\|u\|_{L^2}^2\|u\|_{H^3}. \tag{3.13}$$

It follows from (3.10), (3.11), (3.12), (3.13), (3.6), (3.4) and the Young inequality that

$$\begin{aligned}
\|n\|_{H^3} &\le C\|\partial_t n\|_{H^1} + C + C\|u\|_{H^3} + C\|\nabla u\|_{L^\infty}\|\nabla n\|_{L^2} \\
&\quad + C\|\nabla n\|_{L^2} + C\|\nabla p\|_{L^2}.
\end{aligned} \tag{3.14}$$

Similarly to the p- equation, we have

$$\begin{aligned}
\|p\|_{H^3} &\le C\|\partial_t p\|_{H^1} + C + C\|u\|_{H^3} + C\|\nabla u\|_{L^\infty}\|\nabla p\|_{L^2} \\
&\quad + C\|\nabla n\|_{L^2} + C\|\nabla p\|_{L^2}.
\end{aligned} \tag{3.15}$$

Applying the curl to (1.1), using (1.2), we obtain

$$\partial_t\omega + u \cdot \nabla\omega = \omega \cdot \nabla u + \operatorname{curl}(\Delta\phi\nabla\phi). \tag{3.16}$$

Applying Δ to (3.16), testing by $\Delta\omega$, using (1.2), we find that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int |\Delta\omega|^2 dx &= -\int (\Delta(u\cdot\nabla\omega) - u\nabla\Delta\omega)\Delta\omega dx \\
&\quad + \int \Delta(\omega\cdot\nabla u)\cdot\Delta\omega dx + \int \Delta\operatorname{curl}(\Delta\phi\nabla\phi)\cdot\Delta\omega dx \\
&\le \big(\|\Delta(u\cdot\nabla\omega) - u\nabla\Delta\omega\|_{L^2} + \|\Delta(\omega\cdot\nabla u)\|_{L^2} \\
&\quad + \|\Delta\operatorname{curl}(\Delta\phi\nabla\phi)\|_{L^2}\big)\|\Delta\omega\|_{L^2} \\
&=: (I_1 + I_2 + I_3)\|\Delta\omega\|_{L^2}.
\end{aligned} \tag{3.17}$$

Using (1.2) and Lemma 2.2, I_1 and I_2 can be bounded as follows.

$$\begin{aligned}
I_1 &= \sum_i \|\Delta\partial_i(u_i\omega) - u_i\partial_i\Delta\omega\|_{L^2} \\
&\le C\|\nabla u\|_{L^\infty}\|\Delta\omega\|_{L^2} + C\|\omega\|_{L^\infty}\|\nabla^3 u\|_{L^2} \\
&\le C\|\nabla u\|_{L^\infty}\|u\|_{H^3},
\end{aligned}$$

$$I_2 \le C\|\omega\|_{L^\infty}\|u\|_{H^3} + C\|\nabla u\|_{L^\infty}\|\omega\|_{H^2} \le C\|\nabla u\|_{L^\infty}\|u\|_{H^3}.$$

Noting that

$$\Delta\phi \cdot \nabla\phi = \sum_{i,j} \partial_j(\partial_j\phi\partial_i\phi) - \frac{1}{2}\sum_{i,j} \partial_i(\partial_j\phi)^2,$$

using Lemma 2.2 and (3.7), we have

$$I_3 \le C\|\nabla\phi\|_{L^\infty}\|\nabla\phi\|_{H^4} \le C\|\nabla\phi\|_{H^4} \le C\|\phi\|_{H^5} \le C\|n-p\|_{H^3}.$$

Inserting the above estimates into (3.17), we obtain

$$\frac{1}{2}\frac{d}{dt}\int |\Delta\omega|^2 dx \le C(\|\nabla u\|_{L^\infty}\|u\|_{H^3} + \|n-p\|_{H^3})\|\Delta\omega\|_{L^2}. \tag{3.18}$$

Testing (1.1) by $\partial_t u$, using (1.2), (3.6), (3.7) and (3.13), we infer that

$$\begin{aligned}
\|\partial_t u\|_{L^2} &\le \|\Delta\phi\nabla\phi\|_{L^2} + \|u\cdot\nabla u\|_{L^2}\\
&\le \|\nabla\phi\|_{L^\infty}\|\Delta\phi\|_{L^2} + \|u\|_{L^6}\|\nabla u\|_{L^3}\\
&\le C + C\|u\|_{L^2}^{2/3}\|u\|_{H^3}^{1/3}\|u\|_{L^2}^{1/2}\|u\|_{H^3}^{1/2}\\
&\le C + C\|u\|_{H^3}^{5/6}.
\end{aligned} \tag{3.19}$$

Here we have used the Gagliardo-Nirenberg inequality

$$\|\nabla u\|_{L^3}^2 \le C\|u\|_{L^2}\|u\|_{H^3}.$$

Applying ∂_t to (1.3), we see that

$$\partial_t^2 n + u\cdot\nabla\partial_t n - \Delta\partial_t n = -\partial_t u\cdot\nabla n - \nabla\cdot\partial_t(n\nabla\phi).$$

Testing the above equation by $\partial_t n$, using (1.2), (1.6), (3.4), (3.7), (3.19) and (1.5), we derive

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int (\partial_t n)^2 dx + \int |\nabla\partial_t n|^2 dx\\
&= -\int (\partial_t u\cdot\nabla)n\cdot\partial_t n dx + \int \partial_t(n\nabla\phi)\cdot\nabla\partial_t n dx\\
&= \int \partial_t u\cdot n\nabla\partial_t n dx + \int \partial_t(n\nabla\phi)\cdot\partial_t n dx\\
&\le (\|n\|_{L^\infty}\|\partial_t u\|_{L^2} + \|\nabla\phi\|_{L^\infty}\|\partial_t n\|_{L^2} + \|n\|_{L^\infty}\|\nabla\partial_t\phi\|_{L^2})\,\|\nabla\partial_t n\|_{L^2}\\
&\le C(\|\partial_t u\|_{L^2} + \|\partial_t n\|_{L^2} + \|\partial_t(n-p)\|_{L^2})\|\nabla\partial_t n\|_{L^2}\\
&\le \frac{1}{2}\|\nabla\partial_t n\|_{L^2}^2 + C + C\|u\|_{H^3}^2 + C\|\partial_t n\|_{L^2}^2 + C\|\partial_t p\|_{L^2}^2,
\end{aligned}$$

whence

$$\frac{d}{dt}\int |\partial_t n|^2 dx + \int |\nabla\partial_t n|^2 dx \le C + C\|u\|_{H^3}^2 + C\|\partial_t(n,p)\|_{L^2}^2. \tag{3.20}$$

Similarly, for the p-equation, we have

$$\frac{d}{dt}\int (\partial_t p)^2 dx + \int |\nabla\partial_t p|^2 dx \le C + C\|u\|_{H^3}^2 + C\|\partial_t(n,p)\|_{L^2}^2. \tag{3.21}$$

Combining (3.8), (3.9), (3.14), (3.15), (3.18), (3.20) and (3.21), using (3.6), Lemma 2.1, Lemma 2.3, and Lemma 2.5, we conclude that (3.2) holds. This completes the proof.

Acknowledgements. The author is indebted to the referees for their valuable suggestions. This work is supported by the Natural Science Foundation of Chaohu University (No. XLY-201503), the University Natural Science Foundation of Anhui (No. KJ2015A270).

References

[1] M. Bazant, T. Squires; *Induced-charge electro-kinetic phenomena, theory and microfluidic applications*. Phys. Rev. Lett. 92 (2004), 066101.

[2] R. E. Problestein; *Physicochemical Hydrodynamics, An introduction*. John Wiley and Sons, INC., 1994.

[3] K. Chu, M. Bazant; *Electrochemical thin films at and above the classical limiting current*. SIAM J. Appl. Math. 65 (2005), 1485-1505.

[4] M. Kurokiba, T. Ogawa; *L^p well-posedness for the drift-diffusion system arising from the semiconductor device simulation*. J. Math. Anal. Appl. 342 (2008), 1052-1067.

[5] P. Biler, W. Hebisch, T. Nadzieja; *The Debye system: existence and large time behavior of solutions*. Nonlinear Anal. TMA 23 (1994), 1189-1209.

[6] J. I. Díaz, G. Galiano, A. Jüngel; *On a quasilinear degenerate system arising in semiconductor theory. Part I: Existence and uniqueness of solutions*. Nonlinear Analysis: Real World Applications 2(2001), 305-336.

[7] S. Thamida, H. C. Chang; *Nonlinear electro-kinetic ejection and entrainment due to polarization at nearly insulated wedges*. Phys. Fluids 14 (2002), 4315-4328.

[8] T. Ogawa, Y. Taniuchi; *On blow-up criteria of smooth solutions to the 3D Euler equations in a bounded domain*. J. Differential Equations 190 (2003), 39-63.

[9] Z. Zhang, Z. Yin; *Global well-posedness for the Euler-Nernst-Planck-Poisson system in dimension two*, Nonlinear Anal. 125(2015), 30-53.

[10] M. Vishik; *Incompressible flows of an ideal fluid with unbounded vorticity*. Comm. Math. Phys. 213 (2000), 697-731.

[11] J. P. Bourguignon, H. Brezis; *Remarks on the Euler equation*. J. Funct. Anal. 15 (1974), 341-363.

[12] A. B. Ferrari; *On the blow-up of solutions of 3-D Euler equations in a bounded domain*. Comm. Math. Phys. 155 (1993), 277-294.

[13] Ph. Hartman; *Ordinary Differential Equations*, 2nd Edition. Birkhäuser, Boston, Basel, Stuttgart, 1982.

Miaochao Chen (corresponding author)
School of Applied Mathematics, Chaohu University, Hefei 238000, China
E-mail address: chenmiaochao@chu.edu.cn

Qilin Liu
Department of Mathematics, Southeast University, Nanjing 211189, China
E-mail address: Liuqlseu@126.com