Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 124, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

EXISTENCE OF SOLUTIONS FOR SECOND-ORDER DIFFERENTIAL EQUATIONS WITH DISCONTINUOUS RIGHT-HAND SIDE

ALEXANDER M. KAMACHKIN, DMITRIY K. POTAPOV, VICTORIA V. YEVSTAFYEVA

Abstract. We consider an ordinary differential equation of second order with constant coefficients and a discontinuous right-hand side. First we use the point mapping method defining first return functions, then we use the phase-plane method. We establish both the existence and non-existence of periodic solutions (including stable ones) and oscillatory solutions depending on the coefficients of the equation. By the variational method, we prove the existence of nonzero semiregular solutions for a boundary-value problem.

1. Introduction and statement of the problem

Studies of differential equations with discontinuous right-hand sides go back a long way [1, 4]. Recently, investigation of such equations is also given much attention (see, e.g. [11, 12, 13, 19]). These equations generate interest in theoretical researches and also in many applications. In a number of applied problems, mathematical models are reduced to ODE of second order with discontinuous right-hand sides [9]. These are mathematical models for dynamic systems describing determinate system behaviour of various physical nature. Despite the seeming simplicity of the differential equation, such models are difficult to study fully and in detail, since they are the essentially nonlinear models with a nonanalytical function. It should be noted that linearization of such systems leads to mathematical models describing nonobservable processes in real physical objects. It is known that essential nonlinearity arises as a result of mathematical description of such physical effects as Coulomb friction or ideal relay. Also, such equations describe nonlinear oscillations [2]. Examination of periodic solutions for these equations is of certain interest. Nowadays, there are a number of open questions in this direction.

We consider the differential equation of second order with the discontinuous right-hand side

$$\ddot{x} + A\dot{x} + Bx = C\operatorname{sgn}(x). \tag{1.1}$$

Here $x = x(t)$ is the sought-for function; A, B, and C are real constants ($C \neq 0$). The sign function describes, for example, an ideal relay, and thus equation (1.1) does a non-smooth oscillator. In recent years the ODEs of second order with discontinuous right parts have been studied in [3, 5, 6, 7, 10, 14, 15, 18] Jacquemard

2010 *Mathematics Subject Classification.* 34A34, 34A36, 34C25, 34B15.
Key words and phrases. Point mapping method; first return function; phase-plane method; periodic solution; stability; variational method; semiregular solution.

Submitted March 15, 2016. Published May 16, 2016.

and Teixeira [5] investigate equation (1.1) for the case when $A = B = 0$. The case $A = 0$, $B = C = 1$ is considered by Samoilenko and Nizhnik [18]. An applied problem with a parameter $\mu = -C > 0$ to be the vorticity for equation (1.1) when $A = B = 0$ with zero boundary conditions is discussed in [17]. A boundary value problem to ODE of second order with superlinear convex nonlinearity is given in [20]. Remark that equation (1.1) in the most general form is studied in [7, 10]. This work continues researches mentioned above. In the paper we consider two cases: $A \neq 0$, $B = 0$ and $A = 0$, $B \in \mathbb{R}$. Unlike [10], we investigate the case when $A > 0$ additionally and define the function $f(x) = \operatorname{sgn} x$ as follows:

$$\operatorname{sgn} x = \begin{cases} -1 & \text{if } x < 0, \\ 0 & \text{if } x = 0, \\ 1 & \text{if } x > 0. \end{cases}$$

2. Solution of the problem: case $B = 0$

Let $A \neq 0$ and $B = 0$. Then the motion equations of system (1.1) have the form

$$\ddot{x} + A\dot{x} = -C \text{ for } x < 0, \quad \ddot{x} + A\dot{x} = C \text{ for } x > 0. \tag{2.1}$$

Put $y = \dot{x}$. Note that $y \neq 0$, since $C \neq 0$. Therefore the equations of phase trajectories are the following:

$$\begin{aligned} \frac{dy}{dx} &= -A - \frac{C}{y} \quad \text{for } x < 0, \\ \frac{dy}{dx} &= -A + \frac{C}{y} \quad \text{for } x > 0. \end{aligned}$$

Whence it follows that

$$\begin{aligned} y - y_0 - \frac{C}{A} \ln \frac{-C - Ay}{-C - Ay_0} &= -A(x - x_0) \quad \text{for } x < 0, \\ y - y_0 + \frac{C}{A} \ln \frac{-C + Ay}{-C + Ay_0} &= -A(x - x_0) \quad \text{for } x > 0, \end{aligned} \tag{2.2}$$

where x_0, y_0 are the initial coordinates, and x, y are the current coordinates of the phase trajectories.

The phase trajectories are symmetric with respect to the origin of coordinates and have the following properties:

(1) if $A > 0$, $C > 0$, then at any (x_0, y_0) the image point goes into infinity;
(2) if $A < 0$, $C > 0$, then at any (x_0, y_0) the image point goes into infinity;
(3) if $A > 0$, $C < 0$, then there may exist closed phase trajectories (periodic solution curves);
(4) if $A < 0$, $C < 0$, then the image point tends to the point $(0, 0)$ from any point that belongs to the sufficiently small neighborhood of $(0, 0)$, and additional research is needed.

Let us study the cases (3) and (4) using the point mapping method. Symmetry of the system allows us to carry out point transformations on a half of the phase plane.

For every case we shall define the first return functions $F(x, y) = 0$ and $F(x, |y|) = 0$ as functions of mapping of points on the half-line $x > 0$, $y = 0$ into the points on the half-line $x = 0$, $y > 0$ accordingly, and then of points on the half-line $x > 0$, $y = 0$ into the points on the half-line $x = 0$, $y = -|y| < 0$ in virtue of (2.2).

The first return function $F(x,y)=0$ is defined by the expression

$$y+\frac{C}{A}\ln\big(1-\frac{A}{C}y\big)=-Ax. \tag{2.3}$$

By the definition $F(x,y)=0$, we have

$$\begin{aligned} \frac{dy}{dx}&=-A+\frac{C}{y}\\ \frac{d^2y}{dx^2}&=-\frac{C}{y^2}\frac{dy}{dx}. \end{aligned} \tag{2.4}$$

Mapping of the point set $\{x>0, y=0\}$ into the set $\{x=0, y>0\}$ is realized by (2.3) in the direction opposite to the movement along the phase curves. In addition, the function $y=y(x)$ increases with the growth of x monotonously. Thus, $\frac{dy}{dx}=A-\frac{C}{y}$, contrary to (2.4). If $A>0$, $C<0$, then $\frac{dy}{dx}>0$ and $\frac{d^2y}{dx^2}=\frac{C}{y^2}\frac{dy}{dx}<0$. This means that the function $y=y(x)$ is a monotonously increasing and concave function, $y(0)=0$. If $A<0$, $C<0$, then $y=y(x)$ approaches the line $y=C/A$ from below asymptotically as $x\to+\infty$, $y(0)=0$. The function $y(x)$ is an increasing and concave function, since $\frac{dy}{dx}=A-\frac{C}{y}>0$ when $y<\frac{C}{A}$, $\frac{d^2y}{dx^2}=\frac{C}{y^2}\frac{dy}{dx}<0$.

For the cases above the first return function $F(x,|y|)=0$ is defined by the expression

$$|y|-\frac{C}{A}\ln\big(1+\frac{A}{C}|y|\big)=Ax. \tag{2.5}$$

In this connection,

$$\frac{d|y|}{dx}=A+\frac{C}{|y|},\quad \frac{d^2|y|}{dx^2}=-\frac{C}{|y|^2}\frac{d|y|}{dx}. \tag{2.6}$$

Mapping of the point set $\{x>0, y=0\}$ into the set $\{x=0, y<0\}$ is realized by (2.5) in the direction coinciding with the movement along the phase curves. At the same time the function $y(x)$ decreases monotonously and $y(x)<0$. The function $|y|(x)$ increases monotonously, $|y|(x)>0$, and the sign of derivatives (2.6) changes to the opposite one. If $A<0$, $C<0$, then the function $|y|(x)$ tends to $+\infty$ as $x\to+\infty$ and is concave, $|y|(0)=0$. If $A>0$, $C<0$, then the concave function $|y|(x)$ approaches the line $|y|=-C/A$ asymptotically as $x\to+\infty$, $|y|(0)=0$. Next, let us consider the mutual arrangement of the functions $F(x,y)=0$ and $F(x,|y|)=0$ for each of the two cases.

If $A<0$, $C<0$, then the curves $y(x)$ and $|y|(x)$ are crossed on the plane $(x,(y,|y|))$. This means that there is a point $x=x^*$ such that $y(x^*)=|y|(x^*)$, i.e. $F(x^*,y(x^*))=F(x^*,|y|(x^*))=0$, and also the inequality $|y|(x)<y(x)$ holds at $x\in(0,x^*)$, and the inequality $|y|(x)>y(x)$ is fair for $x>x^*$. A closed periodic trajectory being unstable corresponds to the point $x=x^*$ on the phase plane. In this case, the closed periodic trajectory bounds the domain of the phase plane from any point of which the phase trajectory tends to the point $(0,0)$. If the initial point lies in the contradomain, then the phase trajectory leaves on infinity. Therefore system (2.1) has an unstable periodic solution when $A<0$, $C<0$.

If $A>0$, $C<0$, then the graphs $y(x)$ and $|y|(x)$ are crossed on the plane $(x,(y,|y|))$, i.e. there is also a point $x=x^*$ such that $F(x^*,y(x^*))=F(x^*,|y|(x^*))=0$. In addition, if $x\in(0,x^*)$ then $|y|(x)>y(x)$, and if $x>x^*$ then $|y|(x)<y(x)$. A closed periodic trajectory being stable corresponds to the point $x=x^*$ on the phase

plane. Hence, if $A > 0$, $C < 0$, then system (2.1) has a stable periodic solution to which phase trajectories tend at any initial points.

We establish the following theorem.

Theorem 2.1. *For system* (2.1) *the following statements hold:*
(1) if $A \neq 0$ *and* $C > 0$*, then* (2.1) *has no oscillatory solutions at any initial points except* $(0, 0)$*, the image point leaves on infinity,* $(0, 0)$ *is the unstable equilibrium point;*
(2) if $A < 0$ *and* $C < 0$*, then on the phase plane there exists a unique unstable periodic trajectory, which is a separatrix, separating the domain of attraction of the point* $(0, 0)$ *from the domain in which the image point leaves on infinity at any initial point;*
(3) if $A > 0$ *and* $C < 0$*, then on the phase plane there exists a unique periodic trajectory, which is stable in the large except* $(0, 0)$*, the point* $(0, 0)$ *is the unstable equilibrium point.*

So, we have analysed all possible ratios between A and C in (2.1) for $A \neq 0$, $C \neq 0$. This case is completely considered.

3. Solution of the problem: case $A = 0$

Let $A = 0$ and $B \in \mathbb{R}$. Then equation (1.1) takes the form

$$\ddot{x} + Bx = C \operatorname{sgn}(x(t)). \tag{3.1}$$

For $B = 0$ equation (3.1) is studied in [5]. Equation (3.1) with a discontinuous right part in more general form for $B = 0$ is investigated in [6]. Therefore we shall assume further that $B \neq 0$. The characteristic equation of (3.1) has the form $\lambda^2 + B = 0$.

If $B < 0$, then we have $\lambda_{1,2} = \pm\sqrt{-B}$. The general solution of the nonhomogeneous equation is of the form

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} + \frac{C}{B}$$

(c_1, c_2 are arbitrary constants). If $C > 0$, then $C/B < 0$. For $x > 0$ the proper lines $\dot{x} = \lambda_1\big(x - \frac{C}{B}\big)$, $\dot{x} = \lambda_2\big(x - \frac{C}{B}\big)$ ($\lambda_1 > 0$, $\lambda_2 < 0$) are crossed in the point $(C/B, 0)$ that lies on the left half-plane, since $C/B < 0$. For $x < 0$ these lines are crossed in the point $(-C/B, 0)$ on the right half-plane. Superposition of phase portraits shows that at any initial point $(x_0, \dot{x}_0) \neq (0, 0)$ the phase trajectory leaves on infinity without making oscillating motions. The point $(0, 0)$ is an unstable equilibrium point. Let now $C < 0$. Since $C < 0$, we have $C/B > 0$. For $x > 0$ the proper lines are crossed in the point $(C/B, 0)$ on the right half-plane. For $x < 0$ the proper lines are crossed in the point $(-C/B, 0)$ on the left half-plane. By superposition of the phase portraits, we obtain the closed quadrangle on the plane $(xO\dot{x})$

$$\begin{aligned}\overline{G} = \Big\{&(x, \dot{x}) : -\frac{C}{B} \leq x \leq \frac{C}{B};\\ &\lambda_1\big(x - \frac{C}{B}\big) \leq \dot{x} \leq \lambda_2\big(x - \frac{C}{B}\big) \text{ if } x \geq 0;\\ &\lambda_2\big(x + \frac{C}{B}\big) \leq \dot{x} \leq \lambda_1\big(x + \frac{C}{B}\big) \text{if } x \leq 0\Big\}.\end{aligned} \tag{3.2}$$

The line segments $\dot{x} = \lambda_1(x - \frac{C}{B})$, $\dot{x} = \lambda_2(x - \frac{C}{B})$, $\dot{x} = \lambda_1(x + \frac{C}{B})$, $\dot{x} = \lambda_2(x + \frac{C}{B})$ between the points $(0, -\lambda_1 C/B)$ and $(C/B, 0)$, the points $(0, -\lambda_2 C/B)$ and $(C/B, 0)$, the points $(-C/B,0)$ and $(0,\lambda_1 C/B)$, the points $(-C/B,0)$ and $(0,\lambda_2 C/B)$ respectively make the border ∂G of the set $\overline{G}$ ($\overline{G}=\mathring{G}\cup\partial G$). If the initial point $(x_0,\dot x_0)\notin\overline{G}$,

$$(x_0,\dot x_0)\notin\big\{(x,\dot x):\dot x=\lambda_2(x-\frac{C}{B}),\dot x<0\big\},$$
$$(x_0,\dot x_0)\notin\big\{(x,\dot x):\dot x=\lambda_2(x+\frac{C}{B}),\dot x>0\big\},$$

then the image point goes to infinity without making oscillations. If $(x_0,\dot x_0)\notin\overline{G}$ but

$$(x_0,\dot x_0)\in\big\{(x,\dot x):\dot x=\lambda_2\big(x-\frac{C}{B}\big),\dot x<0\big\},$$

then the image point passes into the point $(C/B,0)$. If $(x_0,\dot x_0)\notin\overline{G}$ but

$$(x_0,\dot x_0)\in\big\{(x,\dot x):\dot x=\lambda_2\big(x+\frac{C}{B}\big),\dot x>0\big\},$$

then the image point passes into the point $(-C/B,0)$. If $(x_0,\dot x_0)\in\partial G$, then for final time the image point passes along the segments into one of the points $(C/B,0)$ or $(-C/B,0)$, which are unstable equilibrium points in every directions except the lines $\dot x=\lambda_2(x-\frac{C}{B})$ and $\dot x=\lambda_2\left(x+\frac{C}{B}\right)$ respectively. If $(x_0,\dot x_0)\in\mathring{G}$ but $(x_0,\dot x_0)\neq(0,0)$, then the image point makes oscillations coming to ∂G asymptotically. Thus oscillations occur in the bounded domain $\mathring{G}$. The point $(0,0)\in\mathring{G}$ is an unstable singularity.

If $B>0$, then $\lambda_{1,2}=\pm i\sqrt{B}$, and the general solution of (3.1) is the following

$$x(t)=c_1\cos\big(\sqrt{B}t\big)+c_2\sin\big(\sqrt{B}t\big)+\frac{C}{B}.$$

If $C>0$, then $C/B>0$. By virtue of (3.1), for $x>0$ there is a circle with the center in the point $(C/B,0)$, which is tangent to the axis $O\dot x$ at the point $(0,0)$. Next we give the equation of this circle

$$B^2\big(x-\frac{C}{B}\big)^2+B\dot x^2=C^2\quad\text{if } x\geq 0. \tag{3.3}$$

Let G_1 be interior of the disk bounded by the circle (3.3). By analogy, for $x<0$ there is a circle with the center in the point $(-C/B,0)$ that is tangent to the axis $O\dot x$ at the point $(0,0)$, i.e.

$$B^2\big(x+\frac{C}{B}\big)^2+B\dot x^2=C^2\quad\text{if } x\leq 0. \tag{3.4}$$

Let G_2 be interior of the disk bounded by the circle (3.4). Hence if the initial point is $(x_0,\dot x_0)\notin\overline{G_1}\cup\overline{G_2}$, then the image point goes around the point $(0,0)$ along the symmetric periodic trajectory. If $(x_0,\dot x_0)\in\partial G_1\cup\partial G_2$, i.e. the initial point belongs to one of the circles (3.3) or (3.4), then for the final time the image point passes into the point $(0,0)$, which is the unstable equilibrium point. If $(x_0,\dot x_0)\in G_1$, then the image point makes a round of the point $(C/B,0)$ along the periodic circular trajectory. If $(x_0,\dot x_0)\in G_2$, then the image point makes a round of the point $(-C/B,0)$ along the periodic circular trajectory. Let next $C<0$, then $C/B<0$. For $x>0$ the center of the circle (3.3) is in the point $(C/B,0)$, i.e. on the left half-plane of the plane $(xO\dot x)$. For $x<0$ the center of the circle (3.4) is transferred to the point $(-C/B,0)$ on the right half-plane. Therefore these centers become virtual for $x>0$ and for $x<0$, and at any initial point $(x_0,\dot x_0)\neq(0,0)$ the image point makes a round of the point $(0,0)$ along the symmetric periodic trajectory. The point $(0,0)$ is an isolated singularity. Therefore the following theorem holds.

Theorem 3.1. *For equation* (3.1) *the following statements hold:*
(1) if $B<0$ and $C>0$, then (3.1) *has no oscillatory solutions at any initial points except* $(0,0)$*, the image point goes into infinity,* $(0,0)$ *is an unstable equilibrium point;*
(2) if $B<0$ and $C<0$, then on the phase plane there exists the bounded set $\overline{G}$ defined by (3.2)*, where $\lambda_1=\sqrt{-B}$, $\lambda_2=-\sqrt{-B}$, and such that*

- (a) *if the initial point $(x_0,\dot{x}_0)\notin\overline{G}$, $(x_0,\dot{x}_0)\notin\left\{(x,\dot{x}):\dot{x}=\lambda_2\left(x-\frac{C}{B}\right),\dot{x}<0\right\}$ and $(x_0,\dot{x}_0)\notin\left\{(x,\dot{x}):\dot{x}=\lambda_2\left(x+\frac{C}{B}\right),\dot{x}>0\right\}$, then the image point goes into infinity without making oscillations;*
- (b) *if $(x_0,\dot{x}_0)\notin\overline{G}$ but $(x_0,\dot{x}_0)\in\left\{(x,\dot{x}):\dot{x}=\lambda_2\left(x-\frac{C}{B}\right),\dot{x}<0\right\}$, then the image point passes into the point $(C/B,0)$; if $(x_0,\dot{x}_0)\notin\overline{G}$ but $(x_0,\dot{x}_0)\in\left\{(x,\dot{x}):\dot{x}=\lambda_2\left(x+\frac{C}{B}\right),\dot{x}>0\right\}$, then the image point passes into the point $(-C/B,0)$;*
- (c) *if $(x_0,\dot{x}_0)\in\partial G$, then for the final time the image point passes into the points $(C/B,0)$ or $(-C/B,0)$ (if $\dot{x}_0>0$, then into $(C/B,0)$, and if $\dot{x}_0<0$, then into $(-C/B,0)$), which are the unstable equilibrium points in every directions except the lines $\dot{x}=\lambda_2\left(x-\frac{C}{B}\right)$ and $\dot{x}=\lambda_2\left(x+\frac{C}{B}\right)$ respectively;*
- (d) *if $(x_0,\dot{x}_0)\in\mathring{G}$ but $(x_0,\dot{x}_0)\neq(0,0)$, then the image point makes oscillations coming to ∂G asymptotically;*
- (e) *the point $(0,0)\in\mathring{G}$ is an unstable singularity;*

(3) if $B>0$ and $C>0$, then on the phase plane there exists a separatrix defined by (3.3), (3.4) *with the following properties:*

- (a) *if the initial point $(x_0,\dot{x}_0)$ belongs to the contradomain bounded by this separatrix, then the image point goes around the point $(0,0)$ along the symmetric periodic trajectory;*
- (b) *if the point $(x_0,\dot{x}_0)$ belongs to the separatrix, then the image point passes into the point $(0,0)$ for the final time; $(0,0)$ is the unstable equilibrium point;*
- (c) *if $(x_0,\dot{x}_0)$ is an inner point of the domain bounded by the separatrix, then along the periodic circular trajectory the image point goes around the point $(C/B,0)$ for $x_0>0$ or the point $(-C/B,0)$ for $x_0<0$; $(C/B,0)$ and $(-C/B,0)$ are isolated singularities;*

(4) if $B>0$ and $C<0$, then at any initial points except $(0,0)$ the image point goes around the point $(0,0)$ along the symmetric periodic trajectory; $(0,0)$ is an isolated singularity.

So, we have considered all possible ratios between B and C in (3.1) when $B\neq 0$, $C\neq 0$. This case is completely studied.

Next let $t\in[a,b]$. In addition, we complement equation (3.1) with the boundary condition

$$x(a)=x(b)=0. \tag{3.5}$$

Let the space $X=H^1_\circ([a,b])$. We assign the functional J^c on X defined by $J^c(x)=J_1(x)+CJ_2(x)$ to the boundary value problem (3.1), (3.5). Here

$$J_1(x)=\frac{1}{2}\int_a^b(x'(t))^2dt-\frac{1}{2}B\int_a^b x^2(t)dt,\quad J_2(x)=\int_a^b dt\int_0^{x(t)}\operatorname{sgn}(s)ds.$$

In forthcoming consideration, we shall use the following definitions.

A *strong solution* to problem (3.1), (3.5) is a function $x \in W_q^2([a,b])$ satisfying equation (3.1) for almost all $t \in [a,b]$ and the boundary conditions (3.5).

A *semiregular solution* to problem (3.1), (3.5) is a strong solution x such that $x(t)$ is a point of continuity of the function $\operatorname{sgn}(\cdot)$ for almost all $t \in [a,b]$.

An *upward jump discontinuity* of a function $f : \mathbb{R} \to \mathbb{R}$ is a point $x \in \mathbb{R}$ such that $f(x-) < f(x+)$, where $f(x\pm) = \lim_{s\to x\pm} f(s)$.

The notion of semiregular solutions for the equations with discontinuous nonlinearities was first introduced in [8].

For any C problem (3.1), (3.5) has a strong trivial solution $x(t) \equiv 0$, which is not semiregular, since $x = 0$ is a point of discontinuity of $\operatorname{sgn}(x)$. Therefore searching nonzero semiregular solutions to problem (3.1), (3.5) is of interest. By the variational method, we have proven the following theorem on the existence of nonzero semiregular solutions to problem (3.1), (3.5) when $B \le 0$ and some $C < 0$.

Theorem 3.2. *Let the coefficient B be nonpositive in* (3.1). *Then there exists a negative number C_0 such that $\inf_{y\in X} J^c(y) < 0$ for each $C < C_0$, there exists an $x_c \in X$ such that $J^c(x_c) = \inf_{y\in X} J^c(y)$, and every x_c satisfying this condition is a nonzero semiregular solution to problem* (3.1), (3.5).

Proof. Problem (3.1), (3.5) is a special case of the boundary value problem investigated in [14]. For this reason, the proof of Theorem 3.2 reduces to verifying the conditions in [14, Theorem].

Since $B \le 0$, we have $J_1(x) \ge \frac{1}{2}||x||^2$ for all $x \in X$. This means that condition (1) in [14, Theorem] (there exists a positive constant γ such that $J_1(x) \ge \gamma||x||^2$ for all $x \in X$) is fair when the constant $\gamma = 1/2 > 0$.

For almost all $t \in [a,b]$, the function $\operatorname{sgn}(\cdot)$ has only upward jump discontinuity $x = 0$ (since $-1 = \operatorname{sgn}(0-) < \operatorname{sgn}(0+) = 1$), $\operatorname{sgn}(0) = 0$, and $|\operatorname{sgn}(x)| \le 1$ for all $x \in \mathbb{R}$, $1 \in L_q([a,b])$, $q > 1$. Therefore [14, condition (2) in Theorem] is satisfied.

As in [16], one can show that there exists an $x_0 \in X$ such that $J_2(x_0) > 0$. Thus [14, condition (3) in Theorem] holds.

In summary, all the conditions in [14, Theorem] for problem (3.1), (3.5) are fulfilled. This implies that there exists a $C_0 < 0$ such that for all $C < C_0$ problem (3.1), (3.5) has a nonzero semiregular solution $x_c \in X$ for which $J^c(x_c) = \inf_{y\in X} J^c(y) < 0$. The proof is complete. $\square$

4. Conclusion

We have considered equations (1.1), (2.1), and (3.1) as mathematical models of real physical processes. Consequently, this is quite justified to study not only transient but also transient-free processes described by these equations. We have investigated all possible combinations of the parameters A and C when $B = 0$ as well as combinations of the parameters B and C when $A = 0$. Despite the seeming simplicity of considered equations, the dynamics of the systems is quite difficult. Moreover, we have obtained a splitting of the parameter space on the domains of the qualitatively various dynamic behavior. The splitting of the phase plane on the phase trajectories is put in correspondence with each such domain, which allows to choose initial or boundary conditions for obtaining demanded dynamics of the processes. These results can be used for modelling and studying dynamics of the systems with discontinuous nonlinearities.

References

[1] A. Ambrosetti, M. Calahorrano, F. Dobarro; *Global branching for discontinuous problems*, Comment. Math. Univ. Carolinae, **31** (1990), no. 2, pp. 213–222.
[2] A. A. Andronov, A. A. Vitt, S. E. Khaikin; *Theory of oscillators*, Dover, New York, 1966.
[3] G. Bonanno, G. D'Agui, P. Winkert; *Sturm–Liouville equations involving discontinuous nonlinearities*, Minimax Theory Appl., **1** (2016), no. 1, pp. 125–143.
[4] A. F. Filippov; *Differential equations with discontinuous righthand sides*, Kluwer Academic Publishers, Dordrecht, 1988.
[5] A. Jacquemard, M. A. Teixeira; *Periodic solutions of a class of non-autonomous second order differential equations with discontinuous right-hand side*, Physica D: Nonlinear Phenomena, **241** (2012), no. 22, pp. 2003–2009.
[6] A. M. Kamachkin, D. K. Potapov, V. V. Yevstafyeva; *Solution to second-order differential equations with discontinuous right-hand side*, Electron. J. Differ. Equ., 2014, no. 221, pp. 1–6.
[7] A. M. Kamachkin, D. K. Potapov, V. V. Yevstafyeva; *Non-existence of periodic solutions to non-autonomous second-order differential equation with discontinuous nonlinearity*, Electron. J. Differ. Equ., 2016, no. 04, pp. 1–8.
[8] M. A. Krasnosel'skii, A. V. Pokrovskii; *Regular solutions of equations with discontinuous nonlinearities*, Soviet Math. Dokl., **17** (1976), no. 1, pp. 128–132.
[9] M. Kunze; *Non-smooth dynamical systems*, Lecture Notes in Mathematics, Vol. 1744, Springer, Berlin, 2000.
[10] I. L. Nyzhnyk, A. O. Krasneeva; *Periodic solutions of second-order differential equations with discontinuous nonlinearity*, J. Math. Sci., **191** (2013), no. 3, pp. 421–430.
[11] V. N. Pavlenko, D. K. Potapov; *The existence of semiregular solutions to elliptic spectral problems with discontinuous nonlinearities*, Sb. Math., **206** (2015), no. 9, pp. 1281–1298.
[12] N. A. Perestyuk, V. A. Plotnikov, A. M. Samoilenko, N. V. Skripnik; *Differential equations with impulse effects: multivalued right-hand sides with discontinuities*, Walter de Gruyter, Berlin, 2011.
[13] D. K. Potapov; *Bifurcation problems for equations of elliptic type with discontinuous nonlinearities*, Math. Notes, **90** (2011), no. 2, pp. 260–264.
[14] D. K. Potapov; *Sturm–Liouville's problem with discontinuous nonlinearity*, Differ. Equ., **50** (2014), no. 9, pp. 1272–1274.
[15] D. K. Potapov; *Existence of solutions, estimates for the differential operator, and a "separating" set in a boundary value problem for a second-order differential equation with a discontinuous nonlinearity*, Differ. Equ., **51** (2015), no. 7, pp. 967–972.
[16] D. K. Potapov; *Continuous approximations of Gol'dshtik's model*, Math. Notes, **87** (2010), no. 2, pp. 244–247.
[17] D. K. Potapov, V. V. Yevstafyeva; *Lavrent'ev problem for separated flows with an external perturbation*, Electron. J. Differ. Equ., 2013, no. 255, pp. 1–6.
[18] A. M. Samoilenko, I. L. Nizhnik; *Differential equations with bistable nonlinearity*, Ukr. Math. J., **67** (2015), no. 4, pp. 584–624.
[19] S. Tiwari; *N-Laplacian critical problem with discontinuous nonlinearities*, Adv. Nonlinear Anal., **4** (2015), no. 2, pp. 109–121.
[20] V. Rădulescu; *Finitely many solutions for a class of boundary value problems with superlinear convex nonlinearity*, Arch. Math., **84** (2005), no. 6, pp. 538–550.

Alexander M. Kamachkin
Saint Petersburg State University, 7/9, Universitetskaya nab., St. Petersburg, 199034, Russia
E-mail address: `a.kamachkin@spbu.ru`

Dmitriy K. Potapov
Saint Petersburg State University, 7/9, Universitetskaya nab., St. Petersburg, 199034, Russia
E-mail address: `d.potapov@spbu.ru`

Victoria V. Yevstafyeva
Saint Petersburg State University, 7/9, Universitetskaya nab., St. Petersburg, 199034, Russia
E-mail address: v.evstafieva@spbu.ru