

RADIAL SOLUTIONS WITH A PRESCRIBED NUMBER OF ZEROS FOR A SUPERLINEAR DIRICHLET PROBLEM IN ANNULAR DOMAIN

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ABSTRACT. In this article we study the existence of radially symmetric solutions to a superlinear Dirichlet problem in annular domain in \mathbb{R}^N . Using fairly straightforward tools of the theory of ordinary differential equations, we show that if k is a sufficiently large nonnegative integer, there is a solution u which has exactly $(k - 1)$ interior zeros.

1. INTRODUCTION

The main goal in this article is to study the existence of radially symmetric solutions $u : \mathbb{R}^N \rightarrow \mathbb{R}$ to the superlinear boundary-value problem

$$\begin{aligned} -\Delta u(x) &= f(u) + g(|x|) \quad \text{if } x \in \Omega \\ u &= 0 \quad \text{if } x \in \partial\Omega, \end{aligned} \tag{1.1}$$

where $|x|$ denotes the standard norm of x in \mathbb{R}^N , $N \geq 3$ and Ω is the annulus of \mathbb{R}^N defined by

$$\Omega = \mathbf{C}(0, R, T) = [x \in \mathbb{R}^N : R < |x| < T]$$

where R and T are two real numbers such that $0 < R < T$, $f : \mathbb{R} \rightarrow \mathbb{R}$ is a nonlinear function and $g \in C^1([R, T], \mathbb{R})$.

We will focus on studying the problem (1.1) with the following hypotheses:

- (H1) f is locally Lipschitzian,
- (H2) f is superlinear i.e.,

$$\lim_{|u| \rightarrow \infty} \frac{f(u)}{u} = +\infty,$$

- (H3) there exists $m > 0$ such that

$$NF(u) - \frac{N-2}{2}uf(u) - \frac{N+2}{2}\|g\||u| - T\|g'\||u| \geq -m$$

where $F(u) = \int_0^u f(s)ds$ and $\|g\| = \sup_{R \leq t \leq T} |g(t)|$,

- (H4) $u \rightarrow f(u)$ is increasing for $|u|$ large.

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From (H2) and L'Hopital's Rule it follows that

$$\lim_{|u| \rightarrow \infty} \frac{F(u)}{u^2} = +\infty. \quad (1.2)$$

In recent decades, the existence of solutions to the superlinear Dirichlet problem (1.1) in general domains has been widely studied. Most of these results are based on variational methods. This requires finding a critical point of some energy functional in Sobolev spaces, by assuming that f is locally Lipschitz and satisfies a growth condition. A standard way to do this is to apply the mountain pass theorem. In this context, we mention as examples the authors Berestyki, Bahri and Struwe. When the growth of the nonlinearity surpasses the critical exponent of the Sobolev embedding theorem and the domain is the ball, Castro and Kurepa [2] proved the superlinear Dirichlet problem (1.1) has infinitely many radially symmetric solutions by offering sufficient condition and using the "shooting method" and "phase-plane angle analysis". However, these arguments are quite difficult and provide no specific information about the solution. In particular we ask whether radial solutions exist with prescribed numbers of zeros. McLeod, Troy and Weissler in [6] studied this question for the following problem

$$\begin{aligned} \Delta u(x) + f(u) &= 0 \quad \text{if } x \in \mathbb{R}^N \\ u(x) &\rightarrow 0 \quad \text{as } |x| \rightarrow +\infty. \end{aligned}$$

Thereafter, in the case of the ball, Iaia and Pudingpeddi [4] answered the question above and give an easy proof by using Bessel functions and proved the problem (1.1) has infinitely many radially symmetric solutions with (H1)–(H4) and adding the additional condition

(H5) There exists a $0 < k \leq 1$, such that

$$\lim_{u \rightarrow \infty} \left(\frac{u}{f(u)} \right)^{N/2} \left(NF(ku) - \frac{N-2}{2} uf(u) - \frac{N+2}{2} \|g\| |u| - T \|g'\| |u| \right) = +\infty.$$

An important contribution was made by Gidas, Li and Nirenberg [3] who showed that if Ω is a ball, then all positive solutions of the problem

$$\begin{aligned} \Delta u(x) + f(u) &= 0 \quad \text{if } x \in \Omega \\ u &= 0 \quad \text{if } x \in \partial\Omega \end{aligned}$$

are radially symmetric. This is not the case in the annulus domain. The difficulty resides with the fact that a positive radial solution in annular domain is not monotonic in the radial direction. Our aim here is to extend the results in [2, 4] to the case in an annular domain, by assuming (H1)–(H4) without adding (H5). Our method is based on the same approach used by Iaia and Pudingpeddi [4, 7]; by approximating the solution of (1.1) with an appropriate linear equation. At last, we note that by (H2) the assumption (H3) is more general than (H5)

Our paper is organized as follows: in Section 2 we begin to establish some preliminary results concerning the existence of radial solutions and by analysing the energy we show that the energy function converges uniformly to infinity. In Section 3 we obtain to localize the zeros of the solution and finally, we shall prove the following theorem.

Theorem 1.1. *If (H1)–(H4) are satisfied then (1.1) has infinitely many radially symmetric solutions u with $u'(R) \neq 0$. For $k \in \mathbb{N}^*$ sufficiently large there exist two*

radially symmetric solutions u_k and w_k of problem (1.1) which have exactly $(k-1)$ zeros on (R, T) such that $w'_k(R) < 0 < u'_k(R)$.

2. PRELIMINARIES

The existence of radially symmetric solution $u(x) = u(r)$ with $r = |x|$ of (1.1) is equivalent to the existence of a solution u of the nonlinear ordinary differential equation

$$u''(r) + \frac{N-1}{r}u'(r) + f(u) + g(r) = 0 \quad \text{if } R < r < T, \quad (2.1)$$

$$u(R) = u(T) = 0. \quad (2.2)$$

To solve (2.1)-(2.2), we apply the shooting method, by considering the initial value problem

$$u''(r) + \frac{N-1}{r}u'(r) + f(u) + g(r) = 0 \quad \text{if } R < r < T, \quad (2.3)$$

$$u(R) = 0 \quad \text{and} \quad u'(R) = d$$

with d an arbitrary nonzero real number. Denote $u(r, d)$ as the solution of (2.3) which depends on parameter d . By varying d , we shall attempt to choose the parameter appropriately to have (2.2) and if k is a sufficiently large nonnegative integer then $u(r, d)$ has exactly $(k-1)$ zeros on (R, T) .

Lemma 2.1. *Let $d > 0$, assume (H1) and (H2) hold. Then (2.3) has a unique solution $u(r, d)$ defined on interval $[R, T]$.*

Proof. The proof is divided into two steps. First we show the existence and uniqueness of the local solution of (2.3). In the second step we prove that a unique solution can be extended to a maximal interval $[R, T]$.

Step 1. We consider the initial value problem

$$u''(r) + \frac{N-1}{r}u'(r) + f(u) + g(r) = 0 \quad \text{if } \rho < r < T \quad (2.4)$$

$$u(\rho) = a, \quad u'(\rho) = b$$

with $R \leq \rho < T$ and $(a, b) \in \mathbb{R}^2$. Let $u(r)$ be a solution of (2.4). Multiplying (2.1) by r^{N-1} and by integrating on (ρ, r) with the initial condition gives

$$u'(r) = \frac{1}{r^{N-1}} \left(b\rho^{N-1} - \int_{\rho}^r t^{N-1}(f(u) + g(t)) dt \right), \quad (2.5)$$

Integrating this, we obtain

$$u(r) = a + \frac{b\rho^{N-1}}{N-2} \left(\frac{1}{\rho^{N-2}} - \frac{1}{r^{N-2}} \right) - \int_{\rho}^r \frac{1}{t^{N-1}} \left(\int_{\rho}^t s^{N-1}(f(u) + g(s)) ds \right) dt. \quad (2.6)$$

Conversely, if $u(r)$ is a continuous function and satisfies (2.6) then u is a solution of (2.4). Let $\varepsilon > 0$ and $\Psi(u)$ be equal to the right hand side of (2.6) where $X = C([\rho, \rho + \varepsilon], \mathbb{R})$ the Banach space of real continuous functions on $[\rho, \rho + \varepsilon]$ with uniform norm. By (H1) we can choose ε sufficiently small such that Ψ is a contraction mapping. This enables us to conclude that the problem (2.3) has a unique solution $u(r, d)$ defined on $[R, R + \varepsilon]$ for ε sufficiently small (we take $a = 0$, $b = d$ and $\rho = R$ in (2.4)).

Step 2. Let $u(r, d) = u(r)$ be the unique solution of (2.3) and denote by $[R, R_1[$ its maximal domain. We will show that $R_1 = T$. Otherwise, we suppose that $R_1 < T$.

Then we claim that u is bounded on $[R, R_1[$. We define the energy function of a solution of (2.3) as

$$E(r, d) = E(r) = \frac{u'^2(r)}{2} + F(u(r)) \quad \forall r \in [R, R_1]. \quad (2.7)$$

Then we see from (1.2) that $F(u) > 0$ for u large enough so there exists a $J > 0$ such that

$$F(u) > -J \quad \forall u \in \mathbb{R}. \quad (2.8)$$

It follows from (1.2), (2.7) and (2.8) that

$$E'(r) = -u'g(r) - \frac{N-1}{r}u'^2 \leq \|g\|\|u'\| \leq \|g\|\sqrt{2(E+J)}.$$

Dividing by $\sqrt{2(E+J)}$ and integrating this on (R, r) we obtain

$$\begin{aligned} \sqrt{2(E+J)} - \sqrt{2(E(R)+J)} &\leq \|g\|(r-R), \\ |u'| &\leq \sqrt{2(E(r)+J)} \leq \|g\|(R_1-R) + \sqrt{d^2+2J}. \end{aligned}$$

It follows that u' is bounded on $[R, R_1[$. Therefore, by the mean value theorem and since $u(R) = 0$ we see that u is bounded on $[R, R_1[$. By using this, (2.5) and (2.6) (we take $a = 0, b = d$ and $\rho = R$) we deduce that $(u(r_n))$ and $(u'(r_n))$ are Cauchy sequences for all sequence (r_n) on $[R, R_1)$ increasing and converging to R_1 which implies the existence of the finite limits

$$\lim_{r \rightarrow R_1^-} u(r) = a, \quad \lim_{r \rightarrow R_1^-} u'(r) = b.$$

Now we consider the initial value problem

$$\begin{aligned} v''(r) + \frac{N-1}{r}v'(r) + f(v) + g(r) &= 0 \quad \text{if } r > R_1 \\ v(R_1) &= a, \quad v'(R_1) = b. \end{aligned}$$

By step 1, there exists a $\varepsilon > 0$ and a solution $v(r)$ defined on $[R_1, R_1 + \varepsilon]$. Then it is easy to see that

$$\tilde{u}(r) = \begin{cases} u(r) & \text{if } R < r < R_1 \\ v(r) & \text{if } R_1 < r < R_1 + \varepsilon \end{cases}$$

is a solution of (2.3) on the interval $[R, R_1 + \varepsilon]$ which contains the maximal domain. This is a contradiction. Hence $R_1 = T$. \square

Remark 2.2. Using the Arzela-Ascoli theorem the solution $u(r, d)$ of (2.3) depends continuously on d in the sense that if the sequence (d_n) converges to d , then the sequence of functions $u(\cdot, d_n)$ converges uniformly to $u(\cdot, d)$ on any bounded interval. A similar property is also true for $u'(\cdot, d_n)$.

Remark 2.3. We can use the standard ODE existence-uniqueness theorem to obtain a local solution of (2.3) on $[R, R + \varepsilon]$ for some $\varepsilon > 0$.

As $u'(R, d) = d > 0$ and by continuity then, there exists $r > R$ such that $u' > 0$ on (R, r) . Denote $r_0(d)$ as the largest $r \in (R, T)$ such that $u' > 0$ on (R, r) .

Lemma 2.4. *Assume (H1) and (H2) hold. Then*

- (1) $\lim_{d \rightarrow +\infty} r_0(d) = R$.
- (2) $\lim_{d \rightarrow +\infty} u(r_0(d), d) = +\infty$.

Proof. For (1), we argue by contradiction. Suppose that there exists $\varepsilon > 0$ such that for all $\gamma > 0$ there exists $d > \gamma$ for which

$$R + \varepsilon \leq r_0(d).$$

Denote $R_0 = R + \varepsilon$. Then there exists a sequence $d_n \rightarrow +\infty$ such that

$$\begin{aligned} r_0(d_n) &\geq R_0 \\ u(r, d_n) &> 0, \quad u'(r, d_n) \geq 0 \quad \forall r \in (R, R_0), \forall n \in \mathbb{N}. \end{aligned} \tag{2.9}$$

We set $\bar{r} = (R + R_0)/2$ and $u(\bar{r}, d_n) = u_n(\bar{r})$. We now show that the sequence $(u_n(\bar{r}))$ is unbounded. Again by contradiction we suppose that there exists $M > 0$ such that for all $n \in \mathbb{N}$, $0 < u_n(\bar{r}) \leq M$. By (2.6) (with $a = 0$, $b = d_n$ and $\rho = R$) and u_n is increasing on $[R, R_0]$ we obtain

$$\begin{aligned} \frac{d_n R^{N-1}}{N-2} \left(\frac{1}{R^{N-2}} - \frac{1}{\bar{r}^{N-2}} \right) &= u_n(\bar{r}) + \int_R^{\bar{r}} \frac{1}{t^{N-1}} \left(\int_R^t s^{N-1} (f(u) + g(s)) \, ds \right) dt \\ &\leq M + \frac{T^2}{N} \sup_{0 \leq \zeta \leq M} (|f(\zeta)| + \|g\|) < \infty \end{aligned}$$

which is a contradiction to $d_n \rightarrow +\infty$. Hence, the sequence $(u_n(\bar{r}))$ is unbounded and passing to subsequence we can suppose that

$$\lim_{n \rightarrow +\infty} u_n(\bar{r}) = +\infty.$$

Now, for all $n \in \mathbb{N}$, we denote

$$M_n = \inf_{\bar{r} \leq r \leq R_0} \left\{ \frac{f(u_n)}{u_n} + \frac{g(r)}{u_n} \right\}.$$

Since, $0 < u_n(\bar{r}) \leq u_n(r)$ for all $r \in [\bar{r}, R_0]$ we see that

$$M_n \geq \inf_{u_n(\bar{r}) \leq u \leq u_n(R_0)} \left\{ \frac{f(u)}{u} \right\} - \frac{\|g\|}{u_n(\bar{r})}.$$

On the other hand, from (H2) and $\lim_{n \rightarrow +\infty} u_n(\bar{r}) = +\infty$ we have $\lim_{n \rightarrow +\infty} M_n = +\infty$. Thus, there exists $n_0 \in \mathbb{N}$ such that $M_{n_0} > \mu_2$ where $\mu_2 > 0$ is the second eigenvalue of $-[\frac{d^2}{dr^2} + \frac{N-1}{r} \frac{d}{dr}]$ in (\bar{r}, R_0) with Dirichlet boundary conditions. It is known that the first eigenfunction of this operator can be chosen to be positive. Then since the second eigenfunction is orthogonal to the first eigenfunction then necessarily the second Φ_2 eigenfunction must be zero somewhere on (\bar{r}, R_0) . Then by Sturm comparison theorem since $\mu_2 < M_{n_0}$ it follows that u_{n_0} has at least one zero in (\bar{r}, R_0) . This is a contradiction with (2.9) and finally we deduce that $\lim_{d \rightarrow +\infty} r_0(d) = R$.

For (2), since $\lim_{d \rightarrow +\infty} r_0(d) = R$ then for $d > 0$ sufficiently large we have $R < r_0(d) < T$. On the other hand, u has a local maximum at $r_0(d)$ then, there exists $r^* \in (r_0(d), T)$ such that u is decreasing and nonnegative on $(r_0(d), r^*)$. Now, we will show that

$$\lim_{d \rightarrow +\infty} u(r_0(d), d) = +\infty.$$

Suppose that there exists a sequence $d_n \rightarrow +\infty$ such that $(u(r_0(d_n), d_n))$ is bounded by M . From (2.5) we obtain that for all $n \in \mathbb{N}$ and for all $r \in (r_0(d_n), r^*)$

$$r^{N-1} u'(r) = d_n R^{N-1} - \int_R^r t^{N-1} (f(u) + g(t)) \, dt \leq 0,$$

$$\begin{aligned} d_n R^{N-1} &\leq \int_R^T t^{N-1} (f(u) + g(t)) dt \quad (0 \leq u \leq M) \\ &\leq \sup_{0 \leq \zeta \leq M} (|f(\zeta)| + \|g\|) \frac{T^N}{N} < \infty. \end{aligned}$$

It follows that (d_n) is bounded which is a contradiction to $d_n \rightarrow +\infty$. \square

Lemma 2.5. *Assume (H1)–(H3) hold. Then*

$$\lim_{d \rightarrow +\infty} \inf_{r \in [R, T]} E(r, d) = +\infty.$$

Proof. Let $r \in [R, T]$. We consider the Pohozaev-type identity

$$\begin{aligned} &\left(r^N E + r^N g(r)u + \frac{N-2}{2} r^{N-1} uu' \right)' \\ &= r^{N-1} \left(NF(u) - \frac{N-2}{2} uf(u) + \frac{N+2}{2} g(r)u + rg'(r)u \right). \end{aligned}$$

From (H3), we have

$$NF(u) - \frac{N-2}{2} uf(u) - \frac{N+2}{2} \|g\| |u| - T \|g'\| |u| \geq -m.$$

Integrating Pohozaev's identity on (R, r) with the initial conditions, gives

$$r^N E + r^N g(r)u + \frac{N-2}{2} r^{N-1} uu' \geq \frac{R^N d^2}{2} - \frac{m}{N} (T^N - R^N). \quad (2.10)$$

Now from (1.2) we deduce there exists $B > 0$ such that for all $|u| > B$,

$$0 < u^2 < F(u) < F(u) + J.$$

If $|u| \leq B$ then from (2.8) we see that

$$u^2 \leq F(u) + J + B^2 \leq E + J + B^2 \quad \forall u \in \mathbb{R}. \quad (2.11)$$

Using Young's inequality we have

$$|uu'| \leq \frac{u^2}{2} + \frac{u'^2}{2} \leq F(u) + J + B^2 + \frac{u'^2}{2},$$

We deduce that

$$|uu'| \leq E + J + B^2. \quad (2.12)$$

Hence using (2.11) and (2.12),

$$\begin{aligned} &r^N E + r^N g(r)u + \frac{N-2}{2} r^{N-1} uu' \\ &\leq T^N E + T^N \|g\| |u| + \frac{N-2}{2} T^{N-1} |uu'| \\ &\leq T^N E + T^N (\|g\|^2 + u^2) + \frac{N-2}{2} T^{N-1} (E + J + B^2) \\ &\leq T^N E + T^N \|g\|^2 + (T^N + \frac{N-2}{2} T^{N-1}) (E + J + B^2) \\ &\leq \left(2T^N + \frac{N-2}{2} T^{N-1} \right) E + T^N \|g\|^2 + (T^N + \frac{N-2}{2} T^{N-1}) (J + B^2) \\ &\leq C_1 E + C_2 \end{aligned}$$

with C_1 and C_2 two positive real numbers depending only on N, T, J and g . From (2.10), then we have

$$\inf_{r \in [R, T]} E \geq \frac{R^N d^2}{2C_1} - \frac{C_2}{C_1} - \frac{m}{NC_1}(T^N - R^N).$$

Finally we deduce that $\lim_{d \rightarrow +\infty} \inf_{r \in [R, T]} E(r, d) = +\infty$. □

Lemma 2.6. *If d is sufficiently large, then*

- (1) *all the zeros of $u(r, d)$ are simple on $[R, T]$.*
- (2) *$u(r, d)$ has a finite number of zeros on $[R, T]$.*

Proof. (1) From Lemma 2.5, for d sufficiently large and all $r \in [R, T]$, we have $E(r, d) > 0$. If t_0 is a zero of $u(r, d)$, then $E(t_0, d) = \frac{u'^2(t_0, d)}{2} > 0$; thus $u'(t_0, d) \neq 0$. Then t_0 is a simple zero of $u(r, d)$.

For (2), we argue by contradiction. Suppose if d is sufficiently large there exists $R < t_1 < \dots < t_n < t_{n+1} \leq T$ and $u(t_n) = 0$ for all $n \in \mathbb{N}$. Using the mean value theorem, there exists $z_n \in (t_n, t_{n+1})$ such that $u'(z_n, d) = 0$ for all $n \in \mathbb{N}$. So (t_n) converges to $t \leq T$ and by continuity of u and u' we deduce that $u(t, d) = u'(t, d) = 0$. This is a contradiction to (1). Thus for d sufficiently large u has a finite number of zeros on $[R, T]$. □

3. SOLUTION WITH A PRESCRIBED NUMBER OF ZEROS

In this section we show the solution $u(r, d)$ has a large number of zeros for d sufficiently large. For this we study the behavior of zeros of $u(r, d)$ for d large enough. Also, assuming (H1)–(H4) hold, it is obvious that the first zero of $u(r, d)$ is $z_0(d) = R$. In the following we focus on finding the zeros of $u(r, d)$ on interval $]R, T]$. From (H2), the mapping $u \mapsto F(u)$ is increasing for large u and decreasing when u is a large negative number. By (1.2), we have $F(u) > 0$ for sufficiently large $|u|$ and from Lemma 2.5 we deduce that for d sufficiently large the equation $F(u) = \frac{1}{2} \inf_{r \in [R, T]} E(r, d)$ has exactly two solutions, which we denote $h_1(d)$ and $h_2(d)$ such that

$$h_2(d) < 0 < h_1(d),$$

$$F(h_i(d)) = \frac{1}{2} \inf_{r \in [R, T]} E(r, d) \quad \text{for } i = 1, 2.$$

From (1.2) and Lemma 2.5 we see that

$$\lim_{d \rightarrow +\infty} h_1(d) = +\infty. \tag{3.1}$$

Also, $\lim_{d \rightarrow +\infty} h_2(d) = -\infty$.

On the other hand by (H2), for d large enough, $u''(r_0(d)) = -f(u(r_0(d)) - g(r_0(d))) < 0$. As $u'(r_0(d)) = 0$ so u is decreasing on $(r_0(d), r)$ for r close enough to $r_0(d)$. Denote for d sufficiently large

$$r^*(d) = \sup \{r \in (r_0(d), T) : u \text{ is decreasing on } (r_0(d), r)\}.$$

There are two cases $r^*(d) = T$ and $r^*(d) < T$.

Lemma 3.1. *If (H1)–(H4) are satisfied, then for d sufficiently large there exist $r_1 \in (r_0(d), T)$ such that $u(r_1) = h_1(d)$ and $h_1(d) < u \leq u(r_0(d))$ on $[r_0(d), r_1)$.*

Proof. Suppose by contradiction there exists a sequence $d_n \rightarrow \infty$ such that for all $n \in \mathbb{N}$,

$$u(r, d_n) = u_n(r) > h_1(d_n) \quad \text{on } (r_0(d_n), T).$$

If $r^*(d_n) = T$ then u_n is decreasing on $[r_0(d_n), T]$ for n large enough. From (3.1), (H2) and (H4) we obtain for n large enough and for all $r \geq r_0(d_n)$

$$u_n(r) > h_1(d_n) \quad \text{and} \quad f(u_n(r)) > f(h_1(d_n)) > \|g\|. \quad (3.2)$$

Let n be large enough and $s \geq r_0(d_n) = r_{0,n}$. From (2.5) we have

$$-u'_n(s) = \frac{1}{s^{N-1}} \int_{r_{0,n}}^s t^{N-1} (f(u_n) + g(t)) dt.$$

Integrating on $(r_{0,n} + \frac{r}{2}, r_{0,n} + r)$ with $r \in (0, T - r_{0,n})$ gives

$$u_n(r_{0,n} + \frac{r}{2}) = u_n(r_{0,n} + r) + \int_{r_{0,n} + \frac{r}{2}}^{r_{0,n} + r} \frac{1}{s^{N-1}} \left(\int_{r_{0,n}}^s t^{N-1} (f(u_n) + g(t)) dt \right) ds.$$

As u_n is decreasing and by (3.2) we have

$$u_n(r_{0,n} + \frac{r}{2}) \geq \frac{f(u_n(r_{0,n} + \frac{r}{2})) - \|g\|}{2NT^{N-1}} \left([r_{0,n} + \frac{r}{2}]^N - r_{0,n}^N \right) r.$$

Taking $r = T - r_{0,n}$ by (3.1), (3.2) and (H2) we see that

$$\left(\left[\frac{r_{0,n} + T}{2} \right]^N - r_{0,n}^N \right) \frac{(T - r_{0,n})}{2NT^{N-1}} \leq \frac{u_n\left(\frac{r_{0,n} + T}{2}\right)}{f(u_n\left(\frac{r_{0,n} + T}{2}\right)) - \|g\|} \rightarrow 0.$$

Since $r_{0,n} \rightarrow R$, it follows that

$$\left(\left[\frac{R + T}{2} \right]^N - R^N \right) \frac{(T - R)}{2NT^{N-1}} = 0$$

which implies $T = R$ which is impossible. Thus it must be that $r^*(d_n) < T$.

For $r^*(d_n) = r^* < T$, we have $u'_n(r^*) = 0$ and $\int_{r_{0,n}}^{r^*} t^{N-1} (f(u_n) + g(t)) dt = 0$. However by (3.2) we deduce that $f(u_n(t)) - g(t) > f(u_n(t)) - \|g\| > 0$ on $[r_{0,n}, r^*]$ and so $\int_{r_{0,n}}^{r^*} t^{N-1} (f(u_n) + g(t)) dt > 0$. This is impossible. End of the proof. \square

Thus, for d sufficiently large we denote by $r_1(d)$ the smallest $r \in (r_0(d), T)$ such that

$$u(r_1(d)) = h_1(d), \quad h_1(d) < u \leq u(r_0(d)) \quad \text{on } [r_0(d), r_1(d)]. \quad (3.3)$$

Lemma 3.2. *If (H1)–(H4) are satisfied, then*

- (1) $\lim_{d \rightarrow +\infty} r_1(d) = R$.
- (2) For d sufficiently large, $u(r, d)$ has a first zero $z_1(d)$ in the interval (R, T) , and $\lim_{d \rightarrow +\infty} z_1(d) = R$.

Proof. For (1), let

$$C(d) = \frac{1}{2} \min_{r \in [r_0(d), r_1(d)]} \frac{f(u)}{u} = \frac{1}{2} \min_{r \in [h_1(d), u(r_0(d))]} \frac{f(s)}{s}.$$

It follows from (3.1) and (H2) that

$$\lim_{d \rightarrow +\infty} C(d) = +\infty. \quad (3.4)$$

We now compare the problem

$$u''(r) + \frac{N-1}{r}u'(r) + \frac{f(u)}{u}u + g(r) = 0 \quad (3.5)$$

with

$$v''(r) + \frac{N-1}{r}v'(r) + C(d)v = 0 \quad (3.6)$$

with the initial conditions

$$u(r_0(d)) = v(r_0(d)) \quad \text{and} \quad u'(r_0(d)) = v'(r_0(d)) = 0. \quad (3.7)$$

Then by (3.4) we see that for d sufficiently large and all $r \in [r_0(d), r_1(d)]$, we have

$$\frac{f(u)}{u} \geq 2C(d) > C(d). \quad (3.8)$$

Claim: for d sufficiently large, $u < v$ on $(r_0(d), r_1(d)]$.

Indeed, multiplying (3.5) by $r^{N-1}v$ and (3.6) by $r^{N-1}u$ and subtracting, gives

$$(r^{N-1}(u'v - uv'))' + r^{N-1}uv\left(\frac{f(u)}{u} + \frac{g(r)}{u} - C(d)\right) = 0.$$

Integrating this on $(r_0(d), r)$ and using the initial conditions, gives

$$r^{N-1}(u'v - uv') = - \int_{r_0(d)}^r t^{N-1}uv\left(\frac{f(u)}{u} + \frac{g(t)}{u} - C(d)\right) dt. \quad (3.9)$$

From (3.1), (3.4) and (3.8) we see that for d sufficiently large,

$$\frac{f(u)}{u} + \frac{g(r)}{u} - C(d) \geq C(d) - \frac{\|g\|}{h_1(d)} > 0. \quad (3.10)$$

For d sufficiently large, let $\mathcal{F} = \{r \in (r_0(d), r_1(d)) : u < v \text{ on } (r_0(d), r)\}$. Then

$$\begin{aligned} u''(r_0(d)) &= -g(r_0(d)) - f(u(r_0(d))) \\ &= u(r_0(d))\left(-\frac{g(r_0(d))}{u(r_0(d))} - \frac{f(u(r_0(d)))}{u(r_0(d))} + C(d)\right) - C(d)u(r_0(d)). \end{aligned}$$

From (H2) and Lemma 2.4 it follows that for d sufficiently large

$$u(r_0(d)) > 0 \quad \text{and} \quad -\frac{g(r_0(d))}{u(r_0(d))} - \frac{f(u(r_0(d)))}{u(r_0(d))} + C(d) < 0.$$

Then, for d sufficiently large we have

$$u''(r_0(d)) < -C(d)u(r_0(d)) = v''(r_0(d)).$$

By continuity there exists $\varepsilon > 0$ such that $(u - v)''(r) < 0$ on $(r_0(d), r_0(d) + \varepsilon)$. Using the initial conditions (3.7) we deduce that $u < v$ on $(r_0(d), r_0(d) + \varepsilon)$. Thus $\mathcal{F} \neq \emptyset$. We denote $\bar{r} = \sup \mathcal{F}$. Now we will show that $\bar{r} = r_1(d)$. Otherwise, suppose that

$$u < v \quad \text{on } (r_0(d), \bar{r}) \quad \text{and} \quad u(\bar{r}) = v(\bar{r}).$$

Since $0 < h_1(d) < u < v$ on $(r_0(d), \bar{r})$ and by (3.10) we see that, for d sufficiently large then

$$r^{N-1}uv\left(\frac{f(u)}{u} + \frac{g(r)}{u} - C(d)\right) > 0.$$

Therefore, by (3.9) $u'(r)v(r) - u(r)v'(r) < 0$ on $(r_0(d), \bar{r}]$. Thus, $u'(\bar{r}) < v'(\bar{r})$. On the other hand, as $u(r) < v(r)$ for $r < \bar{r}$ we have

$$\frac{u(r) - u(\bar{r})}{r - \bar{r}} > \frac{v(r) - v(\bar{r})}{r - \bar{r}}.$$

Hence $u'(\bar{r}) \geq v'(\bar{r})$. This is a contradiction. It follows that $\bar{r} = r_1(d)$ which completes the proof of the claim.

Now, we set

$$z(r) = \left(r/\sqrt{C(d)} \right)^{\frac{N-2}{2}} v \left(r/\sqrt{C(d)} \right).$$

It is easy to verify that $z(r)$ is a solution of Bessel's equation of order $\nu = \frac{N-2}{2} > 0$. i.e.,

$$z'' + \frac{z'}{r} + \left(1 - \frac{\nu^2}{r^2}\right)z = 0.$$

Then there exists a constant $K > 0$ such that every interval of length K has at least one zero of $z(r)$ (see [5]). It follows that every interval of length $K/\sqrt{C(d)}$ contains at least one zero of $v(r)$. Hence by claim for d sufficiently large we have

$$r_0(d) < r_1(d) < r_0(d) + \frac{K}{\sqrt{C(d)}}.$$

Now (1) of this lemma is a consequence of Lemma 2.4 and (3.4).

For (2), suppose not, which means $u > 0$ on $(R, T]$ and consider $r > r_1(d)$. Then $0 < u < u(r_1(d))$. Also as $F(h_1(d)) = \frac{1}{2} \inf_{r \in [R, T]} E(r, d)$ for large d , thus

$$2F(h_1(d)) \leq \frac{u'^2}{2} + F(u) \leq \frac{u'^2}{2} + F(h_1(d)).$$

Therefore

$$-u' = |u'| \geq \sqrt{2F(h_1(d))} \quad \text{for } r_1(d) \leq r \leq T.$$

Integrating on $(r_1(d), r)$ and by (3.3) we obtain

$$h_1(d) - u(r) = u(r_1(d)) - u(r) \geq \sqrt{2F(h_1(d))}(r - r_1(d)),$$

so that

$$h_1(d) - \sqrt{2F(h_1(d))}(r - r_1(d)) \geq u(r) > 0,$$

thus

$$r - r_1(d) \leq \frac{h_1(d)}{\sqrt{2F(h_1(d))}} \tag{3.11}$$

for large d .

Taking $r = T$ and taking the limit as $d \rightarrow \infty$ in (3.11) as well as using (1.2), (3.1) and $r_1(d) \rightarrow R$ we see that

$$0 < T - R \leq \frac{h_1(d)}{\sqrt{2F(h_1(d))}} \rightarrow 0$$

as $d \rightarrow \infty$. This is impossible since $T > R$. Thus u has a first zero $z_1(d)$. Then using a similar argument on $[r_1(d), z_1(d)]$ and letting $r = r_1(d)$ in (3.11) we obtain $\lim_{d \rightarrow +\infty} z_1(d) = R$. The proof is complete. \square

Lemma 3.3. *Let (H1)–(H4) be satisfied. Then for d sufficiently large the solution $u(r, d)$ attains a local minimum at $r_3(d) \in (r_2(d), T)$ and moreover $\lim_{d \rightarrow \infty} r_3(d) = R$.*

Proof. We begin to establish the following claim.

Claim: for d sufficiently large $u(r, d)$ attains the value $h_2(d)$ on $(z_1(d), T)$.

Otherwise, there exists a sequence $d_n \rightarrow \infty$ such that for all $n \in \mathbb{N}$, $u_n(r) > h_2(d_n)$ on $(z_1(d_n), T)$. By Lemma 2.6 we have $u'_n(z_1(d_n)) \neq 0$ for n large enough. As $u'_n < 0$ on $]r_1(d_n), z_1(d_n)[$ therefore $u'_n(z_1(d_n)) < 0$. Then by continuity we see that $u'_n < 0$ on some maximal interval $[z_1(d_n), r^*[$ for n large enough, therefore $h_2(d_n) < u_n$. Thus $F(u_n) < F(h_2(d_n))$ on $[z_1(d_n), r^*[$. Hence by the definition of $h_2(d)$ at the beginning of section 3 we have

$$2F(h_2(d_n)) \leq E(r, d_n) < \frac{u_n'^2}{2} + F(h_2(d_n)).$$

Therefore

$$0 < \sqrt{2F(h_2(d_n))} \leq |u'_n| = -u'_n \quad \forall r \in [z_1(d_n), r^*].$$

In particular $u'_n(r^*) < 0$. This implies $r^* = T$ for if $r^* < T$ then by definition of r^* we would have $u'_n(r^*) = 0$. Now integrating this inequality on $(z_1(d_n), r)$ we obtain, for n large enough

$$h_2(d_n) < u_n(r) \leq -\sqrt{2F(h_2(d_n))}(r - z_1(d_n)) \quad \forall r \in [z_1(d_n), T]. \tag{3.12}$$

Taking $r = T$ we have

$$T - z_1(d_n) \leq \frac{-h_2(d_n)}{\sqrt{2F(h_2(d_n))}}.$$

Since $\lim_{n \rightarrow \infty} h_2(d_n) = -\infty$, by (1.2) we deduce that $\lim_{n \rightarrow \infty} \frac{-h_2(d_n)}{\sqrt{2F(h_2(d_n))}} = 0$. As $\lim_{n \rightarrow \infty} z_1(d_n) = R$ (by Lemma 3.2) then $T = R$. This is a contradiction. End of proof of claim.

We denote by $r_2(d)$ the smallest $r \in (z_1(d), T)$ such that $u(r_2(d)) = h_2(d)$ and $h_2(d) < u(r, d)$ on $[z_1(d), r_2(d)[$. By (3.12) taking $r = r_2(d)$ we see that

$$\lim_{d \rightarrow \infty} r_2(d) = R. \tag{3.13}$$

Now, suppose by contradiction that u is decreasing on $(r_2(d), T)$. Then $u < h_2(d) < 0$ on $(r_2(d), T)$. We set

$$C(d) = \frac{1}{2} \min_{u \leq h_2(d)} \frac{f(u)}{u}.$$

By (H2), we see that

$$\lim_{d \rightarrow +\infty} C(d) = +\infty. \tag{3.14}$$

Now, we compare the problem

$$u''(r) + \frac{N-1}{r}u'(r) + \frac{f(u)}{u}u + g(r) = 0 \tag{3.15}$$

with

$$v''(r) + \frac{N-1}{r}v'(r) + C(d)v = 0 \tag{3.16}$$

and with the initial conditions

$$v(r_2(d)) = u(r_2(d)) = h_2(d) \text{ and } v'(r_2(d)) = u'(r_2(d)). \tag{3.17}$$

As in the proof of Lemma 3.2 we see that $u > v$ on $(r_2(d), T)$, for d large enough. We saw that

$$z(r) = \left(r/\sqrt{C(d)}\right)^{\frac{N-2}{2}} v\left(r/\sqrt{C(d)}\right)$$

is a solution of the Bessel's equation of order $\nu = \frac{N-2}{2}$. Then, there exists $K > 0$ such every interval of length K has at least one zero of $z(r)$. We deduce that for large d , v must have a zero on $(r_2(d), T)$ and since $u > v$ we see that u gets positive which contradicts that u is decreasing on $(r_2(d), T)$. It follows that u has a local minimum at $r_3(d) \in (r_2(d), T)$. Also, for d sufficiently large we have

$$r_2(d) < r_3(d) \leq r_2(d) + \frac{K}{\sqrt{C(d)}}.$$

It follows from (3.14) and (3.13) as $d \rightarrow \infty$ that $r_3(d) \rightarrow R$. This completes the proof. \square

As $F(u(r_3(d))) = E(r_3(d)) \rightarrow \infty$ as $d \rightarrow \infty$ (by Lemma 2.5), in similar way we can show that for d large enough, $u(r, d)$ has a second zero $z_2(d)$ with $r_3(d) < z_2(d) < T$ and moreover $\lim_{d \rightarrow +\infty} z_2(d) = R$. Proceeding in the same way, we can show that for d sufficiently large, $u(r, d)$ has a second local maximum at $r_4(d) \in (z_2(d), T)$ with $\lim_{d \rightarrow +\infty} u(r_4(d)) = +\infty$ and therefore, there exists $z_3(d)$ the third zero of $u(r, d)$ on (R, T) with $\lim_{d \rightarrow +\infty} z_3(d) = R$.

Remark 3.4. Continuing in the same way we can obtain as many zeros of $u(r, d)$ as desired on (R, T) for d large enough.

4. PROOF OF MAIN RESULT

For $d > 0$, let us denote by N_d card{zeros zeros of $u(r, d)$ on (R, T) }. For $k \geq 1$ defined by set

$$S_k = \{d : N_d = k - 1 \text{ and } \inf_{r \in [R, T]} E(r, d) > 0\}.$$

By Lemma 2.5 and remark 3.4, we see that for d sufficiently large, S_k is not empty for some k and $\inf_{r \in [R, T]} E(r, d) > 0$ and we denote $k_0 = \min\{k \in \mathbb{N}^* \mid S_k \neq \emptyset\}$. It follows that S_{k_0} is not empty and is bounded above. Let $d_{k_0} = \sup S_{k_0}$.

Lemma 4.1. $N_{d_{k_0}} = k_0 - 1$.

Proof. By definition of k_0 we have $N_{d_{k_0}} \geq k_0 - 1$. Suppose now that $N_{d_{k_0}} \geq k_0$. Then for d close to d_{k_0} and $d \leq d_{k_0}$ by remark 2.2 with respect to initial conditions and by Lemma 2.6 we see that $N_d \geq k_0$. However, if $d \in S_{k_0}$ and is close to d_{k_0} and $d < d_{k_0}$ then $N_d = k_0 - 1$. This is a contradiction to the definition of d_{k_0} . Hence $N_{d_{k_0}} = k_0 - 1$. \square

Lemma 4.2. $u(T, d_{k_0}) = 0$.

Proof. We argue by contradiction and assume that $u(T, d_{k_0}) \neq 0$, then by remark 2.2 with respect to initial conditions and by Lemma 2.6, we deduce that if d is close to d_{k_0} then $N_d = N_{d_{k_0}}$. Now, for d close to d_{k_0} and $d > d_{k_0}$ then $d \notin S_{k_0}$ therefore, $N_d \neq k_0 - 1$. This is a contradiction with Lemma 4.1. Hence $u(T, d_{k_0}) = 0$. \square

We denote $S_{k_0+1} = \{d > d_{k_0} : N_d = k_0 \text{ and } \inf_{r \in [R, T]} E(r, d) > 0\}$.

Lemma 4.3. $S_{k_0+1} \neq \emptyset$.

Proof. We want to show the following result first.

Claim: If d close to d_{k_0} and $d > d_{k_0}$ then $N_d \leq k_0$.

Suppose by contradiction that there exists a sequence $q_n \rightarrow d_{k_0}$ such that $N_{q_n} \geq k_0 + 1$. For all $1 \leq i \leq k_0$ let us denote z_i^n the i th zero of $u(r, q_n)$ on (R, T) such that

$$R < z_1^n < z_2^n < \dots < z_{k_0}^n < z_{k_0+1}^n < T.$$

For every $1 \leq i \leq k_0 + 1$ the sequence (z_i^n) is bounded and converges to z_i thus, we see that

$$R < z_1 < z_2 < \dots < z_{k_0} < z_{k_0+1} < T.$$

It follows that $N_{d_{k_0}} \geq k_0$, which contradicts Lemma 4.1. Thus the claim is proven.

Finally, if $d > d_{k_0}$ then $N_d \leq k_0$ and $N_d \neq k_0 - 1$ thus, $N_d = k_0$ and $S_{k_0+1} \neq \emptyset$ which completes the proof. □

By remark 3.4 it follows that S_{k_0+1} is not empty and bounded above thus, we denote $d_{k_0+1} = \sup S_{k_0+1}$. We show in a similar way as Lemmas 4.1 and 4.2 that $N_{d_{k_0+1}} = k_0$ and $u(T, d_{k_0+1}) = 0$. Proceeding inductively we can show, for all $k \geq k_0$ there exists a solution $u_k(r) = u(r, d_k)$ of (2.1)-(2.2) which has exactly $(k - 1)$ zeros on (R, T) with $u'_k(R) = d_k > 0$.

Now, in the case $d < 0$ we consider the problem

$$\begin{aligned} u''(r) + \frac{N-1}{r}u'(r) + f(u) + g(r) &= 0 \quad \text{if } R < r < T \\ u(R) = 0, \quad u'(R) &= d < 0. \end{aligned} \tag{4.1}$$

We denote $v(r) = -u(r)$ and $g_1(r) = -g(r)$ on $[R, T]$ and $f_1(s) = -f(-s)$ on \mathbb{R} then the problem (4.1) is equivalent to

$$\begin{aligned} v''(r) + \frac{N-1}{r}v'(r) + f_1(v) + g_1(r) &= 0, \quad \text{if } R < r < T \\ v(R) = 0, \quad v'(R) &= -d > 0. \end{aligned} \tag{4.2}$$

Then g_1 is $C^1([R, T], \mathbb{R})$. It is clear that the assumptions (H1), (H2) and (H4) are satisfied.

It remains to prove (H3). We set $F_1(v) = \int_0^v f_1(s)ds$. Then $F_1(v) = F(-v)$ for all $v \in \mathbb{R}$; thus

$$\begin{aligned} NF_1(v) - \frac{N-2}{2}vf_1(v) - \frac{N+2}{2}\|g_1\||v| - T\|g'_1\||v| \\ = NF(u) - \frac{N-2}{2}uf(u) - \frac{N+2}{2}\|g\||u| - T\|g'\||u| > -m. \end{aligned}$$

Next, according to the case $d > 0$ we deduce that, for k sufficiently large, (2.1)-(2.2) has a solution v_k which has exactly $(k - 1)$ zeros on (R, T) with $v'_k(R) > 0$. Finally, for k sufficiently large, (2.1)-(2.2) has a solution $w_k = -v_k$ which has $(k - 1)$ zeros on (R, T) and $w'_k(R) < 0$. End of proof of the main Theorem 1.1.

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