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LIMIT CYCLES FOR $\mathbb{Z}_{2n}$-EQUIVARIANT SYSTEMS WITHOUT INFINITE EQUILIBRIA

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Abstract. We analyze the dynamics of a class of $\mathbb{Z}_{2n}$-equivariant differential equations of the form $\dot{z} = pz^{n-1}\bar{z}^{n-2} + sz^n\bar{z}^{n-1} - \bar{z}^{2n-1}$, where z is complex, the time t is real, while p and s are complex parameters. This study is the generalisation to $\mathbb{Z}_{2n}$ of previous works with $\mathbb{Z}_4$ and $\mathbb{Z}_6$ symmetry. We reduce the problem of finding limit cycles to an Abel equation, and provide criteria for proving in some cases uniqueness and hyperbolicity of the limit cycle that surrounds either 1, $2n+1$ or $4n+1$ equilibria, the origin being always one of these points.

1. Introduction and statement of main results

Hilbert XVI$^{\text{th}}$ problem was the motivation for a large number of articles over the last century, and remains one of the open questions in mathematics. The study of this problem in the context of equivariant dynamical systems is a new branch of analysis, based on the development of equivariant bifurcation theory, by Golubitsky, Stewart and Schaeffer, [9, 10]. Many other authors, for example Chow and Wang [6], have considered this theory when studying the limit cycles and related phenomena in systems with symmetry.

In this paper we analyze the $\mathbb{Z}_{2n}$-equivariant system

$$\dot{z} = \frac{dz}{dt} = \mathbf{p}z^{n-1}\bar{z}^{n-2} + \mathbf{s}z^n\bar{z}^{n-1} - \bar{z}^{2n-1} = f(z), \tag{1.1}$$

for $n > 3$, where $\mathbf{p} = p_1 + ip_2$, $\mathbf{s} = s_1 + is_2$, $p_1, p_2, s_1, s_2 \in \mathbb{R}$, $t \in \mathbb{R}$.

The general form of the $\mathbb{Z}_q$-equivariant equation is

$$\dot{z} = zA(|z|^2) + B\bar{z}^{q-1} + O(|z|^{q+1}),$$

where A is a polynomial on the variable $|z|^2$ whose degree is the integer part of $(q-1)/2$. This class of equations is studied, for instance in the books [4, 6], when the resonances are strong, *i.e.* $q < 4$ or weak $q > 4$. A partial treatment of the special case $q = 4$ is given, for instance, in the article [13], and in the book [6] that is concerned with normal forms and bifurcations in general. A more complete treatment of the case $q = 4$ appears in the article [1], while the case $q = 6$ appears in [2]. All mentioned articles claim the fact that, since the equivariant term $\bar{z}^{q-1}$ is not dominant with respect to the function on $\bar{z}^2$, they are easier to study than

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other cases. While this argument works for obtaining the bifurcation diagram near the origin, it is no longer helpful for a global analysis or if the analysis is focused on the study of limit cycles. The aim of the present work is to study the global phase portrait of (1.1) on the Poincaré compactifiction of the plane; we devote especial interest to analysing the existence, location and uniqueness of limit cycles surrounding 1, $2n+1$ or $4n+1$ equilibria. Our method uses some of the techniques developed in [1] and includes transforming (1.1) into a scalar Abel equation followed by its analysis.

The main results of this article are Theorems 1.1 and 1.2 below. Consider the quadratic form

$$\mathcal{Q}(p_1,p_2) = p_1^2 + p_2^2 - (p_1 s_2 - p_2 s_1)^2 = (1-s_2^2)p_1^2 + (1-s_1^2)p_2^2 + 2s_1 s_2 p_1 p_2. \quad (1.2)$$

Theorem 1.1. *For $|s_2| > 1$ and for any $s_1 \neq 0$, $\mathbf{p} \neq 0$, if $p_2 s_2 \geq 0$ the only equilibrium of (1.1) is the origin. If $p_2 s_2 < 0$ then the number of equilibria of (1.1) is determined by the quadratic form $\mathcal{Q}(p_1,p_2)$ defined in (1.2) and is:*

(1) *exactly one equilibrium (the origin) if $\mathcal{Q}(p_1,p_2) < 0$;*
(2) *exactly $2n+1$ equilibria (the origin and one saddle-node per region $(k-1)\pi/n \leq \theta < k\pi/n$, $k \in \mathbb{Z}$) if $\mathcal{Q}(p_1,p_2) = 0$;*
(3) *exactly $4n+1$ equilibria (the origin and two equilibria in each region $(k-1)\pi/n \leq \theta < k\pi/n$, $k \in \mathbb{Z}$) if $\mathcal{Q}(p_1,p_2) > 0$.*

Theorem 1.2. *For $|s_2| > 1$ and for any $s_1 \neq 0$, and $\mathbf{p} \neq 0$, consider the conditions:*

(i) $\mathcal{Q}(p_1,p_2) \leq 0$,
(ii) $\mathcal{Q}(2p_1,p_2) \leq 0$.

Then:

(a) *If either condition (i) or (ii) holds, then equation (1.1) has at most one limit cycle surrounding the origin, and when the limit cycle exists it is hyperbolic.*
(b) *There are parameter values where $\mathcal{Q}(p_1,p_2) < 0$ for which there is a stable limit cycle surrounding the origin.*
(c) *There are parameter values where $\mathcal{Q}(p_1,p_2) = 0$ for which there is a limit cycle surrounding the $2n+1$ equilibria given by Theorem 1.1.*
(d) *There are parameter values where $\mathcal{Q}(2p_1,p_2) \leq 0$ for which there is a limit cycle surrounding either the $2n+1$ or the $4n+1$ equilibria given by Theorem 1.1.*

This article is organised as follows. After some preliminary results in Section 2, the number of equilibria is treated in Section 3, as well as the proof of the Theorem 1.1. The Abel equation is obtained in Section 4 and the proof of the Theorem 1.2 is completed in Section 5.

2. Preliminary results

Let Γ be a closed subgroup of $\mathbf{O}(2)$. A system of differential equations $dx/dt = f(x)$ in the plane is said to have symmetry Γ (or to be Γ-equivariant) if $f(\gamma x) = \gamma f(x)$, $\forall \gamma \in \Gamma$. Here we are concerned with $\Gamma = \mathbb{Z}_{2n}$, acting on $\mathbb{C} \sim \mathbb{R}^2$ by multiplication by $\gamma_k = \exp(k\pi i/n)$, $k = 0,1,\dots,2n-1$. For equation (1.1) we obtain the following result.

Proposition 2.1. *Equation (1.1) is $\mathbb{Z}_{2n}$-equivariant.*

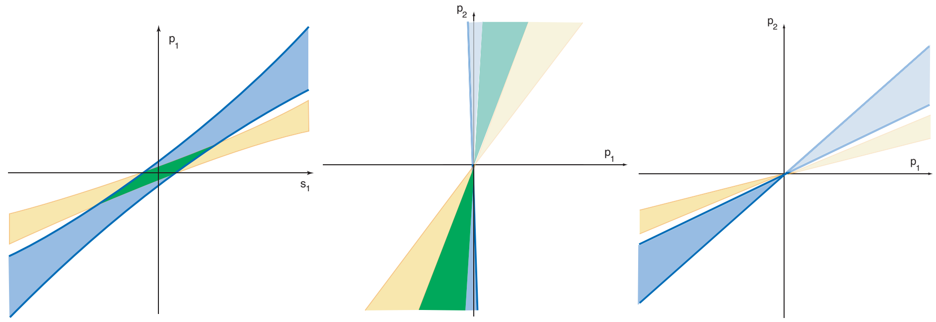

FIGURE 1. The regions defined in Theorems 1.1 and 1.2: blue when $\mathcal{Q}(p1, p_2) \geq 0$, yellow when $\mathcal{Q}(2p1, p_2) \geq 0$, green in the intersection of the two regions. Top: diagram on the (p_1, s_1)-plane, with $p_2 = 1$ and $s_2 = 4$. Bottom: diagrams on the (p_1, p_2)-plane, with $s_1 = 1/2$, $s_2 = 4$ on the left, and with $s_1 = 6$, $s_2 = 4$ on the right. There are $4n + 1$ equilibria on the interior of the blue and green regions when $p_2 s_2 < 0$ (darker colours).

Proof. The monomials in $z, \bar{z}$ that appear in the expression of f are $\bar{z}^{2n-1}$ and $z^{l+1}\bar{z}^l$. The first of these is γ_k-equivariant, while monomials of the form $z^{l+1}\bar{z}^l$ are $\mathbb{Z}_{2n}$-equivariant for all n. □

The next step is to identify the parameter values for which (1.1) is Hamiltonian.

Proposition 2.2. *Equation* (1.1) *is Hamiltonian if and only if* $p_1 = 0 = s_1$.

Proof. The equation $\dot{z} = F(z, \bar{z})$ is Hamiltonian when $\frac{\partial F}{\partial z} + \frac{\partial \bar{F}}{\partial \bar{z}} = 0$. For equation (1.1) we have

$$\frac{\partial F}{\partial z} = (n-1)(p_1 + ip_2)z^{n-2}\bar{z}^{n-2} + n(s_1 + is_2)z^{n-1}\bar{z}^{n-1}$$

$$\frac{\partial \bar{F}}{\partial \bar{z}} = (n-1)(p_1 - ip_2)z^{n-2}\bar{z}^{n-2} + n(s_1 - is_2)z^{n-1}\bar{z}^{n-1}$$

and consequently it is Hamiltonian precisely when $p_1 = s_1 = 0$. □

The expression of equation (1.1) in polar coordinates will be useful. Writing

$$z = \sqrt{r}(\cos(\theta) + i\sin(\theta))$$

and rescaling time as $\frac{dt}{ds} = r^{n-2}$, we obtain

$$\begin{aligned} \dot{r} &= 2r\left(p_1 + rs_1 - r\cos(2n\theta)\right) \\ \dot{\theta} &= p_2 + rs_2 + r\sin(2n\theta). \end{aligned} \tag{2.1}$$

The symmetry means that for most of the time we only need to study the dynamics of (2.1) in the fundamental domain for the $\mathbb{Z}_{2n}$-action, an angular sector of π/n. It will often be convenient to look instead at the behaviour of a rescaled angular variable $\phi = n\theta$ in intervals of length π, where the equation (2.1) takes the form

$$\begin{aligned} \dot{r} &= 2r\left(p_1 + rs_1 - r\cos(2\phi)\right) \\ n\dot{\phi} &= p_2 + rs_2 + r\sin(2\phi). \end{aligned} \tag{2.2}$$

One possible argument for existence of a limit cycle is to show that, in the Poincaré compactification, there are no critical points at infinity and that infinity and the origin have the same stability. The next result is a starting point for this analysis.

Lemma 2.3. *In the Poincaré compactification, equation* (1.1) *satisfies:*

1. *there are no equilibria at infinity if and only if* $|s_2| > 1$*;*
2. *when* $|s_2| > 1$*, infinity is an attractor when* $s_1 s_2 > 0$ *and a repeller when* $s_1 s_2 < 0$*.*

Proof. The proof is similar to that of Lemma 2.2 in [1]. Using the change of variable $R = 1/r$ in (2.1) and reparametrising time by $\frac{dt}{ds} = R$, we obtain

$$R' = \frac{dR}{ds} = -2R\left(s_1 - \cos(2n\theta)\right) - 2p_1 R^2$$
$$\theta' = \frac{d\theta}{ds} = s_2 + \sin(2n\theta) + p_2 R.$$

The invariant set $\{R = 0\}$ corresponds to infinity in (2.1). Hence, there are no equilibria at infinity if and only if $|s_2| > 1$. The stability of infinity in this case (see [11]) is given by the sign of

$$\int_0^{2\pi} \frac{-2(s_1 - \cos(2n\theta))}{s_2 + \sin(2n\theta)} d\theta = \frac{-\mathrm{sgn}(s_2) 4\pi s_1}{\sqrt{s_2^2 - 1}},$$

and the result follows. Note that since we are assuming $|s_2| > 1$, the integral above is always well defined. ☐

3. Analysis of equilibria

In this section we describe the number of equilibria of (1.1). We start with the origin, that is an equilibrium for all values of the parameters. First we show that there is no trajectory of the differential equations that approaches the origin with a definite limit direction: the origin is *monodromic*

Lemma 3.1. *If* $p_2 \neq 0$ *then the origin is a monodromic equilibrium of* (1.1)*. It is unstable if* $p_1 > 0$*, asymptotically stable if* $p_1 < 0$*. If* $p_1 = 0$ *it is unstable if* $s_1 > 1$*, asymptotically stable if* $s_1 < -1$*.*

Note that if $p_1 = s_1 = 0$, equation (1.1) is Hamiltonian. In this case the origin is a centre. In Section 4 below we obtain better estimates for the case $p_1 = 0$.

Proof. To show that the origin is monodromic we compute the arriving directions of the flow to the origin, see [3, Chapter IX] for details. We look for solutions that arrive at the origin tangent to a direction α that are zeros of $r^2\dot{\alpha} = R(x, y) = yP(x, y) + xQ(x, y)$. If the term of lowest degree in the polynomial $R(x, y) = -yP(x, y) + xQ(x, y)$ has no real roots then the origin is monodromic.

The term of lowest degree in (1.1) is $\mathbf{p}z^{n-1}\bar{z}^{n-2} = P(x, y) + iQ(x, y)$, with real part

$$P(x, y) = -p_2 y(x^2 + y^2)^{n-2} + p_1 x(x^2 + y^2)^{n-2} \tag{3.1}$$

and imaginary part

$$Q(x, y) = p_2 x(x^2 + y^2)^{n-2} + p_1 y(x^2 + y^2)^{n-2}. \tag{3.2}$$

Then we have that $R(x,y) = -yP(x,y) + xQ(x,y)$ is given by

$$R(x,y) = p_2(x^2+y^2)(x^2+y^2)^{n-2} = p_2(x^2+y^2)^{n-1} \tag{3.3}$$

which has no nontrivial real roots if $p_2 \neq 0$, so the origin is monodromic.

From the expression for $\dot{r}$ in (2.1) it follows that if $p_1 > 0$ then for r close to 0, we have $\dot{r} > 0$, hence the origin is unstable. Similarly, if $p_1 < 0$ then $\dot{r} < 0$ for r close to 0, and the origin is asymptotically stable. When $p_1 = 0$ the expression for $\dot{r}$ is $\dot{r} = 2r^2(s_1 - \cos(2n\theta))$, the origin is unstable if $s_1 > 1$, stable if $s_1 < -1$. $\square$

We now look for conditions under which (1.1) has nontrivial equilibria. We use equation (2.2) with the variable $\phi = n\theta$ to obtain simpler expressions, and analyse two open sets that cover the fundamental domain $0 \leq \phi < \pi$.

Lemma 3.2. *If $|s_2| > 1$ and $p_1 \neq 0$ then equilibria of* (2.2) *with $r > 0$ exist if and only if $\Delta = p_1^2 + p_2^2 - (p_1 s_2 - p_2 s_1)^2 \geq 0$.*

If $T_+ = p_2 - p_1 s_2 + p_2 s_1 \neq 0$ then equilibria of (2.2) *with $-\pi/2 < \phi < \pi/2$ satisfy:*

$$r_\pm = \frac{-p_2}{s_2 + \sin(2\phi_\pm)}, \quad \tan(\phi_\pm) = \frac{p_1 \pm \sqrt{\Delta}}{p_2 - p_1 s_2 + p_2 s_1}. \tag{3.4}$$

For equilibria with $0 < \phi < \pi$ and $r \neq 0$ the restriction is $T_- = p_2 + p_1 s_2 - p_2 s_1 \neq 0$ and they satisfy

$$r_\pm = \frac{-p_2}{s_2 + \sin(2\phi_\pm)}, \quad \cot(\phi_\pm) = \frac{p_1 \pm \sqrt{\Delta}}{-(p_2 + p_1 s_2 - p_2 s_1)}. \tag{3.5}$$

There is only one equilibrium of (2.2) *with $r \neq 0$ and $-\pi/2 < \phi < \pi/2$ when $T_+ = 0$, and it satisfies $\tan(\phi/2) = T_-/2p_1$. Similarly, for $T_- = 0$, there is only one nontrivial equilibrium with $0 < \phi < \pi$, with $\cot(\phi/2) = T_+/2p_1$.*

Proof. Let $-\pi/2 < \phi < \pi/2$. The equilibria of (2.2), are the solutions of

$$\begin{gathered} 0 = 2rp_1 + 2r^2(s_1 - \cos(2\phi)) \\ 0 = p_2 + r(s_2 + \sin(2\phi)). \end{gathered} \tag{3.6}$$

For $t = \tan(\phi)$, we have

$$\sin(2\phi) = \frac{2t}{1+t^2} \quad \cos(2\phi) = \frac{1-t^2}{1+t^2}. \tag{3.7}$$

Since $s_2 + \sin\phi \neq 0$, we may eliminate $r = -p_2/(s_2 + \sin(2\phi))$ from equations (3.6) to get

$$(-p_2 + p_1 s_2 - p_2 s_1)t^2 + 2p_1 t + p_2 + p_1 s_2 - p_2 s_1 = 0, \tag{3.8}$$

or, equivalently, $T_+ t^2 - 2p_1 t + T_- = 0$. If the coefficient T_+ of t^2 is zero, then equation (3.8) is linear in t and hence has only one solution, $t = T_-/2p_1$. When the coefficient of t^2 is not zero, solving equation (3.8) for t yields the result.

Finally, consider the interval $0 < \phi < \pi$ and let $\tau = \cot\phi$. The expression (3.7) for $\sin 2\phi$ is the same, and that of $\cos 2\phi$ is multiplied by -1. Instead of (3.8) we obtain $T_- t^2 - 2p_1 t + T_+ = 0$ and the result follows by the same arguments. $\square$

Note that when both T_+ and T_- are not zero, the expressions for $\tan(\phi/2)$ and $\cot(\phi/2)$ above define the same angles, since

$$(p_1 \pm \sqrt{\Delta})/T_+ = -T_-/(p_1 \mp \sqrt{\Delta}).$$

In Lemma 3.2 we found the number of equilibria of equation (2.1) with $r \neq 0$ in the regions $-\pi/2 < \phi < \pi/2$ and $0 < \phi < \pi$. To complete the information it remains to deal with the case when, for the same value of the parameters, two equilibria may occur, each in one of these intervals but not in the other.

Lemma 3.3. *If $|s_2| > 1$ and $\mathbf{p} \neq 0$, there are no parameter values for which (2.2) has equilibria with $r > 0$ simultaneously for $\phi = 0$ and $\phi = \pi/2$.*

Proof. If we solve

$$
\begin{aligned}
0 &= p_1 + r(s_1 - \cos(2\phi)) \\
0 &= p_2 + r(s_2 + \sin(2\phi))
\end{aligned}
$$

for $\phi = 0$, we obtain a solution at $r = -p_2/s_2$ subject to the condition $T_- = p_2 + p_1 s_2 - p_2 s_1 = 0$. Solving the same system for $\phi = \pi/2$ yields an equilibrium at $r = -p_2/s_2$ under the restriction $T_+ = p_2 - p_1 s_2 + p_2 s_1 = 0$. The parameter restrictions for $\phi = 0$ and $\phi = \pi/2$ are equivalent to $p_1 s_2 = p_2(s_1 - 1)$ and $p_1 s_2 = p_2(s_1 + 1)$, respectively. Hence, in order to have equilibria at $\phi = 0$ and $\phi = \pi/2$ for the same parameters, it is necessary to have $p_1 = p_2 = 0$. □

In the following we summarize the conditions that the parameters have to fulfill in order that (2.1) has exactly one, $2n+1$ or $4n+1$ equilibria (see Figure 1).

Proposition 3.4. *For $|s_2| > 1$ and $\mathbf{p} \neq 0$, if $p_2 s_2 \geq 0$, then the only equilibrium of (1.1) is the origin. If $p_2 s_2 < 0$ then the number of equilibria of (1.1) is determined by the quadratic form $\mathcal{Q}(p_1, p_2)$ defined in (1.2) and is:*

1. *exactly one equilibrium (the origin) if $\mathcal{Q}(p_1, p_2) < 0$;*
2. *exactly $2n+1$ equilibria if $\mathcal{Q}(p_1, p_2) = 0$;*
3. *exactly $4n+1$ equilibria if $\mathcal{Q}(p_1, p_2) > 0$;*

Since $\mathcal{Q}$ is a quadratic form on p_1, p_2, and since its determinant $1 - s_1^2 - s_2^2$, is negative when $|s_2| > 1$, then for each choice of s_1, s_2 with $s_2 > 1$, the points where $\mathcal{Q}(p_1, p_2)$ is positive lie on two sectors, delimited by the two lines where $\mathcal{Q}(p_1, p_2) = 0$. Also $\mathcal{Q}(p_1, 0) = (1 - s_2^2)p_1^2 < 0$ for $|s_2| > 1$, and thus the sectors where there are two equilibria in each $\theta = \pi/n$ do not include the p_1 axis, as in Figure 1.

Proof. By the $\mathbb{Z}_{2n}$-symmetry, the number of nontrivial equilibria of (1.1) will be $2n$ times the number of equilibria of (2.1) with $r > 0$ and $\theta \in [0, \pi/n)$, or equivalently, $2n$ times the number of equilibria of (2.2) with $r > 0$ and $\phi \in [0, \pi)$.

If p_2 and s_2 have the same sign, then the expression for r in (3.5) is negative, and there are no solutions with $0 < \phi < \pi$. For $\phi = 0$, Lemma 3.3 gives the value $-p_2/s_2$ for r, that would also be negative, so there are no nontrivial equilibria if $p_2 s_2 > 0$.

Suppose now $p_2 s_2 < 0$. By Lemma 3.2, there are no solutions ϕ when the discriminant Δ is negative, corresponding to $\mathcal{Q}(p_1, p_2) < 0$ as in assertion (1). Ignoring for the moment the restriction $R_- \neq 0$, there are exactly 2 solutions $\phi \in (0, \pi)$ if $\Delta > 0$, that corresponds to $\mathcal{Q}(p_1, p_2) < 0$, and this gives us assertion (3).

In order to have exactly $2n$ nontrivial equilibria, two conditions have to be satisfied: $p_2 s_2 < 0$ to ensure positive values of r, and the quantities $\cot(\phi_\pm)$ have to coincide, *i.e.* the discriminant Δ in Lemma 3.2 has to be zero, hence, $r_+ = r_-$ and $\phi_+ = \phi_-$, assertion (2) in the statement.

Finally, if $R_- = 0$, Lemma 3.2 provides only one solution $\phi \in [0, \pi)$, but in this case $\phi = 0$ is also a solution, by Lemma 3.3. When $R_- = 0$ we have $\mathcal{Q}(p_1, p_2) = p_1^2 > 0$, so we are in the situation of assertion (1) if $p_2 s_2 < 0$. $\square$

Lemma 3.5. *For* $|s_2| > 1$, *and* $p_2 s_2 < 0$, *if* $\mathcal{Q}(p_1, p_2) = 0$ *all the nontrivial equilibria of* (1.1) *are saddle-nodes.*

Proof. The Jacobian matrix of (2.1) is

$$J_{(r,\theta)} = \begin{pmatrix} 2p_1 + 4r(s_1 - \cos(2n\theta)) & 4nr^2 \sin(2n\theta) \\ s_2 + \sin(2n\theta) & 2nr\cos(2n\theta) \end{pmatrix}. \tag{3.9}$$

If $\mathcal{Q}(p_1, p_2) = 0$ there is only one nontrivial equilibrium with $-\pi/n < \theta \leq \pi/n$, that we denote by (r_+, θ_+). Substituting the expression (3.4) into the Jacobian matrix (3.9) and taking into account that $\Delta = \mathcal{Q}(p_1, p_2) = 0$, the eigenvalues of the matrix are

$$\lambda_1 = 0 \quad \lambda_2 = 2p_1 - 2p_2 \frac{(2s_1 - n + 2)T_+^2 + p_1^2(2s_1 + n - 2)}{s_2 T_+^2 + 2p_1 T_+ + p_1 s_2}, \tag{3.10}$$

where T_+ was defined in Lemma 3.2.

Therefore (r_+, θ_+) has a zero eigenvalue, and the same holds for its $2n$ copies by the symmetry. To show that these equilibria are saddle-nodes we use the well-known fact that the sum of the indices of all equilibria contained in the interior of a limit cycle of a planar system is $+1$ – see for instance [3]. Since we are assuming $|s_2| > 1$, by Lemma 2.3, there are no equilibria at infinity. Hence, infinity is a limit cycle of the system and it has $2n + 1$ equilibria in its interior: the origin, that is a focus and hence has index $+1$, and $2n$ other equilibria, all of the same type because of the symmetry. Consequently, the index of these equilibria must be 0. As we have proved that they are semi-hyperbolic equilibria then they must be saddle-nodes. $\square$

The proof of Theorem 1.1 is now complete.

4. Reduction to the Abel equation

In this section we address the existence of limit cycles for (1.1).

Lemma 4.1. *The periodic orbits of* (1.1) *that surround the origin are in one-to-one correspondence with the non contractible solutions that satisfy* $x(0) = x(2\pi)$ *of the Abel equation*

$$\frac{dx}{d\theta} = A(\theta)x^3 + B(\theta)x^2 + C(\theta)x \tag{4.1}$$

where

$$\begin{aligned} A(\theta) &= \frac{2}{p_2}\left(p_1 + p_1 s_2^2 - p_2 s_1 s_2 + (-p_2 s_1 + 2p_1 s_2)\sin(2n\theta)\right) \\ &\quad + \frac{2}{p_2}\left(-p_1\cos(2n\theta) + p_2 s_2 + p_2 \sin(2n\theta)\right)\cos(2n\theta), \\ B(\theta) &= \frac{2}{p_2}\left(-2p_1 s_2 + p_2 s_1 - 2p_1 \sin(2n\theta) - p_2\cos(2n\theta)\right), \\ C(\theta) &= \frac{2p_1}{p_2}. \end{aligned} \tag{4.2}$$

Proof. From (2.1) we obtain

$$\frac{dr}{d\theta} = \frac{2r(p_1 + r\,(s_1 + \cos(2n\theta)))}{p_2 + r\,(s_2 + \sin(2n\theta))}.$$

Applying the Cherkas transformation $x = \frac{r}{p_2 + r(s_2+\sin(2n\theta))}$, see [5], we obtain the scalar equation (4.1). The limit cycles that surround the origin of equation (1.1) are transformed into non contractible periodic orbits of equation (4.1), as they cannot intersect the set $\{\dot\theta = 0\}$, where the denominators of $dr/d\theta$ and of the Cherkas transformation vanish. For more details see [7]. $\square$

Corollary 4.2. *If $p_2 \neq 0$ and $p_1 = 0$ then the origin is an asymptotically stable equilibrium of* (1.1) *if $s_1 < 0$, unstable if $s_1 > 0$, .*

Proof. The stability of the origin can be determined from the two first Lyapunov constants. For an Abel equation they are given by

$$V_1 = \exp\Big(\int_0^{2\pi} C(\theta)d\theta\Big) - 1, \quad V_2 = \int_0^{2\pi} B(\theta)d\theta.$$

Using this, we obtain from the expressions given in (4.2) that if $p_1 = 0$ then $C(\theta) = 0$ implying $V_1 = 0$. On the other hand $V_2 = 4\pi s_1$, and we obtain the result. $\square$

Lemma 4.3. *For $|s_2| > 1$ the function $A(\theta)$ of Lemma 4.1 changes sign if and only if $\mathcal{Q}(p_1, p_2) > 0$, where $\mathcal{Q}$ is the quadratic form defined in* (1.2).

Proof. Writing $x = \sin(2n\theta),\ y = \cos(2n\theta)$, the function $A(\theta)$ in (4.2) becomes

$$A(x, y) = \frac{2}{p_2}\big(p_1 - p_2 s_1 s_2 + p_1 s_2^2 + (2p_1 s_2 - p_2 s_1)x + (p_2 x - p_1 y + p_2 s_2)y\big),$$

and we solve the set of equations

$$\begin{gathered} A(x, y) = 0, \\ x^2 + y^2 = 1. \end{gathered}$$

to obtain the solutions

$$\begin{gathered} x_1 = -s_2, \quad y_1 = \sqrt{1 - s_2^2} \\ x_2 = -s_2, \quad y_2 = -\sqrt{1 - s_2^2} \\ x_\pm = \frac{p_1 p_2 s_1 - p_1^2 s_2 \pm p_2\sqrt{\mathcal{Q}(p_1, p_2)}}{p_1^2 + p_2^2} \\ y_\pm = \frac{p_2^2 s_1 - p_1 p_2 s_2 \mp p_1\sqrt{\mathcal{Q}(p_1, p_2)}}{p_1^2 + p_2^2} \end{gathered}$$

The first two pairs of solutions $(x_1, y_1), (x_2, y_2)$ cannot be solutions of $A(\theta) = 0$ since $x = \sin(2n\theta) = -s_2$ and we are assuming $|s_2| > 1$.

If we look for the intervals where the expression $A(x, y)$ does not change sign we have two possibilities: either $|x_\pm| > 1$ (again not compatible with $x = \sin(2n\theta)$ nor with $x^2 + y^2 = 1$) or the discriminant $\mathcal{Q}(p_1, p_2)$ is negative or zero. In the case $\mathcal{Q}(p_1, p_2) < 0$, there will be no real solutions x, y. If $\mathcal{Q}(p_1, p_2) = 0$, the function $A(\theta)$ will have a double zero and will not change sign. So the only possibility is to have $\mathcal{Q}(p_1, p_2) > 0$. $\square$

Lemma 4.4. *For $|s_2| > 1$ the function $B(\theta)$ of Lemma 4.1 changes sign if and only if $\mathcal{Q}(2p_1, p_2) > 0$, where $\mathcal{Q}$ is the quadratic form defined in* (1.2).

Proof. Using the substitution of the proof of the previous lemma, $x = \sin(2n\theta)$, $y = \cos(2n\theta)$, we obtain that the solutions of the system

$$\begin{gathered} B(x,y) = 0, \\ x^2 + y^2 = 1, \end{gathered}$$

are

$$\begin{aligned} x_\pm &= \frac{2p_1p_2s_1 - 4p_1^2s_2 \pm p_2\sqrt{\mathcal{Q}(2p_1,p_2)}}{4p_1^2 + p_2^2}, \\ y_\pm &= \frac{p_2^2s_1 - 2p_1p_2s_2 \mp 2p_1\sqrt{\mathcal{Q}(2p_1,p_2)}}{4p_1^2 + p_2^2}. \end{aligned}$$

By the same arguments of the previous proof, we obtain that the function $B(\theta)$ will not change sign if and only if $\mathcal{Q}(2p_1, p_2) \leq 0$ $\square$

To complete the proof of Theorem 1.2 in this section, we need some results on Abel equations proved in [12] and [8], that we summarise in a theorem.

Theorem 4.5 (Pliss 1966, Gasull & Llibre 1990)**.** *Consider the Abel equation* (4.1) *and assume that either $A(\theta) \not\equiv 0$ or $B(\theta) \not\equiv 0$ does not change sign. Then it has at most three solutions satisfying $x(0) = x(2\pi)$, taking into account their multiplicities.*

5. Analysis of limit cycles

Proof of Theorem 1.2. For assertion (a), define the function $c(\theta)$ by $c(\theta) = s_2 + \sin(2n\theta)$. Since $|s_2| > 1$, we have $c(\theta) \neq 0$ for all $\theta \in [0, 2\pi]$, and a simple calculation shows that the curve $x = 1/c(\theta)$ is a solution of (4.1) satisfying $x(0) = x(2\pi)$. As shown in [1], doing the Cherkas transformation backwards we obtain that $x = 1/c(\theta)$ is mapped into infinity of the original differential equation.

Assume that one of conditions (i) or (ii) is satisfied. By Lemma 4.1, we reduce the study of the periodic orbits of equation (1.1) to the analysis of the non contractible periodic orbits of the Abel equation (4.1). If $\mathcal{Q}(p_1, p_2) \leq 0$, by Lemma 4.3, the function $A(\theta)$ in the Abel equation does not change sign. If $\mathcal{Q}(2p_1, p_2) \leq 0$ then $B(\theta)$ does not change sign, by Lemma 4.4. In both cases, Theorem 4.5 ensures that there are at most three solutions, counted with multiplicities, of (4.1) satisfying $x(0) = x(2\pi)$. One of them is trivially $x = 0$. A second one is $x = 1/c(\theta)$. Hence, there is at most one more contractible solution of (4.1), and by Theorem 4.5, the maximum number of limit cycles of equation (1.1) is one. Moreover, from the same theorem it follows that when the limit cycle exists it has multiplicity one and hence it is hyperbolic. This completes the proof of assertion (a) in Theorem 1.2.

For assertion (b), let $s_2 > 1$, $s_1 < 0$ and choose $p_1 > 0$ and $p_2 \neq 0$ in the region $\mathcal{Q}(p_1, p_2) < 0$ (for instance, $s_1 = -1/2$, $s_2 = 2$, $p_1 = p_2 = 1$, $\mathcal{Q}(p_1, p_2) = -17/4 < 0$). By Theorem 1.1 the only equilibrium is the origin, and by Lemma 3.1 it is a repeller since $p_2 \neq 0$ and $p_1 > 0$. Infinity is also a repeller by Lemma 2.3, because $s_1s_2 < 0$. By the Poincaré-Bendixson Theorem and by the first part of the proof of this theorem, there is exactly one hyperbolic limit cycle surrounding the origin. Moreover, this limit cycle is stable. An unstable limit cycle may be obtained changing the signs of p_1, s_1 and s_2.

For assertion (c), we take $s_2 > 1$, $s_1 < 0$ and choose $p_1 > 0$ and $p_2 < 0$ in one of the lines $\mathcal{Q}(p_1, p_2) = 0$ (for instance, $s_1 = -1/2$, $s_2 = 2$, $p_1 = (2 + \sqrt{13})/6$, $p_2 = -1$, $\mathcal{Q}(p_1, p_2) = 0$). By the same arguments above, both the origin and infinity are repellers, since $s_1 s_2 < 0$, $p_2 \neq 0$ and $p_1 > 0$. Also, since $p_2 s_2 < 0$, by Theorem 1.1 there is exactly one equilibrium, a saddle-node, in each region $(k-1)\pi/n \leq \theta < k\pi/n$, $k \in \mathbb{Z}$. Again by (a) there is at most one limit cycle. In order to show that this cycle exists and encircles the saddle-nodes, we will construct a polygonal line from the origin to the saddle-node (r_*, θ_*), $-\pi/n < \theta_* < 0$ where the vector field points outwards, away from the saddle-node, see Figure 2. Copies of the poligonal by the symmetries will join the origin to the other saddle-nodes and the union of all these will form a polygon where the vector field points outwards, away from the saddle-nodes. Since infinity is a repeller and there are no equilibria outside the polygon, by the Poincaré-Bendixson Theorem there will be a limit cycle encircling the saddle-nodes.

For the construction of the polygon we need some information on the location of the saddle-node $z_* = (r_*, \theta_*)$. Solving $\mathcal{Q}(p_1, p_2) = 0$ for p_1 yields

$$p_1 = -p_2 \frac{s_1 s_2 \pm \sqrt{s_1^2 + s_2^2 - 1}}{1 - s2^2}.$$

Choosing the solution with the minus sign and substituting into (3.4), we obtain

$$\frac{1}{\tan(n\theta_*)} = \frac{p_2}{p_1}(1 + s_1) - s_2 = \frac{(s_2^2 - 1)(1 + s_1)}{s_1 s_2 - \sqrt{s_1^2 + s_2^2 - 1}} - s_2 < -1.$$

Therefore $-1 < \tan(n\theta_*) < 0$ and hence $-\pi/4n < \theta_* < 0$.

For the first piece of the polygonal we look at the ray $\theta = -\pi/4n$, where $\dot{\theta} < 0$ if $0 < r < r_0 = -p_2/(s_2 + \sqrt{2}/2)$. Therefore on the segment $0 < r < r_0$ the vector field points away from the saddle-node z_*.

Another piece of the polygonal will be contained in the line $z_* + xv$ where $x \in \mathbb{R}$ and v is an eigenvector corresponding to the non-zero eigenvalue of z_*. This line is tangent to the separatrix of the saddle region of the saddle-node. Let x_0 be the smallest positive value of x for which the vector field is not transverse to this line.

If the ray intersects the tangent to the separatrix at a point with $0 < r < r_0$ and with $0 < x < x_0$, then the polygonal is the union of the two segments, from the origin to the intersection and from there to z_*. Otherwise, the segment joining the point $z_1 = z_* + x_0 v$ to the point z_2 with $r = r_0$, $\theta = -\pi/4n$ will also be transverse to the vector field, and the polygonal will consist of the three segments from the origin to z_2, from there to z_1, and whence to z_*. This completes the construction of the polygonal, and hence, the proof of assertion (c).

Finally, for assertion (d) we start with parameters for which (c) holds with $\mathcal{Q}(2p_1, p_2) < 0$. The example given above, $s_1 = -1/2$, $s_2 = 2$, $p_1 = (2 + \sqrt{13})/6$, $p_2 = -1$, $\mathcal{Q}(p_1, p_2) = 0$ satisfies $\mathcal{Q}(2p_1, p2) = -\left(13 + 8\sqrt{13}\right)/12 < 0$. By Lemma 4.4, the function $B(\theta)$ does not change sign. The hyperbolic limit cycle persists under small changes of parameters, and $\mathcal{Q}(2p_1, p_2)$ is still negative, while moving the parameters away from the line $\mathcal{Q}(p_1, p_2) = 0$. When the parameters move into the region where $\mathcal{Q}(p_1, p_2) > 0$, each saddle-node splits into two equilibria that are still encircled by the limit cycle. Moving in the opposite direction, int $\mathcal{Q}(p_1, p_2) < 0$ destroys all the non-trivial equilibria, and only the origin remains inside the limit cycle. Thus, all situations of assertion (d) occur. $\square$

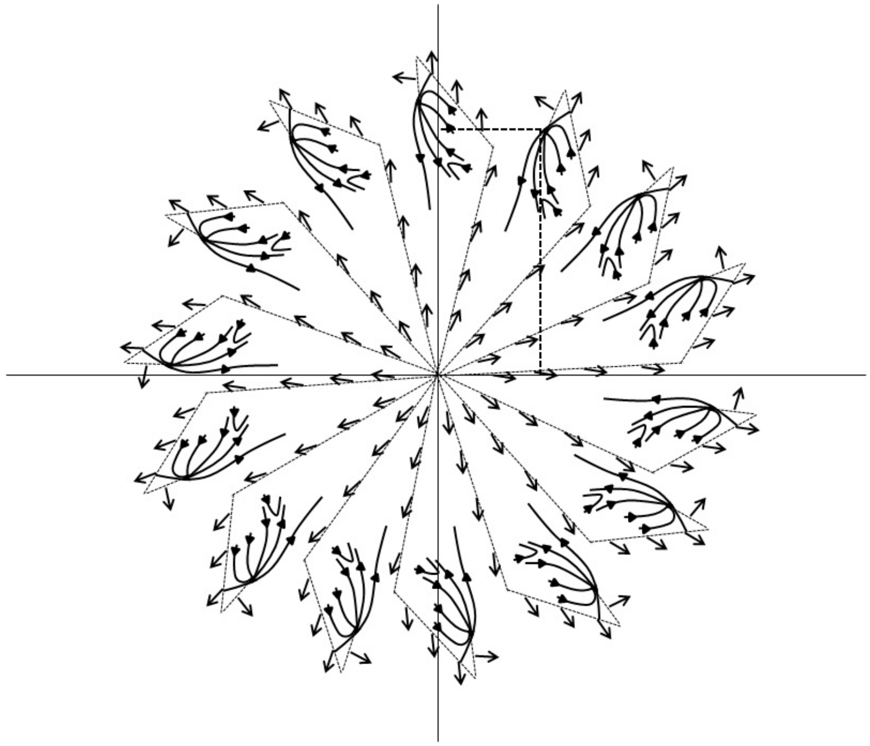

FIGURE 2. The polygonal curve transverse to the flow of the differential equation and the separatrices of the saddle-nodes of (1.1), for $n = 7$.

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