Electronic Journal of Differential Equations, Vol. 2016 (2016), No. 110, pp. 1–16.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu

ON THE HIGH-ORDER TOPOLOGICAL ASYMPTOTIC EXPANSION FOR SHAPE FUNCTIONS

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Abstract. This article concerns the topological sensitivity analysis for the Laplace operator with respect to the presence of a Dirichlet geometry perturbation. Two main results are presented in this work. In the first result we discuss the influence of the considered geometry perturbation on the Laplace solution. In the second result we study the high-order topological derivatives. We derive a high-order topological asymptotic expansion for a large class of shape functions.

1. Introduction

The topological sensitivity analysis consists in studying the variation of a shape functional with respect to the presence of a small geometry perturbation at an arbitrary point of the domain; see [1, 7, 9, 15, 16, 17, 19, 21, 24]. To present the basic idea, we consider an open and bounded domain $\Omega \subset \mathbb{R}^3$ and a shape function $j(\Omega) = J(u_\Omega)$ to be minimized, where u_Ω is the solution to a given partial differential equation defined in Ω. For $\varepsilon > 0$, let $\Omega_{z,\varepsilon} = \Omega \setminus \overline{\omega_{z,\varepsilon}}$ be the perturbed domain obtained by removing a small part $\omega_{z,\varepsilon} = z + \varepsilon\omega$ from the domain Ω, where $z \in \Omega$ and $\omega \subset \mathbb{R}^3$ is a given fixed and bounded domain containing the origin. The topological sensitivity analysis leads to an asymptotic expansion of the function j in the form

$$j(\Omega_{z,\varepsilon}) = j(\Omega) + f(\varepsilon)\delta j(z) + o(f(\varepsilon)),$$

where $f(\varepsilon)$ is a scalar positive function approaching zero as ε approaches zero. The function δj is called the topological gradient. It gives us the best locations in Ω of the geometry perturbations for which the shape function j decrease most, i.e. the topological gradient δj is as negative as possible. In fact, if $\delta j(z) < 0$, we have $j(\Omega_{z,\varepsilon}) < j(\Omega)$ for small ε.

The topological gradient δj has been used as a descent direction to solve various problems; fluid flow optimal shape design [1, 2, 6], structural mechanics [14, 15], geometry inverse problems [5, 7, 20], image processing [8], and many other applications.

The majority of the optimization algorithms dealing with the topological derivative are based on the first-order asymptotic expansion. This provides interesting

2010 *Mathematics Subject Classification.* 35A15, 35B25, 35B40, 49K40.
Key words and phrases. Laplace equation; calculus of variations; sensitivity analysis; topological derivative; topology optimization.

Submitted January 3, 2016. Published April 26, 2016.

optimization results in some particular configurations like the case when the unknown domain is small and not close to the boundary $\partial\Omega$, one can consult *detection of small cavities in Stokes flow* in BenAbda et al [7].

Classically, the topological gradient δj described by the leading term of the first-order asymptotic expansion, dealing only with infinitesimal geometry perturbations. However, for practical applications, we need to detect domains of finite size. Therefore, as a natural extension of the topological derivative concept we consider high-order terms in the asymptotic expansion. In this context, Novotny et al. [11, 12, 10] was derived a second-order topological asymptotic for the Laplace operator. The obtained results are limited to the two dimensional case.

In this work, we consider the three dimensional case and we derive a high-order topological asymptotic expansion for the Laplace operator with respect to the presence of Dirichlet geometric perturbations. The proposed approach is based on two main steps.

In the first one, we derive a high-order asymptotic expansion for the solution of the perturbed Laplace equation with respect to ε. This question has been investigated by Ammari and Kang [3] in the inhomogeneities case where the perturbed solution is computed in the entire domain Ω using continuity condition on the boundary $\partial\omega_{z,\varepsilon}$. In this work, we deal with more singular geometric perturbation. The solution of the perturbed Laplace equation is computed in $\Omega_{z,\varepsilon} = \Omega\setminus\overline{\omega_{z,\varepsilon}}$ with Dirichlet condition on $\partial\omega_{z,\varepsilon}$. As we will show in Section 3, this type of perturbations leads to an asymptotic behavior with respect to ε different from that obtained in [3].

In the second step, we derive a high-order topological asymptotic expansion for the Laplace operator. More precisely, we derive an asymptotic expansion of a given shape functional j in the form

$$j(\Omega_{z,\varepsilon}) = j(\Omega) + \sum_{k=1}^{N} f_k(\varepsilon)\delta^k j(z) + o(f_N(\varepsilon)),$$

where,

- f_k, $1 \le k \le N$ are positive scalar functions satisfying $f_{k+1}(\varepsilon) = o(f_k(\varepsilon))$ and $\lim_{\varepsilon\to 0} f_k(\varepsilon) = 0$.
- $\delta^k j$ denotes the k-th topological derivative of the shape function j.

The topological asymptotic expansion has been derived for various operators and has been applied for many applications; one can see [16] for the Laplace equation, [17, 19] for the Stokes system, [15, 19] for the elasticity problem, [23, 24] for the Helmhotz equation, etc. In all theses works, the optimization algorithms are based on the first-order topological derivative which is only valid for small geometry perturbation size. The use of higher-order terms in the topological asymptotic expansion of the shape function may certainly be decisive in improving the topological optimization algorithms without restrictions on the perturbations sizes. The high-order topological derivative are essential when the first-order topological derivative δj vanishes at some critical points inside Ω.

The present work can be considered as a generalization of the topological gradient notion. The obtained results are valid for a large class of shape functions. The mathematic analysis is general and can be easily adapted to other partial differential equations.

This article is organized as follows. The formulation of the problem is presented in Section 2. In Section 3, we discuss the influence of the geometry perturbation on the Laplace equation solution. We derive an asymptotic expansion for the perturbed solution with respect to ε. The Section 4 is devoted to the high-order topological derivatives. A high-order topological asymptotic expansion is derived for a large class of shape functions. Two particular examples of shape functions are considered in Section 5. Some concluding remarks are presented in Section 6.

2. Formulation of the problem

Let Ω be a bounded domain of $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. We consider the case in which Ω contains a small geometry perturbation $\omega_{z,\varepsilon}$ that is centered at $z \in \Omega$ and has the form $\omega_{z,\varepsilon} = z + \varepsilon\omega$, where $\omega \subset \mathbb{R}^3$ is a given fixed and bounded regular domain containing the origin.

Consider now a shape function

$$j(\Omega \setminus \overline{\omega_{z,\varepsilon}}) = J_\varepsilon(u_\varepsilon),$$

where J_ε is defined on $H^1(\Omega\setminus\overline{\omega_{z,\varepsilon}})$ and u_ε is the solution to Laplace problem in the perturbed domain $\Omega_{z,\varepsilon} = \Omega \setminus \overline{\omega_{z,\varepsilon}}$ with homogeneous Dirichlet condition on $\partial\omega_{z,\varepsilon}$

$$\begin{gathered} -\Delta u_\varepsilon = 0 \quad \text{in } \Omega_{z,\varepsilon}, \\ \nabla u_\varepsilon \cdot n = \Phi_n \quad \text{on } \Gamma_n, \\ u_\varepsilon = \Phi_d \quad \text{on } \Gamma_d, \\ u_\varepsilon = 0 \quad \text{on } \partial\omega_{z,\varepsilon}, \end{gathered} \tag{2.1}$$

where $\Phi_n \in H^{-1/2}(\Gamma_n)$ and $\Phi_d \in H^{1/2}(\Gamma_d)$ are two given data, with Γ_n and Γ_d are two parts of the boundary $\partial\Omega$ satisfying $\overline{\partial\Omega} = \overline{\Gamma_n} \cup \overline{\Gamma_d}$ and $\Gamma_d \cap \Gamma_n = \emptyset$.

Note that for $\varepsilon = 0$, we have $\Omega_0 = \Omega$ and u_0 is the solution to

$$\begin{gathered} -\Delta u_0 = 0 \quad \text{in } \Omega, \\ \nabla u_0 \cdot n = \Phi_n \quad \text{on } \Gamma_n, \\ u_0 = \Phi_d \quad \text{on } \Gamma_d. \end{gathered} \tag{2.2}$$

Using the weak formulation of (2.1), one can deduce that u_ε is the unique solution to the variational problem find $u_\varepsilon \in H^1(\Omega_{z,\varepsilon})$ such that

$$\begin{gathered} a_\varepsilon(u_\varepsilon, w) = l_\varepsilon(w), \quad \forall w \in \mathcal{V}_\varepsilon, \\ u_\varepsilon = \Phi_d \quad \text{on } \Gamma_d \end{gathered} \tag{2.3}$$

where the function space $\mathcal{V}_\varepsilon$, the bilinear form a_ε, and the linear form l_ε are defined by:

$$\begin{gathered} \mathcal{V}_\varepsilon = \big\{u \in H^1(\Omega_{z,\varepsilon});\, u = 0 \quad \text{on } \Gamma_d \cup \partial\omega_{z,\varepsilon}\big\}, \\ a_\varepsilon(v, w) = \int_{\Omega_{z,\varepsilon}} \nabla v \cdot \nabla w \, dx, \quad \forall v, w \in \mathcal{V}_\varepsilon, \\ l_\varepsilon(w) = \int_{\Gamma_n} \Phi_n w ds, \quad \forall w \in \mathcal{V}_\varepsilon. \end{gathered}$$

In the absence of any perturbation (i.e. $\varepsilon = 0$), the weak formulation of problem (2.2) consists in finding $u_0 \in H^1(\Omega)$ such that

$$\begin{gathered} a_0(u_0, w) = l_0(w), \quad \forall w \in \mathcal{V}_0 \\ u_0 = \Phi_d \quad \text{on } \Gamma_d. \end{gathered}$$

As we have mentioned in the introduction, the aim of this work is to derive a high-order topological asymptotic expansion for the shape function j with respect to the presence of the geometry perturbation $\omega_{z,\varepsilon}$ in the domain Ω. It consists in studying the variation $j(\Omega_{z,\varepsilon})-j(\Omega)$ with respect to ε and establishing an asymptotic formula of the form

$$j(\Omega_{z,\varepsilon}) - j(\Omega) = \sum_{k=1}^{N} f_k(\varepsilon)\delta^k j(z) + o(f_N(\varepsilon)).$$

To derive the expected formula, we will proceed in two steps. Firstly, we will give a topological sensitivity analysis for the Laplace operator in Section 3. It consists in studying the asymptotic behavior of the solution u_ε with respect to ε. Secondly, we will study the variation of a shape function j with respect to the presence of a geometry perturbation $\omega_{z,\varepsilon}$ in Ω. The general case, which is valid for a large class of shape functions, will be discussed in Section 4. In Section 5, we will present the asymptotic formulas for two shape functionals examples.

3. Sensitivity analysis for the Laplace operator

In this section, we give a sensitivity analysis for the Laplace operator with respect to the presence of a geometry perturbation $\omega_{z,\varepsilon}$ in the domain Ω. More precisely, we derive an asymptotic expansion for the solution u_ε with respect to ε. Our procedure is based on the successive approximations of the variation $u_\varepsilon - u_0$. We start our analysis by the following estimate.

Lemma 3.1. *Let $\omega_{z,\varepsilon} = z + \varepsilon\omega$ be a topological perturbation inside the domain Ω. If $\omega_{z,\varepsilon} \subset \Omega$ is not close to the boundary $\partial\Omega$, then the variation $u_\varepsilon - u_0$ admits the estimate*

$$u_\varepsilon(x) - u_0(x) = W_0((x-z)/\varepsilon) + O(\varepsilon) \quad \text{in } \Omega_{z,\varepsilon},$$

where the function $x \mapsto W_0((x-z)/\varepsilon)$ is the unique solution to the Laplace exterior problem

$$\begin{gathered} -\Delta W_0 = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\omega}, \\ W_0 \to 0 \quad \text{at } \infty \\ W_0 = -u_0(z) \quad \text{on } \partial\omega. \end{gathered} \tag{3.1}$$

Proof. The existence of the function W_0 is most easily established using a single layer potential [13]

$$W_0(y) = \int_{\partial\omega} G(y-t)\, q_0(t) ds(t), \quad \forall y \in \mathbb{R}^3 \setminus \overline{\omega},$$

where G is the fundamental solution of the Laplace equation in $\mathbb{R}^3$,

$$G(y) = \frac{1}{4\pi\|y\|}.$$

The function $q_0 \in H^{-1/2}(\partial\omega)$ is the solution to the boundary integral equation

$$\int_{\partial\omega} G(y-t)\, q_0(t) ds(t) = -u_0(z), \forall y \in \partial\omega.$$

Posing $R_{0,\varepsilon}(x) = u_\varepsilon(x) - u_0(x) - W_0((x-z)/\varepsilon)$. One can easily remark that $R_{0,\varepsilon}$ is solution to the system

$$-\Delta R_{0,\varepsilon} = 0 \quad \text{in } \Omega_{z,\varepsilon},$$

$$\begin{gathered}\nabla R_{0,\varepsilon} \cdot n = -\nabla W_0((x-z)/\varepsilon) \cdot n \quad \text{on } \Gamma_n, \\ R_{0,\varepsilon} = -W_0((x-z)/\varepsilon) \quad \text{on } \Gamma_d, \\ R_{0,\varepsilon} = -(u_0 - u_0(z)) \quad \text{on } \partial\omega_{z,\varepsilon}.\end{gathered}$$

Since the perturbation $\omega_{z,\varepsilon}$ is not close to the boundary $\partial\Omega$, the function $x \mapsto W_0((x-z)/\varepsilon)$ is regular in the neighborhood of Γ_d and Γ_n. It satisfies the following asymptotic behavior: for all $x \in \Omega_{z,\varepsilon}$,

$$\begin{aligned} W_0((x-z)/\varepsilon) &= \varepsilon \int_{\partial\omega} G(x-z-\varepsilon\, t)\, q_0(t) ds(t) \\ &= \varepsilon\, G(x-z) \int_{\partial\omega} q_0(t) ds(t) + O(\varepsilon). \end{aligned}$$

Similarly, the smoothness of u_0 near z leads to $u(x) - u_0(z) = O(\varepsilon)$ on $\partial\omega_{z,\varepsilon}$. By elliptic variational inequality, one can deduce the estimate

$$R_{0,\varepsilon} = O(\varepsilon) \quad \text{in } \Omega_{z,\varepsilon}.$$

Consequently, the solution u_ε of the Laplace equation in the perturbed domain admits the following asymptotic expansion

$$u_\varepsilon(x) = u_0(x) + W_0((x-z)/\varepsilon) + O(\varepsilon) \quad \text{in } \Omega_{z,\varepsilon}.$$

□

This result was proved in [1, Proposition 3.1] for the Stokes system. It has been used to describe the variation of the velocity field with respect to the presence of a small obstacle.

We are now ready to present the main result of this section. We will derive a high-order asymptotic expansion of u_ε with respect to ε. The obtained result is described by the following theorem.

Theorem 3.2. *Let $\omega_{z,\varepsilon} = z + \varepsilon\omega$ be a topological perturbation inside the domain Ω. If $\omega_{z,\varepsilon} \subset \Omega$ is not close to the boundary $\partial\Omega$, then the Laplace equation solution u_ε in the perturbed domain $\Omega_{z,\varepsilon}$ admits the following asymptotic expansion*

$$u_\varepsilon(x) = \sum_{k=0}^{N} \varepsilon^k [U_k(x) + W_k((x-z)/\varepsilon))] + O(\varepsilon^{N+1}) \qquad \text{in } \Omega_{z,\varepsilon},$$

where

- *U_k, $0 \le k \le N$ are smooth functions defined in Ω, obtained as the solutions to a sequence of interior Laplace problems.*
- *W_k, $0 \le k \le N$ are smooth functions defined in $\mathbb{R}^3 \setminus \overline{\omega}$, obtained as the solutions to a sequence of exterior Laplace problems.*

Proof. The sequences of functions $(U_k)_{0\le k\le N}$ and $(W_k)_{0\le k\le N}$ are constructed using an iterative process with $U_0 = u_0$ and W_0 is the solution to (3.1). As we will show later, for all $1 \le k \le N$:

• The term U_k will be defined as the solution of the Laplace equation in Ω with boundaries conditions depending on the function $x \mapsto W_l((x-z)/\varepsilon)$, $0 \le l \le k-1$.

• The term W_k, will be defined as the solution of the Laplace equation in $\mathbb{R}^3 \setminus \overline{\omega}$ with a boundary condition depending on the functions U_l, $0 \le l \le k$.

Using a single layer potential [13], the functions W_k, $0 \le k \le N$ can be written on the following general form

$$W_k(y) = \int_{\partial\omega} G(y-t)\, q_k(t) ds(t), \quad \forall y \in \mathbb{R}^3 \setminus \overline{\omega},$$

where q_k is the solution to a boundary integral equation defined on $\partial\omega$.

To present our construction process, we start our analysis by studying the variation of the function $x \mapsto W_k((x-z)/\varepsilon)$ with respect to ε. For each $x \in \mathbb{R}^3 \setminus \overline{\omega_{z,\varepsilon}}$ we have

$$\begin{aligned} W_k((x-z)/\varepsilon) &= \int_{\partial\omega} G((x-z)/\varepsilon - t)\, q_k(t) ds(t) \\ &= \varepsilon \int_{\partial\omega} G((x-z) - \varepsilon\, t)\, q_k(t) ds(t). \end{aligned}$$

Using the fact that the perturbation $\omega_{z,\varepsilon}$ is not close to the boundary $\partial\Omega$, one can remark that for all $t \in \partial\omega$ and for all $x \in \Omega_{z,\varepsilon}$, the function $\varphi_{x-z,t} : \varepsilon \mapsto \varphi_{x-z,t}(\varepsilon) = \varepsilon\, G((x-z) - \varepsilon\, t)$ is smooth with respect to ε and satisfies

$$\varphi_{x-z,t}(\varepsilon) = \sum_{p=1}^{N} \frac{\varepsilon^p}{p!} \varphi_{x-z,t}^{(p)}(0) + O(\varepsilon^{N+1}),$$

where $\varphi_{x-z,t}^{(p)}(0)$ is the p-th derivative of $\varphi_{x-z,t}$ at $\varepsilon = 0$. It depends on the p-th derivative of the function G at the point $x - z$.

Consequently, the function $x \mapsto W_k((x-z)/\varepsilon)$ admits the asymptotic expansion

$$W_k((x-z)/\varepsilon) = \sum_{p=1}^{N} \varepsilon^p\, W_k^{(p)}(x-z) + O(\varepsilon^{N+1}), \tag{3.2}$$

where $W_k^{(p)}$ is the smooth function defined in $\mathbb{R}^3 \setminus \{z\}$ by

$$W_k^{(p)}(x-z) = \frac{1}{p!} \int_{\partial\omega} \varphi_{x-z,t}^{(p)}(0) q_k(t) ds(t), \quad \forall x \in \mathbb{R}^3 \setminus \{z\}. \tag{3.3}$$

We are now ready to present the main steps of our construction procedure.

First order term: It is described by the function $x \mapsto U_1(x) + W_1((x-z)/\varepsilon)$, $x \in \Omega_{z,\varepsilon}$ which is constructed as follows:

- The term U_1 depends on W_0 and solves the interior problem

$$\begin{aligned} -\Delta U_1 &= 0 \quad \text{in } \Omega, \\ \nabla U_1 \cdot n &= -\nabla W_0^{(1)}(x-z) \cdot n \quad \text{on } \Gamma_n, \\ U_1 &= -W_0^{(1)}(x-z) \quad \text{on } \Gamma_d, \end{aligned} \tag{3.4}$$

  with $W_0^{(1)}$ is defined by (3.3) in the particular case $k = 0$ and $p = 1$. One can easily check that

$$W_0^{(1)}(x-z) = G(x-z) \int_{\partial\omega} q_0(t) ds(t),$$

  where q_0 is the density associated to W_0.

- The term W_1 depends on U_0 and U_1, and solves the following exterior problem
$$\begin{gathered} -\Delta W_1 = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\omega}, \\ W_1 \to 0 \quad \text{at } \infty \\ W_1 = -U_1(z) - DU_0(z)(y) \quad \text{on } \partial\omega. \end{gathered} \tag{3.5}$$

Higher-order terms: Let us assume that we have already calculated the first $k-1$ terms. The k-th order term is described by the function $x \mapsto U_k(x)+W_k((x-z)/\varepsilon)$, $x \in \Omega_{z,\varepsilon}$ which is defined as follows:

- The term U_k depends on W_j, $0 \le j \le k-1$ and solves the interior problem
$$\begin{gathered} -\Delta U_k = 0 \quad \text{in } \Omega, \\ \nabla U_k \cdot n = -\sum_{p=1}^{k} \nabla W_{k-p}^{(p)}(x-z) \cdot n \quad \text{on } \Gamma_n, \\ U_k = -\sum_{p=1}^{k} W_{k-p}^{(p)}(x-z) \quad \text{on } \Gamma_d, \end{gathered} \tag{3.6}$$
with $W_j^{(p)}$ is defined by (3.3).
- The term W_k depends on U_j, $0 \le j \le k$ and solves the exterior problem
$$\begin{gathered} -\Delta W_k = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\omega}, \\ W_k \to 0 \quad \text{at } \infty \\ W_k = -U_k(z) - \sum_{p=1}^{k} \frac{1}{p!} D^p U_{k-p}(z)(y^p) \quad \text{on } \partial\omega, \end{gathered} \tag{3.7}$$
where $D^p U_{k-p}(z)$ is the p-th derivative of the harmonic function U_{k-p} at the point $z \in \Omega$ and $y^p = (y, \dots, y) \in (\mathbb{R}^3)^p$.

To prove the desired estimate, we introduce the function $R_{N,\varepsilon}$ defined in $\Omega_{z,\varepsilon}$ by
$$\begin{aligned} R_{N,\varepsilon}(x) &= U_0(x) + W_0((x-z)/\varepsilon) + \varepsilon\,(U_1(x) + W_1((x-z)/\varepsilon)) + \dots \\ &\quad + \varepsilon^N (U_N(x) + W_N((x-z)/\varepsilon) - u_\varepsilon(x). \end{aligned}$$
It is easy to see that $R_{N,\varepsilon}$ is harmonic in $\Omega_{z,\varepsilon}$ and satisfies the following boundary conditions:

On $\partial\omega_{z,\varepsilon}$:
$$\begin{aligned} R_{N,\varepsilon}(x) &= U_0(x) + W_0((x-z)/\varepsilon) + \sum_{k=1}^{N} \varepsilon^k [U_k(x) + W_k((x-z)/\varepsilon)] \\ &= \sum_{k=0}^{N} \varepsilon^k U_k(x) - \sum_{k=0}^{N} \varepsilon^k \Big[\sum_{p=0}^{k} \frac{1}{p!} D^p U_{k-p}(z)(((x-z)/\varepsilon)^p)\Big]. \end{aligned} \tag{3.8}$$
Using the multi-linearity of $D^p U_{k-p}(z)$, it follows
$$\begin{aligned} \sum_{k=1}^{N} \varepsilon^k \Big[\sum_{p=0}^{k} \frac{1}{p!} D^p U_{k-p}(z)(((x-z)/\varepsilon)^p)\Big] &= \sum_{k=0}^{N} \sum_{p=0}^{k} \frac{\varepsilon^{k-p}}{p!} D^p U_{k-p}(z)((x-z)^p) \\ &= \sum_{k=0}^{N} \varepsilon^k \sum_{p=0}^{N-k} \frac{1}{p!} D^p U_k(z)((x-z)^p). \end{aligned}$$

Then, one can deduce

$$R_{N,\varepsilon}(x) = \sum_{k=0}^{N} \varepsilon^k \Big[U_k(x) - \sum_{p=0}^{N-k} \frac{1}{p!} D^p U_k(z)((x-z)^p)\Big].$$

Due to Taylor's Theorem and the fact that $\|x - z\| = O(\varepsilon)$ on $\partial\omega_{z,\varepsilon}$, we obtain

$$R_{N,\varepsilon}(x) = O(\varepsilon^{N+1}) \quad \text{on } \partial\omega_{z,\varepsilon}.$$

On Γ_d:

$$\begin{aligned} R_{N,\varepsilon}(x) &= \sum_{k=0}^{N} \varepsilon^k W_k((x-z)/\varepsilon) - \sum_{k=1}^{N} \varepsilon^k \big[\sum_{p=1}^{k} W_{k-p}^{(p)}(x-z)\big] \\ &= \sum_{k=0}^{N} \varepsilon^k W_k((x-z)/\varepsilon) - \sum_{k=0}^{N-1} \varepsilon^k \big[\sum_{p=1}^{N-k} \varepsilon^p W_k^{(p)}(x-z)\big]. \end{aligned}$$

The last equality can be rewritten as

$$R_{N,\varepsilon}(x) = \varepsilon^N W_N((x-z)/\varepsilon) + \sum_{k=0}^{N-1} \varepsilon^k \big[W_k((x-z)/\varepsilon) - \sum_{p=1}^{N-k} \varepsilon^p W_k^{(p)}(x-z)\big].$$

Then, using the asymptotic expansion (3.2) we obtain

$$R_{N,\varepsilon}(x) = O(\varepsilon^{N+1}) \quad \text{on } \Gamma_d.$$

On Γ_n: using the same analysis, one can derive

$$\nabla R_{N,\varepsilon} \cdot n = O(\varepsilon^{N+1}) \quad \text{on } \Gamma_n.$$

□

4. High-order topological asymptotic expansion

This section is focused on high-order topological derivatives. It consists in studying the variation of a shape function j with respect to the topology perturbation of the domain. The topology perturbation is described by the hole $\omega_{z,\varepsilon}$ created at an arbitrary point $z \in \Omega$ and having the form $\omega_{z,\varepsilon} = z + \varepsilon\omega$. We derive a high-order topological asymptotic expansion for a large class of shape functions. More precisely, the obtained results are valid for all shape function j having the form

$$j(\Omega_{z,\varepsilon}) = J_\varepsilon(u_\varepsilon),$$

with J_ε is a scalar function defined on $H^1(\Omega_{z,\varepsilon})$ and satisfying the assumptions:

(A1) The function J_0 is differentiable with respect to u.

(A2) There exist real numbers $\delta^1 J(z), \dots, \delta^N J(z)$, such that for all $\varepsilon > 0$,

$$J(u_\varepsilon) - J_0(u_0) = DJ_0(u_0)(u_\varepsilon - u_0) + \sum_{k=1}^{N} \varepsilon^k \delta^k J(z) + o(\varepsilon^N).$$

In the last equality, the solution u_ε is extended by zero inside the domain $\omega_{z,\varepsilon}$. Its extension will be denoted by u_ε throughout the rest of the paper.

Under the considered assumptions, the variation of the shape function j reads

$$j(\Omega_{z,\varepsilon}) - j(\Omega) = a_0(u_0 - u_\varepsilon, v_0) + \sum_{k=1}^{N} \varepsilon^k \delta^k J(z) + o(\varepsilon^N),$$

where $v_0 \in \mathcal{V}_0$ is the solution to the adjoint problem

$$a_0(w, v_0) = -DJ_0(u_0)(w), \quad \forall w \in \mathcal{V}_0. \tag{4.1}$$

Next, we will derive an asymptotic expansion of the term $a_0(u_0 - u_\varepsilon, v_0)$ which can be written as

$$\begin{aligned} a_0(u_0 - u_\varepsilon, v_0) &= \int_\Omega (\nabla u_0 - \nabla u_\varepsilon) \cdot \nabla v_0 dx \\ &= \int_{\omega_{z,\varepsilon}} \nabla u_0 \cdot \nabla v_0 dx + \int_{\Omega_{z,\varepsilon}} (\nabla u_0 - \nabla u_\varepsilon) \cdot \nabla v_0 dx. \end{aligned}$$

Using Green formula, it follows that

$$a_0(u_0 - u_\varepsilon, v_0) = \int_{\omega_{z,\varepsilon}} \nabla u_0 \cdot \nabla v_0 dx + \int_{\partial\omega_{z,\varepsilon}} \nabla(u_0 - u_\varepsilon) \cdot n v_0 ds. \tag{4.2}$$

By Theorem 3.2, we have

$$\begin{aligned} \int_{\partial\omega_{z,\varepsilon}} \nabla(u_0 - u_\varepsilon) \cdot n v_0 ds = &- \sum_{k=1}^{N} \varepsilon^k \int_{\partial\omega_{z,\varepsilon}} \nabla U_k(x) \cdot n(x)\, v_0(x) ds \\ &- \sum_{k=0}^{N} \varepsilon^k \int_{\partial\omega_{z,\varepsilon}} \nabla_x W_k((x-z)/\varepsilon)) \cdot n\, v_0 ds + O(\varepsilon^{N+1}). \end{aligned}$$

Consequently, the term $a_0(u_0 - u_\varepsilon, v_0)$ can be decomposed as

$$\begin{aligned} &a_0(u_0 - u_\varepsilon, v_0) \\ &= \int_{\omega_{z,\varepsilon}} \nabla u_0 \cdot \nabla v_0 dx - \sum_{k=0}^{N} \varepsilon^k \int_{\partial\omega_{z,\varepsilon}} \nabla_x W_k((x-z)/\varepsilon)) \cdot n\, v_0 ds \\ &\quad - \sum_{k=1}^{N} \varepsilon^k \int_{\partial\omega_{z,\varepsilon}} \nabla U_k(x) \cdot n(x)\, v_0(x) ds + O(\varepsilon^{N+1}). \end{aligned} \tag{4.3}$$

In the next section, we will derive an estimate for each term on the right-hand-side of the equality (4.3).

4.1. **Preliminary estimates.** The following lemma gives an estimate for the first term.

Lemma 4.1. *The first term on the right-hand-side of the equality* (4.3) *admits the asymptotic expansion*

$$\int_{\omega_{z,\varepsilon}} \nabla u_0 \cdot \nabla v_0 dx = \sum_{k=3}^{N} \varepsilon^k\, \mathcal{T}^{1,k-3}_{u_0,v_0}(z) + O(\varepsilon^{N+1}),$$

where the functions $z \mapsto \mathcal{T}^{1,k}_{u_0,v_0}(z)$, $0 \le k \le N$ are defined in Ω by

$$\mathcal{T}^{1,k}_{u_0,v_0}(z) = \sum_{p=0}^{k} \frac{1}{p!(k-p)!} \int_\omega \nabla^{(p+1)} u_0(z)(y^p) \cdot \nabla^{(k-p+1)} v_0(z)(y^{k-p}) dy,$$

with $y^k = (y, \dots, y) \in (\mathbb{R}^3)^k$ and $\nabla^{(k)} w(z)$ denotes the k-th derivative of the function w at the point z.

Proof. The proof of this lemma is based on the well known Taylor-Young formula. Since u_0 and v_0 are sufficiently regular in $\omega_{z,\varepsilon}$, we have

$$\nabla u_0(z+\varepsilon y) = \nabla u_0(z) + \sum_{k=1}^{N-1} \frac{\varepsilon^k}{k!} \nabla^{(k+1)} u_0(z)(y^k) + O(\varepsilon^N)$$
$$\nabla v_0(z+\varepsilon y) = \nabla v_0(z) + \sum_{k=1}^{N-1} \frac{\varepsilon^k}{k!} \nabla^{(k+1)} v_0(z)(y^k) + O(\varepsilon^N).$$

Using the change of variable $x = z + \varepsilon y$, we derive

$$\begin{aligned} &\int_{\omega_{z,\varepsilon}} \nabla u_0 \cdot \nabla v_0 dx \\ &= \varepsilon^3 \int_\omega \nabla u_0(z+\varepsilon y) \cdot \nabla v_0(z+\varepsilon y) dy \\ &= \varepsilon^3 \int_\omega \Big[\sum_{k=0}^{N-1} \frac{\varepsilon^k}{k!} \nabla^{(k+1)} u_0(z)(y^k) \Big] \Big[\sum_{k=0}^{N-1} \frac{\varepsilon^k}{k!} \nabla^{(k+1)} v_0(z)(y^k) \Big] dy + O(\varepsilon^{N+1}). \end{aligned}$$

Using the Cauchy product formula, we obtain the desired result

$$\begin{aligned} &\int_{\omega_{z,\varepsilon}} \nabla u_0 \cdot \nabla v_0 dx \\ &= \sum_{k=0}^{N-3} \varepsilon^{k+3} \Big(\sum_{p=0}^{k} \frac{1}{p!(k-p)!} \int_\omega \nabla^{(p+1)} u_0(z)(y^p) \cdot \nabla^{(k-p+1)} v_0(z)(y^{k-p}) dy \Big) \\ &\quad + O(\varepsilon^{N+1}). \end{aligned}$$

□

Lemma 4.2. *The second term on the right-hand-side of the equality* (4.3) *admits the asymptotic expansion*

$$\sum_{k=0}^{N} \varepsilon^k \int_{\partial\omega_{z,\varepsilon}} \nabla_x W_k((x-z)/\varepsilon)) \cdot n\, v_0 ds = -\sum_{k=1}^{N} \varepsilon^k \mathcal{T}^{2,k-1}_{W,v_0}(z) + O(\varepsilon^{N+1}),$$

where the functions $z \mapsto \mathcal{T}^{2,k}_{W,v_0}(z)$, $0 \le k \le N$ are defined in Ω by

$$\mathcal{T}^{2,k}_{W,v_0}(z) = -\sum_{p=0}^{k} \frac{1}{p!} \int_{\partial\omega} \nabla_y W_{k-p}(y) \cdot n(y) [\nabla^{(p)} v_0(z)(y^p)] ds(y).$$

Proof. Using the change of variable $x = z + \varepsilon y$, we obtain

$$\int_{\partial\omega_{z,\varepsilon}} \nabla_x W_k((x-z)/\varepsilon)) \cdot n(x) v_0(x) ds = \varepsilon \int_{\partial\omega} \nabla_y W_k(y) \cdot n(y)\, v_0(z+\varepsilon y) ds(y). \quad (4.4)$$

Using the fact that v_0 is smooth in a neighborhood of z, one can derive

$$\begin{aligned} v_0(z+\varepsilon y) &= v_0(z) + \sum_{p=1}^{N-1} \frac{\varepsilon^p}{p!} \nabla^{(p)} v_0(z)(y^p) + O(\varepsilon^N) \\ &= \sum_{p=0}^{N-1} \frac{\varepsilon^p}{p!} \nabla^{(p)} v_0(z)(y^p) + O(\varepsilon^N). \end{aligned}$$

It leads to the asymptotic expansion of the term (4.4),

$$\begin{aligned}&\int_{\partial\omega_{z,\varepsilon}} \nabla_x W_k((x-z)/\varepsilon))\cdot n(x)v_0(x)ds\\&=\sum_{p=0}^{N-1}\frac{\varepsilon^{p+1}}{p!}\int_{\partial\omega}\nabla_y W_k(y)\cdot n(y)[\nabla^{(p)}v_0(z)(y^p)]ds(y)+O(\varepsilon^{N+1}).\end{aligned}$$

Consequently,

$$\begin{aligned}&\sum_{k=0}^{N}\varepsilon^k\int_{\partial\omega_{z,\varepsilon}}\nabla_x W_k((x-z)/\varepsilon))\cdot n\,v_0\,ds\\&=\sum_{k=0}^{N}\varepsilon^k\sum_{p=0}^{N-1}\frac{\varepsilon^{p+1}}{p!}\int_{\partial\omega}\nabla_y W_k(y)\cdot n(y)[\nabla^{(p)}v_0(z)(y^p)]ds(y)+O(\varepsilon^{N+1})\\&=\sum_{k=1}^{N}\varepsilon^k\sum_{p=0}^{k-1}\frac{1}{p!}\int_{\partial\omega}\nabla_y W_{k-p-1}(y)\cdot n(y)[\nabla^{(p)}v_0(z)(y^p)]ds(y)+O(\varepsilon^{N+1}).\end{aligned}$$

□

Lemma 4.3. *The third term on the right-hand-side of the equality* (4.3) *admits the following expansion*

$$\sum_{k=1}^{N}\varepsilon^k\int_{\partial\omega_{z,\varepsilon}}\nabla U_k(x)\cdot n(x)v_0(x)ds=-\sum_{k=3}^{N}\varepsilon^k\mathcal{T}^{3,k-3}_{U,v_0}(z)+O(\varepsilon^{N+1}).$$

where the functions $z\mapsto\mathcal{T}^{3,k}_{U,v_0}(z)$, $0\le k\le N$ *are defined in* Ω *by*

$$\begin{aligned}&\mathcal{T}^{3,k}_{U,v_0}(z)\\&=-\sum_{p=0}^{k}\sum_{q=0}^{p}\frac{1}{q!(p-q)!}\int_{\partial\omega}[\nabla^{(q+1)}U_{k-p+1}(z)(y^q)]\cdot n(y)[\nabla^{(p-q)}v_0(z)(y^{p-q})]ds(y).\end{aligned}$$

Proof. Using the change of variable $x=z+\varepsilon y$, we obtain

$$\int_{\partial\omega_{z,\varepsilon}}\nabla U_k(x)\cdot n(x)v_0(x)ds=\varepsilon^2\int_{\partial\omega}\nabla U_k(z+\varepsilon y)\cdot n(z+\varepsilon y)v_0(z+\varepsilon y)ds(y).\quad(4.5)$$

From the fact that v_0 is smooth in a neighborhood of z, one can derive

$$\begin{aligned}v_0(z+\varepsilon y)&=v_0(z)+\sum_{p=1}^{N-1}\frac{\varepsilon^p}{p!}\nabla^{(p)}v_0(z)(y^p)+O(\varepsilon^N)\\&=\sum_{p=0}^{N-1}\frac{\varepsilon^p}{p!}\nabla^{(p)}v_0(z)(y^p)+O(\varepsilon^N).\end{aligned}$$

Similarly, U_k is smooth in a neighborhood of z, it can be estimated as

$$\nabla U_k(z+\varepsilon y)=\sum_{q=0}^{N-1}\frac{\varepsilon^q}{q!}\nabla^{(q+1)}U_k(z)(y^q)+O(\varepsilon^N).$$

Then, it follows that

$$\int_{\partial\omega_{z,\varepsilon}}\nabla U_k(x)\cdot n(x)v_0(x)ds$$

$$= \varepsilon^2 \int_{\partial\omega} [\sum_{q=0}^{N-1} \frac{\varepsilon^q}{q!} \nabla^{(q+1)} U_k(z)(y^q)] \cdot n(y) \big[\sum_{p=0}^{N-1} \frac{\varepsilon^p}{p!} \nabla^{(p)} v_0(z)(y^p) \big] ds(y) + O(\varepsilon^{N+1}).$$

Using the Cauchy product formula, one can check the following asymptotic expansion of the term (4.5),

$$\begin{aligned} &\int_{\partial\omega_{z,\varepsilon}} \nabla U_k(x) \cdot n(x) v_0(x) ds \\ &= \sum_{p=0}^{N-2} \varepsilon^{p+2} \sum_{q=0}^{p} \frac{1}{q!(p-q)!} \\ &\quad \times \int_{\partial\omega} [\nabla^{(q+1)} U_k(z)(y^q)] \cdot n(y) [\nabla^{(p-q)} v_0(z)(y^{p-q})] ds(y) + O(\varepsilon^{N+1}). \end{aligned}$$

Consequently,

$$\begin{aligned} &\sum_{k=1}^{N} \varepsilon^k \int_{\partial\omega_{z,\varepsilon}} \nabla U_k(x) \cdot n(x)\, v_0(x) ds \\ &= \sum_{k=1}^{N} \sum_{p=0}^{N-2} \varepsilon^{k+p+2} \sum_{q=0}^{p} \frac{1}{q!(p-q)!} \\ &\quad \times \int_{\partial\omega} [\nabla^{(q+1)} U_k(z)(y^q)] \cdot n(y) [\nabla^{(p-q)} v_0(z)(y^{p-q})] ds(y) + O(\varepsilon^{N+1}) \\ &= \sum_{k=3}^{N} \varepsilon^k \sum_{p=0}^{k-3} \sum_{q=0}^{p} \frac{1}{q!(p-q)!} \\ &\quad \times \int_{\partial\omega} [\nabla^{(q+1)} U_{k-p-2}(z)(y^q)] \cdot n(y) [\nabla^{p-q} v_0(z)(y^{(p-q)})] ds(y) + O(\varepsilon^{N+1}). \end{aligned}$$

□

4.2. **Asymptotic expansion.** We are now ready to present the main results of this section. Based on the previous estimates, we derive a high-order topological asymptotic expansion for all shape function satisfying the assumptions (A1) and (A2).

Theorem 4.4. *Let $\omega_{z,\varepsilon} = z + \varepsilon\omega$ be a small topological perturbation in Ω and j a shape function of the form $j(\Omega_{z,\varepsilon}) = J_\varepsilon(u_\varepsilon)$. If J_ε satisfies the assumptions* (A1) and (A2)*, then j admits the asymptotic expansion*

$$j(\Omega_{z,\varepsilon}) - j(\Omega) = \sum_{k=1}^{N} \varepsilon^k \delta^k j(z) + o(\varepsilon^N),$$

where $\delta^k j$ is the k-th topological derivative defined in Ω by

$$\delta^k j(z) = \begin{cases} \mathcal{T}^{2,k-1}_{W,v_0}(z) + \delta^k J(z) & \text{if } k = 1, 2 \\ \mathcal{T}^{1,k-3}_{u_0,v_0}(z) + \mathcal{T}^{2,k-1}_{W,v_0}(z) + \mathcal{T}^{3,k-3}_{U,v_0}(z) + \delta^k J(z) & \text{if } 3 \le k \le N. \end{cases}$$

Proof. Using the fact that j satisfies assumptions (A1) and (A2), we have

$$J_\varepsilon(u_\varepsilon) - J_0(u_0) = DJ_0(u_0)(u_\varepsilon - u_0) + \sum_{k=1}^{N} \varepsilon^k \delta^k J(z) + o(\varepsilon^N).$$

Using (4.1), we derive

$$DJ_0(u_0)(u_\varepsilon - u_0) = a_0(u_0 - u_\varepsilon, v_0),$$

Using the decomposition (4.3) and according to Lemmas 4.1, 4.2 and 4.3, we derive

$$\begin{aligned} DJ_0(u_0)(u_\varepsilon - u_0) &= \sum_{k=3}^{N} \varepsilon^k \mathcal{T}^{1,k-3}_{u_0,v_0}(z) + \sum_{k=1}^{N} \varepsilon^k \mathcal{T}^{2,k-1}_{W,v_0}(z) \\ &\quad + \sum_{k=3}^{N} \varepsilon^k \mathcal{T}^{3,k-3}_{U,v_0}(z) + O(\varepsilon^{N+1}). \end{aligned}$$

By combining the above equalities we obtain the desired result. $\square$

5. Shape function examples

We now discuss the assumptions (A1) and (A2). We present two examples of shape functions satisfying the considered assumptions and we calculate their variations $\delta^1 J$, $\delta^2 J$, ..., and $\delta^N J$.

5.1. **First example.** We consider the linear function

$$J_\varepsilon(u) = \int_{\Omega_{z,\varepsilon}} g\, u dx, \quad \forall u \in H^1(\Omega_{z,\varepsilon}), \tag{5.1}$$

with $g \in H^1(\Omega)$ is a given function.

Proposition 5.1. *The function J_ε satisfies the assumptions* (A1) *and* (A2) *with*

$$DJ_0(w) = \int_\Omega gw dx, \quad \forall w \in \mathcal{V}_0, \text{ and for any } 1 \le k \le N, \quad \delta^k J(z) = 0 \quad \text{in } \Omega.$$

Then the associated shape function

$$j(\Omega_{z,\varepsilon}) = \int_{\Omega_{z,\varepsilon}} g\, u_\varepsilon dx$$

admits the high-order asymptotic expansion

$$j(\Omega_{z,\varepsilon}) - j(\Omega) = \sum_{k=1}^{N} \varepsilon^k \delta^k j(z) + o(\varepsilon^N),$$

where $\delta^k j$ is the k-th topological derivative of j defined in Ω by

$$\delta^k j(z) = \begin{cases} \mathcal{T}^{2,k-1}_{W,v_0}(z) & \text{if } k = 1,2 \\ \mathcal{T}^{1,k-3}_{u_0,v_0}(z) + \mathcal{T}^{2,k-1}_{W,v_0}(z) + \mathcal{T}^{3,k-3}_{U,v_0}(z) & \text{if } 3 \le k \le N. \end{cases} \tag{5.2}$$

Proof. The function J_0 is differentiable and we have

$$DJ_0(w) = \int_\Omega gw\, dx, \quad \forall w \in \mathcal{V}_0.$$

The variation of j is given by

$$j(\Omega_{z,\varepsilon}) - j(\Omega) = \int_{\Omega_{z,\varepsilon}} gu_\varepsilon\, dx - \int_\Omega gu_0\, dx = DJ_0(u_0)(u_\varepsilon - u_0).$$

Hence the function J_ε satisfies the assumptions (A1) and (A2) with

$$DJ_0(w) = \int_\Omega gw\, dx \quad \forall w \in \mathcal{V}_0,$$

$$\delta^k J(z) = 0 \quad \text{for each } 1 \le k \le N \text{ and all } z \in \Omega.$$

The asymptotic expansion of j follows immediately from Theorem 4.4. □

5.2. **Second example.** We consider the semi-norm function associated to the H^1 Sobolev space

$$J_\varepsilon(u) = \int_{\Omega_{z,\varepsilon}} |\nabla u - \nabla U_d|^2 dx, \quad \forall u \in H^1(\Omega_{z,\varepsilon}) \tag{5.3}$$

with $U_d \in H^1(\Omega)$ is a given desired (objective) state, smooth in a neighborhood of z.

Proposition 5.2. *The function J_ε satisfies the assumptions* (A1) *and* (A2) *with*

$$DJ_0(w) = 2\int_\Omega \nabla(u_0 - U_d) \cdot \nabla w dx, \quad \forall w \in \mathcal{V}_0,$$

where

$$\delta^k J(z) = \begin{cases} \mathcal{T}^{2,k-1}_{W,u_0}(z) & \text{if } k = 1, 2 \\ \mathcal{T}^{2,k-1}_{W,u_0}(z) + \mathcal{T}^{1,k-3}_{u_0,u_0}(z) + \mathcal{T}^{1,k-3}_{U_d,U_d}(z) + \mathcal{T}^{3,k-3}_{U,u_0}(z) & \text{if } 3 \le k \le N. \end{cases}$$

Proof. The function J_0 is differentiable and we have

$$DJ_0(u_0)(w) = 2\int_\Omega [\nabla u_0 - \nabla U_d] \cdot \nabla w dx,$$

and

$$\begin{aligned} j(\Omega_{z,\varepsilon}) - j(\Omega) &= \int_{\Omega_{z,\varepsilon}} |\nabla u_\varepsilon - \nabla U_d|^2 dx - \int_\Omega |\nabla u_0 - \nabla U_d|^2 dx \\ &= DJ_0(u_0)(u_\varepsilon - u_0) + \int_{\omega_{z,\varepsilon}} |\nabla u_0|^2 dx \\ &\quad - \int_{\omega_{z,\varepsilon}} |\nabla U_d|^2 dx + \int_{\Omega_{z,\varepsilon}} |\nabla u_0 - \nabla u_\varepsilon|^2 dx. \end{aligned}$$

Thanks to the regularity of u_0 and U_d in $\omega_{z,\varepsilon}$, one obtains

$$\begin{aligned} \int_{\omega_{z,\varepsilon}} |\nabla u_0|^2 dx &= \sum_{k=3}^{N} \varepsilon^k \mathcal{T}^{1,k-3}_{u_0,u_0}(z) + O(\varepsilon^{N+1}), \\ \int_{\omega_{z,\varepsilon}} |\nabla U_d|^2 dx &= \sum_{k=3}^{N} \varepsilon^k \mathcal{T}^{1,k-3}_{U_d,U_d}(z) + O(\varepsilon^{N+1}). \end{aligned}$$

By the Green formula, it follows that

$$\int_{\Omega_{z,\varepsilon}} |\nabla u_0 - \nabla u_\varepsilon|^2 dx = -\int_{\partial\omega_{z,\varepsilon}} \nabla(u_0 - u_\varepsilon) \cdot n u_0 ds.$$

Applying the technique developed in Section 4, one can derive

$$\int_{\Omega_{z,\varepsilon}} |\nabla u_0 - \nabla u_\varepsilon|^2 dx = \sum_{k=1}^{N} \varepsilon^k \mathcal{T}^{2,k-1}_{W,u_0}(z) + \sum_{k=3}^{N} \varepsilon^k \mathcal{T}^{3,k-3}_{U,u_0}(z) + O(\varepsilon^{N+1}).$$

By combining the above equalities we obtain the desired result. □

Concluding remarks. Two main results are presented in this paper.

The first result is devoted to a high-order asymptotic expansion for the Laplace equation solution with respect to the presence of a Dirichlet geometry perturbation. This question has been investigated by Ammari and Kang [3] in the inhomogeneities case. Here, we extend this result for a more singular case described by a Dirichlet perturbation.

The second result deals with the high-order topological derivatives. A high-order topological asymptotic expansion is derived for a large class of shape functions. The use of higher-order terms in the topological asymptotic expansion of the shape function may certainly be decisive in improving the topological optimization algorithms without restrictions on the perturbations sizes. The high-order topological derivative are essential when the first-order topological derivative δj vanishes at some critical points inside Ω.

The present work can be considered as a generalization of the topological gradient notion. The mathematic analysis is general and can be easily adapted to other partial differential equations.

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