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STRUCTURAL STABILITY OF SOLUTIONS TO THE RIEMANN PROBLEM FOR A NON-STRICTLY HYPERBOLIC SYSTEM WITH FLUX APPROXIMATION

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Abstract. We study the Riemann problem for a non-strictly hyperbolic system of conservation laws under the linear approximations of flux functions with three parameters. The approximated system also belongs to the type of triangular systems of conservation laws and this approximation does not change the structure of Riemann solutions to the original system. Furthermore, it is proven that the Riemann solutions to the approximated system converge to the corresponding ones to the original system as the perturbation parameter tends to zero.

1. Introduction

Non-strictly hyperbolic systems of conservation laws have not only important physical background but also special interest and difficulty in mathematics. It is well known that the Cauchy problem usually does not have a weak L^∞-solution for some non-strictly hyperbolic systems of conservation laws. Thus, the measure-valued solution should be introduced into this nonclassical situation, such as delta shock wave [1, 21, 25] and singular shock wave [10, 15], which can often provide a reasonable explanation for some physical phenomena. However, the mechanism for the formation of delta shock wave cannot be fully understood, though the necessity of delta shock wave is obvious for the solutions of Riemann problems for some non-strictly hyperbolic systems of conservation laws.

In this article, we are concerned with the non-strictly hyperbolic system of conservation laws

$$\begin{aligned} &u_t + (u^2)_x = 0,\\ &v_t + (uv)_x = 0. \end{aligned} \tag{1.1}$$

The system (1.1) can be derived in [25] directly from the system of Euler gas dynamics by letting both the density and the pressure to be constants in the momentum equation. The system (1.1) arises in several fields which can be used to model conservation laws for some specific situations, such as magnetohydrodynamics, elasticity and oil recovery process [16, 23]. The first equation in (1.1) is just the inviscid Burgers equation and the solutions of the Riemann problem are the classical entropy solutions. The Dirac delta function is introduced as a part for v

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in the second equation in (1.1) when the characteristic velocity u is discontinuous. In 1994, Tan, Zhang and Zheng [25] considered the Riemann problem for (1.1) and they discovered that the form of the standard Dirac delta function supported on a shock wave was used as a part in their Riemann solutions for some specific initial data. Since then, the delta shock wave solution for (1.1) has been widely investigated such as in [19, 20, 28].

The formation of delta shock wave has been extensively studied by using the vanishing pressure approximation for the systems of pressureless gas dynamics [1, 2, 12, 13, 14, 18, 29, 30] and Chaplygin gas dynamics [3, 21, 27], which is a particular case of flux function approximation. Recently, the flux function approximation with two parameters [17] and three parameters [26] has also been carried out for the systems of pressureless gas dynamics. In the present paper, we consider the linear approximations of flux functions in (1.1) as follows:

$$\begin{aligned} &u_t + (u^2 + \varepsilon\alpha u)_x = 0, \\ &v_t + (uv + \varepsilon\beta u + \varepsilon\gamma v)_x = 0, \end{aligned} \tag{1.2}$$

where α, β, γ are arbitrary real constant numbers and ε is a sufficiently small positive number. More precisely, we are only concerned with the Riemann problem here, which is a special Cauchy problem with initial data

$$(u, v)(x, 0) = \begin{cases} (u_-, v_-), & x < 0, \\ (u_+, v_+), & x > 0, \end{cases} \tag{1.3}$$

where $u_\pm$ and $v_\pm$ are all given constants.

It is remarkable that system (1.1) is a particular example for the triangular systems of conservation laws due to the special structure where the evolution of the unknown variable u does not depend on the succeeding unknown variable v. It can be seen that system (1.2) also belongs to the type of triangular systems of conservation laws under the triangular linear approximations of flux functions. It can be discovered that the delta shock wave also appears in the solution of the Riemann problem (1.2) and (1.3) for some specific initial data. Furthermore, it is proven rigorously that the limits of solutions to the Riemann problem (1.2) and (1.3) converge to the corresponding ones of the Riemann problem (1.1) and (1.3) when the perturbation parameter ε tends to zero. In other words, the Riemann solutions of (1.1) and (1.3) is stable with respect to the triangular linear approximations of flux functions in the form of (1.2). Actually, one can see that the Riemann solutions of (1.1) and (1.3) just translate in the (x, t) plane under the triangular linear approximations of flux functions in the form of (1.2). Thus, this triangular linear approximations of flux functions in the form of (1.2) does not change the structure of solutions to the Riemann problem (1.1) and (1.3).

In fact, the concept of Dirac delta function was first introduced into the classical weak solution of hyperbolic conservation laws by Korchinski [11] in 1975 when he considered the Riemann problem for the system

$$\begin{aligned} &u_t + (\frac{1}{2}u^2)_x = 0, \\ &v_t + (\frac{1}{2}uv)_x = 0, \end{aligned} \tag{1.4}$$

which has the trivial difference $u \to 2u$ from (1.1). Since 1994, there are numerous excellent papers about the concept of delta shock wave for the related equations and results, see [4, 7, 10, 15, 22] for instance. At present, the vanishing pressure approach is one of the popular approaches to study the formation of delta shock wave appearing in the Riemann solution for some hyperbolic systems of conservation laws. In the present paper, we consider the linear approximations of flux functions with three parameters in the form of (1.2) for a particular triangular hyperbolic system of conservation laws which has not been paid attention before.

This article is organized as follows. In section 2, we describe simply the solutions of the Riemann problem (1.1) and (1.3) for completeness. In section 3, the Riemann problem for the approximated system (1.2) is considered and the Riemann solutions are constructed completely for six different cases. In section 4, the limit of Riemann solutions to the approximated system (1.2) is taken by letting the perturbation parameter ε tend to zero, which is identical with the corresponding ones to the original system. Finally, the conclusion and discussion are drawn in section 5.

2. Preliminaries

In this section, we simply describe the results on the Riemann problem (1.1) and (1.3), which can be seen such as in [25]. The eigenvalues of (1.1) are $\lambda_1 = u$ and $\lambda_2 = 2u$ and the corresponding right eigenvectors are $\overrightarrow{r}_1 = (0,1)^T$ and $\overrightarrow{r}_2 = (1, v/u)^T$, respectively. It is noted that $\lambda_1 < \lambda_2$ for $u > 0$ and $\lambda_1 > \lambda_2$ for $u < 0$ here. Thus, (1.1) is a non-strictly hyperbolic system. It can be obtained directly that the characteristic field for λ_1 is linearly degenerate and the characteristic field for λ_2 is genuinely nonlinear.

Besides the constant state, it can be seen from [25] that the self-similar waves $(u,v)(\xi)$ $(\xi = x/t)$ of the first family are contact discontinuities denoted by J as

$$J : \xi = u_l = u_r,$$

and those of the second family are rarefaction waves denoted by R as

$$R : \xi = 2u, \quad \frac{u}{v} = \frac{u_l}{v_l}, \quad u_l < u_r,$$

or shock waves denoted by S as

$$S : \xi = u_l + u_r, \quad \frac{u_r}{v_r} = \frac{u_l}{v_l}, \quad u_l > u_r > 0, \quad \text{or } 0 > u_l > u_r,$$

in which the indices l and r stand for the left and right states respectively. All the waves J, R and S are called as classical waves here.

For the case $u_+ \le 0 \le u_-$, a solution containing a weighted delta measure supported on a line should be constructed. In order to define the delta shock wave solution to the Riemann problem (1.1) and (1.3), let us introduce the following definitions below.

Definition 2.1. To define the measure solutions, the two-dimensional weighted delta measure $w(s)\delta_\Gamma$ supported on a smooth curve $\Gamma = \{(x(s), t(s)) : a < s < b\}$ is defined by

$$\langle w(s)\delta_\Gamma, \psi(x,t)\rangle = \int_a^b w(s)\psi(x(s), t(s))ds, \tag{2.1}$$

for any test function $\psi(x,t) \in C_0^\infty(R \times R_+)$.

Now, let us introduce the definition of delta shock wave solution in the framework introduced by Danilov and Shelkovich [5, 6] and developed by Kalisch and Mitrovic [8, 9] below. Suppose that $\Gamma = \{\gamma_i | i \in I\}$ is a graph in the closed upper half-plane

$\{(x,t)|(x,t)\in(-\infty,\infty)\times[0,\infty)\}$ which contains Lipschitz continuous arcs γ_i with $i\in I$ in which I is a finite index set. Suppose that I_0 is a subset of I which contains all indices of arcs linking to the x-axis and $\Gamma_0=\{x_k^0|k\in I_0\}$ is the set of initial points of γ_k with $k\in I_0$.

Definition 2.2. Let (u,v) be a pair of distributions where v has the form

$$v(x,t)=\hat{v}(x,t)+\alpha(x,t)\delta(\Gamma), \tag{2.2}$$

in which $u,\hat{v}\in L^\infty(R\times R_+)$ and the singular part is defined by

$$\alpha(x,t)\delta(\Gamma)=\sum_{i\in I}\alpha_i(x,t)\delta(\gamma_i). \tag{2.3}$$

Let us consider the delta shock wave type initial data

$$(u,v)(x,0)=\Big(u_0(x),\hat{v}_0(x)+\sum_{k\in I_0}\alpha_k(x_k^0,0)\delta(x-x_k^0)\Big), \tag{2.4}$$

in which $u_0,\hat{v}_0\in L^\infty(R)$, then the pair of distributions (u,v) are called as a generalized delta shock wave solution for (1.1) with the delta shock wave type initial data (2.4) if the following integral identities

$$\int_{R_+}\int_R\left(u\psi_t+u^2\psi_x\right)\,dx\,dt+\int_R u_0(x)\psi(x,0)dx=0, \tag{2.5}$$

$$\begin{aligned}&\int_{R_+}\int_R\left(\hat{v}\psi_t+u\hat{v}\psi_x\right)\,dx\,dt+\sum_{i\in I}\int_{\gamma_i}\alpha_i(x,t)\frac{\partial\psi(x,t)}{\partial l}\\&+\int_R\hat{v}_0(x)\psi(x,0)dx+\sum_{k\in I_0}\alpha_k(x_k^0,0)\psi(x_k^0,0)=0,\end{aligned} \tag{2.6}$$

hold for all test functions $\psi\in C_c^\infty(R\times R_+)$, in which $\frac{\partial\psi(x,t)}{\partial l}$ stands for the tangential derivative of ψ on the graph γ_i and $\int_{\gamma_i}$ expresses the line integral along γ_i.

With the above definition, a piecewise smooth solution of (1.1) and (1.3) can be constructed for the case $u_+\le 0\le u_-$ in the form

$$u(x,t)=\begin{cases}u_-, & x<\sigma_\delta t,\\ u_+, & x>\sigma_\delta t,\end{cases}\quad v(x,t)=\begin{cases}v_-, & x<\sigma_\delta t\\ v_+, & x>\sigma_\delta t\end{cases}+w(t)\delta(x-\sigma_\delta t) \tag{2.7}$$

where

$$\sigma_\delta=u_-+u_+,\quad w(t)=(u_-v_+-u_+v_-)t. \tag{2.8}$$

The functions $w(t)$ and σ_δ express the strength and propagation speed of delta shock wave, respectively.

The delta shock wave solution (2.7) with (2.8) satisfies the generalized Rankine-Hugoniot condition

$$\begin{gathered}\frac{dx}{dt}=\sigma_\delta,\\ \frac{dw(t)}{dt}=\sigma_\delta[v]-[uv],\\ [u^2]=\sigma_\delta[u],\end{gathered} \tag{2.9}$$

where $[u]=u(x(t)+0,t)-u(x(t)-0,t)$, etc. In order to ensure the uniqueness, the entropy condition of delta shock wave should be proposed as $\lambda_{2r}\le\lambda_{1r}\le$

$\sigma_\delta \leq \lambda_{1l} \leq \lambda_{2l}$. It is an over-compressive condition which implies that all the characteristics on both sides of the delta shock wave curve are incoming.

The above constructed delta shock wave solution (2.7) with (2.8) should satisfy

$$\begin{aligned} \langle u, \psi_t \rangle + \langle u^2, \psi_x \rangle = 0, \\ \langle v, \psi_t \rangle + \langle uv, \psi_x \rangle = 0, \end{aligned} \tag{2.10}$$

for any test function $\psi(x,t) \in C_0^\infty(R \times R_+)$. In the above formula (2.10), as in [1, 22, 25], we have

$$\langle v, \psi \rangle = \int_0^\infty \int_{-\infty}^\infty v_0 \psi \, dx \, dt + \langle w(t)\delta_S, \psi \rangle, \tag{2.11}$$

$$\langle uv, \psi \rangle = \int_0^\infty \int_{-\infty}^\infty u_0 v_0 \psi \, dx \, dt + \langle u_\delta w(t)\delta_S, \psi \rangle, \tag{2.12}$$

in which $u_0 = u_- + [u]H(x - \sigma_\delta t)$, $v_0 = v_- + [v]H(x - \sigma_\delta t)$ and $u_0 v_0 = u_- v_- + [uv]H(x - \sigma_\delta t)$. In order to require the solution (2.7) with (2.8) to satisfy (2.10) in the sense of distributions, it is necessary to specify the value of velocity u along the trajectory of singularity. Thus, u_δ is introduced in the formula (2.12) which stands for the assignment of u on this delta shock wave curve $x = \sigma_\delta t$, although the reasonable physical explanation may not be given clearly. Then, the solution (2.7) can be rewritten as

$$(u, v)(x,t) = \begin{cases} (u_-, v_-), & x < \sigma_\delta t, \\ (u_\delta, w(t)\delta(x - \sigma_\delta t)), & x = \sigma_\delta t, \\ (u_+, v_+), & x > \sigma_\delta t, \end{cases} \tag{2.13}$$

where $u_\delta = \sigma_\delta = u_- + u_+$. In fact, it can be seen from [25] that the delta shock wave solution (2.13) with (2.8) indeed satisfies (2.10) in the sense of distributions.

With the entropy conditions of shock wave and delta shock wave above, there exist six different configurations of solutions to the Riemann problem (1.1) and (1.3) according to the values of u_- and u_+ as follows:

$$\delta S(u_+ \leq 0 \leq u_-), \quad J + \overrightarrow{S}(0 < u_+ < u_-), \quad J + \overrightarrow{R}(0 \leq u_- < u_+),$$
$$\overleftarrow{R} + \overrightarrow{R}(u_- < 0 < u_+), \quad \overleftarrow{R} + J(u_- < u_+ \leq 0), \quad \overleftarrow{S} + J(u_+ < u_- < 0).$$

3. Riemann problems (1.2) and (1.3)

In this section, we consider (1.2) and (1.3) for any given sufficiently small parameter $\varepsilon > 0$. System (1.2) can be rewritten in the quasi-linear form

$$\begin{pmatrix} u \\ v \end{pmatrix}_t + \begin{pmatrix} 2u + \varepsilon\alpha & 0 \\ v + \varepsilon\beta & u + \varepsilon\gamma \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}_x = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.1}$$

It can be derived directly from (3.1) that the two eigenvalues of system (1.2) are

$$\lambda_1(u,v) = u + \epsilon\gamma, \quad \lambda_2(u,v) = 2u + \varepsilon\alpha. \tag{3.2}$$

It is clear that (1.2) is non-strictly hyperbolic for the reason that $\lambda_1 < \lambda_2$ when $u > \varepsilon(\gamma - \alpha)$ and $\lambda_1 > \lambda_2$ when $u < \varepsilon(\gamma - \alpha)$. The corresponding right eigenvectors of λ_i $(i = 1, 2)$ can be expressed respectively as

$$\overrightarrow{r}_1 = (0, 1)^T, \quad \overrightarrow{r}_2 = (u + \varepsilon\alpha - \varepsilon\gamma, v + \varepsilon\beta)^T. \tag{3.3}$$

By a direct calculation, we have $\nabla\lambda_1 \cdot \overrightarrow{r}_1 = 0$ and $\nabla\lambda_2 \cdot \overrightarrow{r}_2 = 2(u + \varepsilon\alpha - \varepsilon\gamma)$, in which ∇ denotes the gradient with respect to (u, v). Thus, it can be concluded that the characteristic field for λ_1 is always linearly degenerate and the associated wave is the contact discontinuity denoted by J and the characteristic field for λ_2 is genuinely nonlinear provided that $u \neq \varepsilon(\gamma-\alpha)$ and the associated wave is the shock wave denoted by S or the rarefaction wave denoted by R. The Riemann invariants along the characteristic fields are

$$w = u, \quad z = \frac{v+\varepsilon\beta}{u+\varepsilon\alpha-\varepsilon\gamma}. \tag{3.4}$$

Since both system (1.2) and the Riemann initial data (1.3) are unchanged under the coordinate transformations in the (x,t) plane: $(x,t) \to (kx, kt)$ where k is a constant, we need to look for the self-similar solutions of the form

$$(u,v)(x,t) = (u,v)(\xi), \quad \xi = x/t. \tag{3.5}$$

Hence, the Riemann problem (1.2) and (1.3) is reduced to the boundary-value problem of ordinary differential equations

$$\begin{gathered} -\xi u_\xi + (u^2+\varepsilon\alpha u)_\xi = 0, \\ -\xi v_\xi + (uv+\varepsilon\beta u+\varepsilon\gamma v)_\xi = 0, \end{gathered} \tag{3.6}$$

with the boundary condition $(u,v)(\pm\infty) = (u_\pm, v_\pm)$.

Let us denote $U = (u,v)^T$ and consider the smooth solutions of the above boundary-value problem, then (3.6) may be rewritten as

$$A(U)U_\xi = 0, \tag{3.7}$$

in which

$$A(u,v) = \begin{pmatrix} -\xi+2u+\varepsilon\alpha & 0 \\ v+\varepsilon\beta & -\xi+u+\varepsilon\gamma \end{pmatrix}. \tag{3.8}$$

Besides the constant state solution, it provides a rarefaction wave which is a continuous solution of (3.7) in the form $(u,v)(\xi)$ which is a function of the single variable $\xi = \frac{x}{t}$. For a fixed left state (u_-, v_-), the rarefaction curves in the (u,v) phase plane, which are the sets of states that may be connected on the right by a rarefaction wave, are as follows:

$$R(u_-, v_-) : \begin{cases} \xi = \lambda_2(u,v) = 2u+\varepsilon\alpha, \\ \frac{v+\varepsilon\beta}{u+\varepsilon\alpha-\varepsilon\gamma} = \frac{v_-+\varepsilon\beta}{u_-+\varepsilon\alpha-\varepsilon\gamma}, \\ u_- < u. \end{cases} \tag{3.9}$$

It is worthwhile to notice that it is a 1-rarefaction wave for $u_- < u < \varepsilon(\gamma-\alpha)$ and a 2-rarefaction wave for $\varepsilon(\gamma-\alpha) < u_- < u$.

Let us turn to the study of shock wave curves. For a bounded discontinuity at $x = x(t)$, the Rankine-Hugoniot conditions can be expressed as

$$\begin{gathered} \sigma[u] = [u^2+\varepsilon\alpha u], \\ \sigma[v] = [uv+\varepsilon\beta u+\varepsilon\gamma v], \end{gathered} \tag{3.10}$$

where $\sigma = \frac{dx}{dt}$ and $[u] = u_r - u_l$ with $u_l = u(x(t)-0,t)$ and $u_r = u(x(t)+0,t)$, etc.

It follows from the first equation in (3.10) that

$$(\sigma - u_l - u_r - \varepsilon\alpha)(u_r - u_l) = 0. \tag{3.11}$$

If $u_l = u_r$, then it follows from the second equation in (3.10) that

$$\sigma = u_l + \varepsilon\gamma = u_r + \varepsilon\gamma, \tag{3.12}$$

which responds to the contact discontinuity. Thus, for a fixed left state (u_-, v_-), the contact discontinuity curves in the (u, v) phase plane are as follows:

$$J(u_-, v_-) : \begin{cases} \tau = u + \varepsilon\gamma = u_- + \varepsilon\gamma, \\ u = u_-. \end{cases} \tag{3.13}$$

On the other hand, if $\sigma = u_l + u_r + \varepsilon\alpha$, then it follows from the second equation in (3.10) that the relation

$$\frac{v_r - v_l}{u_r - u_l} = \frac{v_r + \varepsilon\beta}{u_r + \varepsilon(\alpha - \gamma)} = \frac{v_l + \varepsilon\beta}{u_l + \varepsilon(\alpha - \gamma)} \tag{3.14}$$

should be satisfied. It is well known that an admissibility condition should be added in order to rule out non-physical shock waves. Here the Lax entropy conditions deduce that $u < u_- < \varepsilon(\gamma - \alpha)$ for the 1-shock wave and $\varepsilon(\gamma - \alpha) < u < u_-$ for the 2-shock wave should be satisfied. Through the above analysis, for a given left state (u_-, v_-), the shock curves in the (u, v) phase plane are as follows:

$$S(u_-, v_-) : \begin{cases} \sigma = u_- + u + \varepsilon\alpha, \\ \frac{v+\varepsilon\beta}{u+\varepsilon\alpha-\varepsilon\gamma} = \frac{v_-+\varepsilon\beta}{u_-+\varepsilon\alpha-\varepsilon\gamma}, \\ u < u_-. \end{cases} \tag{3.15}$$

It is worthwhile to notice that the shock curves coincide with the rarefaction curves in the (u, v) phase plane, thus (1.2) belongs to the so-called Temple class [24].

Using the elementary waves discussed above, one is in a position to construct the solutions of the Riemann problem (1.2) and (1.3) in the following six different cases according to the values of u_- and u_+.

(1) If $\varepsilon(\gamma - \alpha) < u_+ < u_-$, then the Riemann solution to (1.2) and (1.3) is $J + S$ and the intermediate state between J and S is determined by

$$\begin{gathered} u_* = u_-, \\ \frac{v_* + \varepsilon\beta}{u_* + \varepsilon\alpha - \varepsilon\gamma} = \frac{v_+ + \varepsilon\beta}{u_+ + \varepsilon\alpha - \varepsilon\gamma}, \end{gathered} \tag{3.16}$$

which enables us to have

$$(u_*, v_*) = \Big(u_-, v_+ - \frac{(u_+ - u_-)(v_+ + \varepsilon\beta)}{u_+ + \varepsilon\alpha - \varepsilon\gamma}\Big). \tag{3.17}$$

Thus, when $\varepsilon(\gamma - \alpha) < u_+ < u_-$, the Riemann solution to (1.2) and (1.3) can be expressed as:

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & \xi < \tau_1, \\ (u_*, v_*), & \tau_1 < \xi < \sigma_2, \\ (u_+, v_+), & \xi > \sigma_2, \end{cases} \tag{3.18}$$

in which the intermediate state (u_*, v_*) is given by (3.17) and the propagation speeds of J_1 and S_2 can be calculated by $\tau_1 = u_- + \varepsilon\gamma$ and $\sigma_2 = u_- + u_+ + \varepsilon\alpha$ respectively.

(2) If $\varepsilon(\gamma - \alpha) < u_- < u_+$, then the Riemann solution to (1.2) and (1.3) is $J + R$ and the intermediate state between J and R can also be calculated by (3.17). The state (u, v) in R_2 is determined by

$$\begin{gathered} \xi = 2u + \varepsilon\alpha, \\ \frac{v + \varepsilon\beta}{u + \varepsilon\alpha - \varepsilon\gamma} = \frac{v_+ + \varepsilon\beta}{u_+ + \varepsilon\alpha - \varepsilon\gamma}, \end{gathered} \tag{3.19}$$

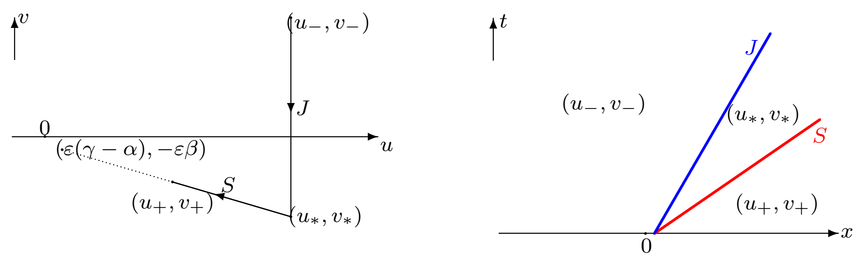

FIGURE 1. The Riemann solution of (1.2) and (1.3) is $J+S$ when $\varepsilon(\gamma-\alpha) < u_+ < u_-$, where $\beta > 0$, $\gamma > \alpha$ and ε is a sufficiently small positive number.

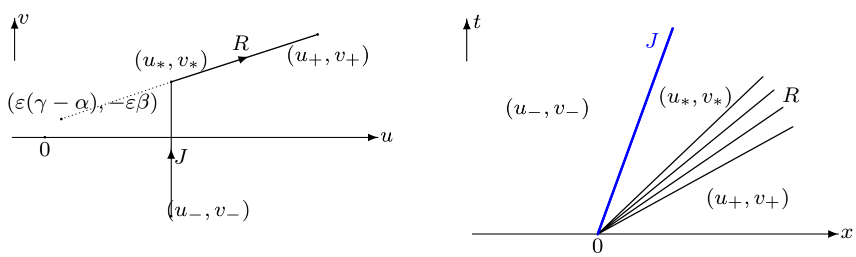

FIGURE 2. The Riemann solution of (1.2) and (1.3) is $J+R$ when $\varepsilon(\gamma-\alpha) < u_- < u_+$, where $\beta < 0$, $\gamma > \alpha$ and ε is a sufficiently small positive number.

such that we have

$$(u, v) = \Big(\frac{\xi - \varepsilon\alpha}{2}, v_+ - \frac{(u_+ - \frac{\xi-\varepsilon\alpha}{2})(v_+ + \varepsilon\beta)}{u_+ + \varepsilon\alpha - \varepsilon\gamma}\Big). \tag{3.20}$$

Thus, when $\varepsilon(\gamma-\alpha) < u_- < u_+$, the Riemann solution to (1.2) and (1.3) is

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & \xi < u_- + \varepsilon\gamma, \\ (u_*, v_*), & u_- + \varepsilon\gamma < \xi < 2u_- + \varepsilon\alpha, \\ R_2, & 2u_- + \varepsilon\alpha \le \xi \le 2u_+ + \varepsilon\alpha, \\ (u_+, v_+), & \xi > 2u_+ + \varepsilon\alpha, \end{cases} \tag{3.21}$$

in which the intermediate state (u_*, v_*) and the state (u, v) in R_2 are given by (3.17) and (3.20) respectively.

(3) If $u_+ < u_- < \varepsilon(\gamma-\alpha)$, then the Riemann solution to (1.2) and (1.3) is $S+J$ and the intermediate state between S and J is determined by

$$\begin{gathered} \frac{v_* + \varepsilon\beta}{u_* + \varepsilon\alpha - \varepsilon\gamma} = \frac{v_- + \varepsilon\beta}{u_- + \varepsilon\alpha - \varepsilon\gamma}, \\ u_* = u_+, \end{gathered} \tag{3.22}$$

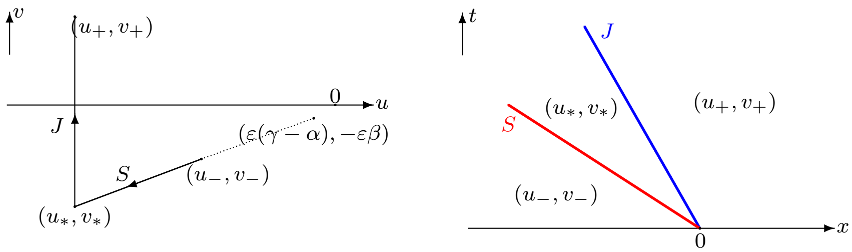

FIGURE 3. The Riemann solution of (1.2) and (1.3) is $S + J$ when $u_+ < u_- < \varepsilon(\gamma - \alpha)$, where $\beta > 0$, $\gamma < \alpha$ and ε is a sufficiently small positive number.

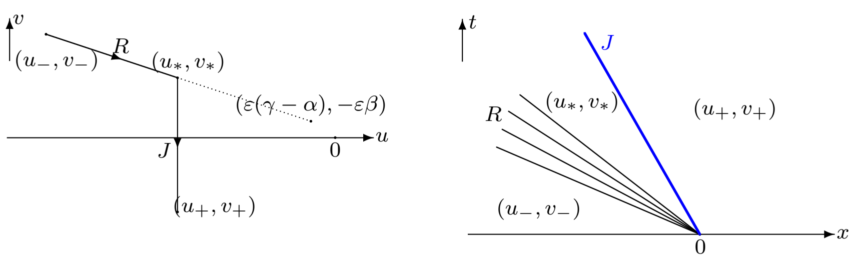

FIGURE 4. The Riemann solution of (1.2) and (1.3) is $R + J$ when $u_- < u_+ < \varepsilon(\gamma - \alpha)$, where $\beta < 0$, $\gamma < \alpha$ and ε is a sufficiently small positive number.

which implies that

$$(u_*, v_*) = \Big(u_+, v_- + \frac{(u_+ - u_-)(v_- + \varepsilon\beta)}{u_- + \varepsilon\alpha - \varepsilon\gamma}\Big). \tag{3.23}$$

Thus, when $u_+ < u_- < \varepsilon(\gamma - \alpha)$, the Riemann solution to (1.2) and (1.3) ie represented as

$$(u, v)(x, t) = \begin{cases} (u_-, v_-), & \xi < \sigma_1, \\ (u_*, v_*), & \sigma_1 < \xi < \tau_2, \\ (u_+, v_+), & \xi > \tau_2, \end{cases} \tag{3.24}$$

in which the intermediate state (u_*, v_*) is given by (3.23) and the propagation speeds of S_1 and J_2 can be calculated by $\sigma_1 = u_- + u_+ + \varepsilon\alpha$ and $\tau_2 = u_+ + \varepsilon\gamma$ respectively.

(4) If $u_- < u_+ < \varepsilon(\gamma - \alpha)$, then the Riemann solution to (1.2) and (1.3) is $R + J$ and the intermediate state between R and J can also be calculated by (3.23). The state (u, v) in R_1 is determined by

$$\begin{gathered} \xi = 2u + \varepsilon\alpha, \\ \frac{v + \varepsilon\beta}{u + \varepsilon\alpha - \varepsilon\gamma} = \frac{v_- + \varepsilon\beta}{u_- + \varepsilon\alpha - \varepsilon\gamma}, \end{gathered} \tag{3.25}$$

such that we have

$$(u,v)=\Big(\frac{\xi-\varepsilon\alpha}{2},v_- - \frac{(u_- - \frac{\xi-\varepsilon\alpha}{2})(v_- + \varepsilon\beta)}{u_- + \varepsilon\alpha - \varepsilon\gamma}\Big). \tag{3.26}$$

Thus, when $u_- < u_+ < \varepsilon(\gamma-\alpha)$, the Riemann solution to (1.2) and (1.3) is

$$(u,v)(x,t)=\begin{cases}(u_-,v_-), & \xi<2u_-+\varepsilon\alpha,\\ R_1, & 2u_-+\varepsilon\alpha\le\xi\le 2u_+ +\varepsilon\alpha,\\ (u_*,v_*), & 2u_+ +\varepsilon\alpha<\xi<u_+ +\varepsilon\gamma,\\ (u_+,v_+), & \xi>u_+ +\varepsilon\gamma,\end{cases} \tag{3.27}$$

in which the state (u,v) in R_1 and the intermediate state (u_*,v_*) are given by (3.26) and (3.23) respectively.

(5) If $u_- < \varepsilon(\gamma-\alpha) < u_+$, then the Riemann solution to (1.2) and (1.3) is R_1+J+R_2 which can be expressed as

$$(u,v)(x,t)=\begin{cases}(u_-,v_-), & \xi<2u_-+\varepsilon\alpha,\\ R_1, & 2u_-+\varepsilon\alpha\le\xi<\varepsilon(2\gamma-\alpha),\\ J, & \xi=\varepsilon(2\gamma-\alpha),\\ R_2, & \varepsilon(2\gamma-\alpha)<\xi\le 2u_+ +\varepsilon\alpha,\\ (u_+,v_+), & \xi>2u_+ +\varepsilon\alpha,\end{cases} \tag{3.28}$$

where the states (u_1,v_1) in R_1 and (u_2,v_2) in R_2 are given by (3.26) and (3.20) respectively. It is remarkable that the two rarefaction waves R_1 and R_2 are connected by the contact discontinuity J directly.

(6) If $u_+ < \varepsilon(\gamma-\alpha) < u_-$, then one can see that the singularity is impossible to be a jump with finite amplitude, which implies that there is no solution which is piecewise smooth and bounded. Motivated by [25], when $u_+ < \varepsilon(\gamma-\alpha) < u_-$, a solution containing a weighted delta measure supported on a curve should be introduced into the solution to the Riemann problem (1.2) and (1.3).

For the case $u_+ < \varepsilon(\gamma-\alpha) < u_-$, it can be concluded from Definitions 2.1 and 2.2 that the delta shock wave solution to the Riemann problem (1.2) and (1.3) can also be constructed in the form

$$(u,v)(x,t)=\begin{cases}(u_-,v_-), & \xi<\sigma_\delta,\\ (u_\delta,w(t)\delta(x-\sigma_\delta t)), & \xi=\sigma_\delta,\\ (u_+,v_+), & \xi>\sigma_\delta,\end{cases} \tag{3.29}$$

where $w(t)$ and σ_δ denote the strength and propagation speed of delta shock wave and u_δ indicates the assignment of u on this delta shock wave curve, respectively. In fact, the delta shock wave solution in the form (3.29) to the Riemann problem (1.2) and (1.3) should also satisfy

$$\begin{gathered}\langle u,\psi_t\rangle+\langle u^2+\varepsilon\alpha u,\psi_x\rangle=0,\\ \langle v,\psi_t\rangle+\langle uv+\varepsilon\beta u+\varepsilon\gamma v,\psi_x\rangle=0,\end{gathered} \tag{3.30}$$

for all test functions $\psi(x,t)\in C_0^\infty(R\times R_+)$. Actually, we have the following theorem to describe completely the delta shock wave solution to the Riemann problem (1.2) and (1.3) for the case $u_+ < \varepsilon(\gamma-\alpha) < u_-$.

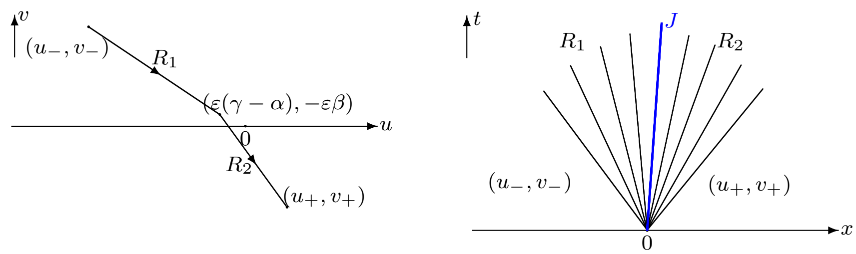

FIGURE 5. The Riemann solution of (1.2) and (1.3) is R_1+J+R_2 when $u_- < \varepsilon(\gamma-\alpha) < u_+$, where $\beta < 0$, $\gamma < \alpha$ and ε is a sufficiently small positive number.

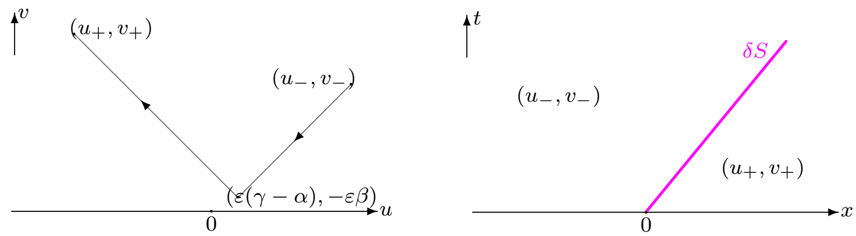

FIGURE 6. The Riemann solution of (1.2) and (1.3) is a delta shock wave δS when $u_+ < \varepsilon(\gamma-\alpha) < u_-$, where $\beta < 0$, $\gamma > \alpha$ and ε is a sufficiently small positive number.

Theorem 3.1. *If $u_+ < \varepsilon(\gamma-\alpha) < u_-$, then the delta shock wave solution to the Riemann problem* (1.2) *and* (1.3) *can be expressed in the form of* (3.29) *where*

$$\begin{gathered} u_\delta = u_- + u_+ + \varepsilon(\alpha-\gamma), \\ \sigma_\delta = u_- + u_+ + \varepsilon\alpha, \\ w(t) = (u_-v_+ - u_+v_- + \varepsilon(\alpha-\gamma)(v_+-v_-) - \varepsilon\beta(u_+-u_-))t. \end{gathered} \tag{3.31}$$

Furthermore, the delta shock wave solution (3.29) *with* (3.31) *should also satisfy the following generalized Rankine-Hugoniot condition*

$$\begin{gathered} \frac{dx}{dt} = \sigma_\delta, \\ \frac{dw(t)}{dt} = \sigma_\delta[v] - [uv + \varepsilon\beta u + \varepsilon\gamma v], \\ [u^2 + \varepsilon\alpha u] = \sigma_\delta[u], \end{gathered} \tag{3.32}$$

and the over-compressive entropy condition

$$\lambda_2(u_+,v_+) < \lambda_1(u_+,v_+) < \sigma_\delta < \lambda_1(u_-,v_-) < \lambda_2(u_-,v_-). \tag{3.33}$$

Proof. It is to check that the delta shock wave solution (3.29) with (3.31) should satisfy (1.2) in the sense of distributions. Based on the definition of Schwarz distributions, it is equivalent to proving that (3.29) with (3.31) should satisfy

$$\begin{gathered}\int_0^\infty\int_{-\infty}^\infty\Big(u\psi_t+(u^2+\varepsilon\alpha u)\psi_x\Big)\,dx\,dt=0,\\ \int_0^\infty\int_{-\infty}^\infty\Big(v\psi_t+(uv+\varepsilon\beta u+\varepsilon\gamma v)\psi_x\Big)\,dx\,dt=0,\end{gathered}\tag{3.34}$$

which is a weak form of system (1.2).

Without loss of generality, let us assume $\sigma_\delta>0$, noticing the fact that ψ is compact support in R_+^2, then we have

$$\begin{aligned} I&=\int_0^\infty\int_{-\infty}^\infty\Big(u\psi_t+(u^2+\varepsilon\alpha u)\psi_x\Big)\,dx\,dt\\ &=\int_0^\infty\int_{-\infty}^{x(t)}\Big(u_-\psi_t+(u_-^2+\varepsilon\alpha u_-)\psi_x\Big)\,dx\,dt+\int_0^\infty\int_{x(t)}^\infty\Big(u_+\psi_t+(u_+^2+\varepsilon\alpha u_+)\psi_x\Big)\,dx\,dt\\ &=\int_0^\infty\int_{t(x)}^\infty u_-\psi_t dtdx+\int_0^\infty\int_0^{t(x)}u_+\psi_t dtdx+\int_0^\infty(u_-^2+\varepsilon\alpha u_--u_+^2-\varepsilon\alpha u_+)\psi(x(t),t)dt\\ &=\int_0^\infty(u_+-u_-)\psi(x,t(x))dx+\int_0^\infty(u_-^2+\varepsilon\alpha u_--u_+^2-\varepsilon\alpha u_+)\psi(x(t),t)dt, \end{aligned}$$

where $t=t(x)=\frac{x}{\sigma_\delta}$ is the inverse function of $x=x(t)=\sigma_\delta t$. Thus, one can arrive at $\sigma_\delta=\frac{dx}{dt}=u_-+u_++\varepsilon\alpha$ which satisfies $\sigma_\delta[u]=[u^2+\varepsilon\alpha u]$ for I vanishes for any test function $\psi(x,t)\in C_c^\infty(R\times R_+)$.

Analogously, we also have

$$\begin{aligned} II&=\int_0^\infty\int_{-\infty}^\infty\Big(v\psi_t+(uv+\varepsilon\beta u+\varepsilon\gamma v)\psi_x\Big)\,dx\,dt\\ &=\int_0^\infty\int_{-\infty}^{x(t)}\Big(v_-\psi_t+(u_-v_-+\varepsilon\beta u_-+\varepsilon\gamma v_-)\psi_x\Big)\,dx\,dt\\ &\quad+\int_0^\infty\int_{x(t)}^\infty\Big(v_+\psi_t+(u_+v_++\varepsilon\beta u_++\varepsilon\gamma v_+)\psi_x\Big)\,dx\,dt\\ &\quad+\int_0^\infty w(t)\Big(\psi_t(x(t),t)+(u_\delta+\varepsilon\gamma)\psi_x(x(t),t)\Big)dt\\ &=\int_0^\infty\Big(u_-v_-+\varepsilon\beta u_-+\varepsilon\gamma v_--u_+v_+-\varepsilon\beta u_+-\varepsilon\gamma v_+\Big)\psi(x(t),t)dt\\ &\quad+\int_0^\infty(v_+-v_-)\psi(x,t(x))dx+\int_0^\infty w(t)d\psi(x(t),t), \end{aligned}$$

in which $u_\delta+\varepsilon\gamma=\sigma_\delta$ has been used. Thus, one can see that the second equality in (3.32) should be satisfied since II vanishes for any test function $\psi(x,t)\in C_c^\infty(R\times R_+)$, such that one can get the strength $w(t)$ of delta shock wave.

To ensure the uniqueness of delta shock wave solution to the Riemann problem (1.2) and (1.3) when $u_+<\varepsilon(\gamma-\alpha)<u_-$, the over-compressive entropy condition should be proposed. Remember that $\lambda_1<\lambda_2$ when $u>\varepsilon(\gamma-\alpha)$ and $\lambda_1>\lambda_2$ when $u<\varepsilon(\gamma-\alpha)$. If $u_+<\varepsilon(\gamma-\alpha)<u_-$, then the over-compressive entropy

condition for delta shock wave (3.33) should be proposed, which implies that all the characteristics enter the delta shock wave curve from both sides. □

4. Limits of Riemann solutions as $\varepsilon \to 0$

In this section, we are concerned that the limits of solutions to the Riemann problem (1.2) and (1.3) converge to the corresponding ones of the Riemann problem (1.1) and (1.3) or not when the perturbation parameter ε tends to zero. In what follows, we have the theorem to depict the limit problem fully.

Theorem 4.1. *The limits of solutions to the Riemann problem* (1.2) *with* (1.3) *converge to the corresponding ones for the non-strictly hyperbolic system* (1.1) *with the same Riemann initial data as* $\varepsilon \to 0$ *in all kinds of situations.*

Proof. The proof should also be divided into the following six cases according to the values of u_- and u_+.

(1) If $\varepsilon(\gamma-\alpha) < u_+ < u_-$, then it tends to $0 < u_+ < u_-$ by taking the limit $\varepsilon \to 0$. It is clear to see from (3.17) and (3.18) that the limit $\varepsilon \to 0$ of Riemann solution to (1.2) and (1.3) is also $J + S$ which can be expressed as

$$\lim_{\varepsilon\to 0}(u,v)(x,t) = \begin{cases} (u_-, v_-), & \xi < u_-, \\ (u_-, \frac{u_- v_+}{u_+}), & u_- < \xi < u_- + u_+, \\ (u_+, v_+), & \xi > u_- + u_+. \end{cases} \tag{4.1}$$

(2) If $\varepsilon(\gamma-\alpha) < u_- < u_+$, then we have $0 < u_- < u_+$ in the limit situation. Now it follows from (3.21) together with (3.17) and (3.20) that the limit $\varepsilon \to 0$ of Riemann solution to (1.2) and (1.3) is $J + R$ which can be expressed as

$$\lim_{\varepsilon\to 0}(u,v)(x,t) = \begin{cases} (u_-, v_-), & \xi < u_-, \\ (u_-, \frac{u_- v_+}{u_+}), & u_- < \xi < 2u_-, \\ (\frac{\xi}{2}, \frac{\xi v_+}{2u_+}), & 2u_- \le \xi \le 2u_+, \\ (u_+, v_+), & \xi > 2u_+. \end{cases} \tag{4.2}$$

(3) If $u_+ < u_- < \varepsilon(\gamma-\alpha)$, then we have $u_+ < u_- < 0$ in the limit situation. Then, it can be derived from (3.23) and (3.24) that the limit $\varepsilon \to 0$ of Riemann solution to (1.2) and (1.3) is also $S + J$ given by

$$\lim_{\varepsilon\to 0}(u,v)(x,t) = \begin{cases} (u_-, v_-), & \xi < u_- + u_+, \\ (u_+, \frac{u_+ v_-}{u_-}), & u_- + u_+ < \xi < u_+, \\ (u_+, v_+), & \xi > u_+. \end{cases} \tag{4.3}$$

(4) If $u_- < u_+ < \varepsilon(\gamma-\alpha)$, then it is cleat that $u_- < u_+ < 0$ in the limit situation. Then, it follows from (3.27) together with (3.23) and (3.26) that the limit $\varepsilon \to 0$ of Riemann solution to (1.2) and (1.3) is also $R + J$ given by

$$\lim_{\varepsilon\to 0}(u,v)(x,t) = \begin{cases} (u_-, v_-), & \xi < 2u_-, \\ (\frac{\xi}{2}, \frac{\xi v_-}{2u_-}), & 2u_- \le \xi \le 2u_+, \\ (u_+, \frac{u_+ v_-}{u_-}), & 2u_+ < \xi < u_+, \\ (u_+, v_+), & \xi > u_+. \end{cases} \tag{4.4}$$

(5) It is clear to get $u_- < 0 < u_+$ by taking the limit $\varepsilon \to 0$ in the inequality $u_- < \varepsilon(\gamma-\alpha) < u_+$. Then, it follows from (3.28) together with (3.20) and (3.26)

that the limit $\varepsilon \to 0$ of Riemann solution to (1.2) and (1.3) is also $R+J+R$ given by

$$\lim_{\varepsilon\to 0}(u,v)(x,t)=\begin{cases}(u_-,v_-), & \xi<2u_-,\\ (\frac{\xi}{2},\frac{\xi v_-}{2u_-}), & 2u_-\le\xi<0,\\ (0,0), & \xi=0,\\ (\frac{\xi}{2},\frac{\xi v_+}{2u_+}), & 0<\xi\le 2u_+,\\ (u_+,v_+), & \xi>2u_+.\end{cases} \tag{4.5}$$

(6) Finally, we also have $u_+<0<u_-$ by taking the limit $\varepsilon\to 0$ in the inequality $u_+<\varepsilon(\gamma-\alpha)<u_-$. Then, it follows from (3.29) and (3.31) that the limit $\varepsilon\to 0$ of Riemann solution to (1.2) and (1.3) is also a delta shock wave given by

$$\begin{aligned}&\lim_{\varepsilon\to 0}(u,v)(x,t)\\ &=\begin{cases}(u_-,v_-), & \xi<u_-+u_+,\\ (u_-+u_+,(u_-v_+-u_+v_-)t\delta(x-(u_-+u_+)t)), & \xi=u_-+u_+,\\ (u_+,v_+), \quad \xi>u_-+u_+.\end{cases}\end{aligned} \tag{4.6}$$

Thus, the conclusion of the theorem can be drawn by gathering the results for the six different cases together. □

5. Conclusions and discussions

It can be seen from the above discussions that the limits of solutions to the Riemann problem (1.2) and (1.3) converge to the corresponding ones of the Riemann problem (1.1) and (1.3) as $\varepsilon\to 0$. The reason lies in that the approximated system (1.2) is still non-strictly hyperbolic and the characteristic field for λ_1 is still linearly degenerate and the characteristic field for λ_2 is still genuinely nonlinear. Thus, this perturbation does not change the structure of Riemann solutions.

In addition, let us turn our attentions on the simplified system of pressureless gas dynamics as follows:

$$\begin{gathered}u_t+(\frac{u^2}{2})_x=0,\\ v_t+(uv)_x=0.\end{gathered} \tag{5.1}$$

It is easy to check that (5.1) has a double eigenvalue $\lambda=u$ and only one right eigenvector $\overrightarrow{r}=(0,1)^T$. Then, we can get $\nabla\lambda\cdot\overrightarrow{r}=0$, which implies that the characteristic field for λ is always linearly degenerate. Thus, the solutions of Riemann problem (5.1) and (1.3) can be constructed by contact discontinuities, vacuum or delta shock wave connecting two constant states $(u_\pm,v_\pm)$.

If we also consider the linear approximations of flux functions for (5.1) in the form

$$\begin{gathered}u_t+(\frac{u^2}{2}+\varepsilon\alpha u)_x=0,\\ v_t+(uv+\varepsilon\beta u+\varepsilon\gamma v)_x=0.\end{gathered} \tag{5.2}$$

We can check that if $\alpha=\gamma$, then (5.2) has also a double eigenvalue $\lambda=u+\varepsilon\alpha$ and only one right eigenvector $\overrightarrow{r}=(0,1)^T$. Furthermore, one can see that the characteristic field for λ is always linearly degenerate. In other words, (5.2) also belongs to the Temple class when $\alpha=\gamma$. Through a simple calculation, it can be concluded that the Riemann solutions for the approximated system (5.2) just translate the ones for the original system (5.1) in the (x,t) plane and do not change the structure. Thus, when $\alpha = \gamma$, we can also see that the limits of solutions to the Riemann problem (5.2) and (1.3) converge to the corresponding ones of the Riemann problem (5.1) and (1.3) as $\varepsilon \to 0$ in all the situations.

Otherwise, if $\alpha \neq \gamma$, then (5.2) has two different eigenvalues $\lambda_1 = u + \varepsilon\alpha$ and $\lambda_2 = u + \varepsilon\gamma$. Thus, (5.2) is strictly hyperbolic for $\alpha \neq \gamma$. It is easy to get that the right eigenvectors are $\overrightarrow{r}_1 = (\varepsilon(\alpha - \gamma), v + \varepsilon\beta)^T$ and $\overrightarrow{r}_2 = (0, 1)^T$ respectively, such that we have $\nabla\lambda_1 \cdot \overrightarrow{r}_1 = \varepsilon(\alpha - \gamma) \neq 0$ and $\nabla\lambda_2 \cdot \overrightarrow{r}_2 = 0$. Hence, the characteristic field for λ_1 is genuinely nonlinear and the characteristic field for λ_2 is linearly degenerate. By a simple calculation, it can be seen that (5.2) does not belong to the Temple class any more when $\alpha \neq \gamma$. It is clear to see that if $\alpha \neq \gamma$, then the Riemann solutions for the approximated system (5.2) have completely different structures with those for the original system (5.1). If we introduce the substitutions of state variables $u + \varepsilon\alpha \to u$ and $v + \varepsilon\beta \to v$, then (5.2) becomes

$$
\begin{gathered}
u_t + (\frac{u^2}{2})_x = 0, \\
v_t + (uv + \varepsilon(\gamma - \alpha)v)_x = 0.
\end{gathered} \tag{5.3}
$$

It is well known that this form of the system does not change for the reason the variable substitutions are linear in the conserved quantities. If $\gamma < \alpha$, then it can be concluded from [20] that the limits of solutions to the Riemann problem (5.3) and (1.3) also converge to the corresponding ones of the Riemann problem (5.1) and (1.3) as $\varepsilon \to 0$ in all the situations. On the other hand, if $\gamma > \alpha$, then the similar calculation can be carried out and the same conclusion can be drawn. Thus, if $\alpha \neq \gamma$, then one can conclude that the limits of solutions to the Riemann problem (5.2) and (1.3) converge to the corresponding ones of the Riemann problem (5.1) and (1.3) as $\varepsilon \to 0$ in all the situations.

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