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INVERSE SPECTRAL PROBLEMS FOR ENERGY-DEPENDENT STURM-LIOUVILLE EQUATIONS WITH FINITELY MANY POINT δ-INTERACTIONS

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In memory of M. G. Gasymov (1939-2008)

Abstract. In this study, inverse spectral problems for a energy-dependent Sturm-Liouville equations with finitely many point δ-interactions. The uniqueness theorems for the inverse problems of reconstruction of the boundary value problem from the Weyl function, from the spectral data and from two spectra are proved and a constructive procedure for finding its solution are obtained.

1. Introduction

We consider inverse problems for the boundary value problem L (BVP L) generated by the differential equation

$$-y'' + q(x)y = \lambda^2 y, \quad x \in \cup_{s=0}^{m}(a_s, a_{s+1}) \quad (0 = a_0,\ a_{m+1} = \pi) \tag{1.1}$$

with the Robin boundary conditions

$$\begin{aligned} U(y) &:= y'(0) - hy(0) = 0, \\ V(y) &:= y'(\pi) + Hy(\pi) = 0, \end{aligned} \tag{1.2}$$

and the conditions at the points $x = a_s$:

$$I(y) := \{ \begin{aligned} y(a_s + 0) = y(a_s - 0) = y(a_s) \\ y'(a_s + 0) - y'(a_s - 0) = 2\alpha_s \lambda y(a_s), \end{aligned} \tag{1.3}$$

$q(x)$ is a complex-valued function in $L_2(0,\pi)$; h, H and α_s are complex numbers, $s = \overline{1,m} := 1, 2, \dots m$; and λ is a spectral parameter.

Note that, we can understand problem (1.1) and (1.3) as studying the equation

$$y'' + (\lambda^2 - 2\lambda p(x) - q(x))y = 0, \quad x \in (0,\pi), \tag{1.4}$$

when $p(x) = \sum_{s=1}^{m} \alpha_s \delta(x - a_s)$, where $\delta(x)$ is the Dirac function (see [1]).

Sturm-Liouville spectral problems with potentials depending on the spectral parameter arise in various models of quantum and classical mechanics. For instance,

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the evolution equations that are used to model interactions between colliding relativistic spineless particles can be reduced to the form (1.1). Then λ^2 is associated with the energy of the system (see [14, 16]).

Spectral problems of differential operators are studied in two main branches, namely, direct and inverse problems. Direct problems of spectral analysis consist in investigating the spectral properties of an operator. On the other hand, inverse problems aim at recovering operators from their spectral characteristics. One takes for the main spectral data, for instance, one, two, or more spectra, the spectral function, the spectrum, and the normalized constants, the Weyl function. Direct and inverse problems for the classical Sturm-Liouville operators have been extensively studied (see [8, 15, 17] and the references therein). Some classes of direct and inverse problems for discontinuous $BVPs$ (special case $p(x) \equiv 0$) in various statements have been considered in [2, 11, 13, 20, 21] and for further discussion see the references therein. Notice that, inverse spectral problems for non-selfadjoint Sturm-Liouville operators on a finite interval with possibly multiple spectra have been investigated in [4, 5].

Non-linear dependence of (1.4) on the spectral parameter λ should be regarded as a spectral problem for a quadratic operator pensil. The problem with $p(x) \in W_2^1(0,1)$ and $q(x) \in L_2(0,1)$ and with Robin boundary conditions was discussed in [10]. Such problems for separated and non-separated boundary conditions were considered (see [9, 12, 22] and the references therein). In this aspect, various inverse spectral problems for the equation (1.4) have been investigated in [18, 19] for the case $m = 1$.

In this article we establish various uniqueness results for inverse spectral problems of energy-dependent Sturm-Liouville equations with finitely many point δ-interactions, and obtain a process for finding its solution.

2. Properties of the spectral characteristics of BVP L

In this section, we provide the spectral characteristics of the BVP L and present the relationship among these spectral characteristics. Moreover, we formulate the inverse problem of the reconstruction of BVP L: from the Weyl function, from the spectral data, and from two spectra. The technique employed is similar to those used in [8].

Let $y(x)$ and $z(x)$ be continuously differentiable functions on the intervals $(0, a_1)$, $(a_1, a_2), \dots, (a_{m-1}, a_m)$ and (a_m, π). Denote $\langle y, z\rangle := yz' - y'z$. If $y(x)$ and $z(x)$ satisfy the conditions (1.3), then

$$\langle y, z\rangle_{x=a_s-0} = \langle y, z\rangle_{x=a_s+0}, \quad s = \overline{1,m}, \tag{2.1}$$

i.e. the function $\langle y, z\rangle$ is continuous on $(0, \pi)$.

Let $\varphi(x,\lambda)$, $\psi(x,\lambda)$ be solutions of (1.1) under the conditions

$$\begin{gathered} \varphi(0,\lambda) = \psi(\pi,\lambda) = 1 \\ \varphi'(0,\lambda) = h, \quad \psi'(\pi,\lambda) = -H, \end{gathered} \tag{2.2}$$

and under condition (1.3).

Denote $\Delta(\lambda) := \langle \varphi(x,\lambda), \psi(x,\lambda)\rangle$. By (2.1) and the Ostrogradskii-Liouville theorem (see [6, p.83]), $\Delta(\lambda)$ does not depend on x. The function $\Delta(\lambda)$ is called the characteristic function of L. Clearly,

$$\Delta(\lambda) = -V(\varphi) = U(\psi). \tag{2.3}$$

Obviously, the function $\Delta(\lambda)$ is entire in λ and it has at most a countable set of zeros $\{\lambda_n\}$.

We confine ourselves, for simplicity, to the case when all zeros of the characteristic function $\Delta(\lambda)$ are simple.

Lemma 2.1. *The eigenvalues $\{\lambda_n^2\}_{n\geq 0}$ of the BVP L coincide with the zeros of the characteristic function. The functions $\varphi(x,\lambda_n)$ and $\psi(x,\lambda_n)$ are eigenfunctions, and there exists a sequence $\{\beta_n\}$ such that*

$$\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n),\quad \beta_n\neq 0.$$

Proof. Let $\Delta(\lambda_0)=0$. Then by $\langle\varphi(x,\lambda_0),\psi(x,\lambda_0)\rangle=0$, we have $\psi(x,\lambda_0)=C\varphi(x,\lambda_0)$ for some constant C. Hence λ_0 is an eigenvalue and $\varphi(x,\lambda_0)$, $\psi(x,\lambda_0)$ are eigenfunctions related to λ_0.

Conversely, let λ_0 be an eigenvalue of L. We shall show that $\Delta(\lambda_0)=0$. Assuming the converse, suppose that $\Delta(\lambda_0)\neq 0$. In this case the functions $\varphi(x,\lambda_0)$ and $\psi(x,\lambda_0)$ are linearly independent. Then $y(x,\lambda_0)=c_1\varphi(x,\lambda_0)+c_2\psi(x,\lambda_0)$ is a general solution of the problem L. If $c_1\neq 0$, we can write

$$\varphi(x,\lambda_0)=\frac{1}{c_1}y(x,\lambda_0)-\frac{c_2}{c_1}\psi(x,\lambda_0).$$

Then we have

$$\langle\varphi(x,\lambda_0),\psi(x,\lambda_0)\rangle=-\frac{1}{c_1}[y'(\pi,\lambda_0)+Hy(\pi,\lambda_0)]=0,$$

which is a contradiction.

Note that for each eigenvalue there exists only one eigenfunction. Therefore there exists sequence β_n such that $\psi(x,\lambda_n)=\beta_n\varphi(x,\lambda_n)$. □

Denote

$$\gamma_n=\int_0^\pi\varphi^2(x,\lambda_n)dx-\frac{1}{\lambda_n}\sum_{s=1}^m\alpha_s\varphi^2(a_s,\lambda_n).$$

The set $\{\lambda_n,\gamma_n\}_{n=0,\pm1,\pm2,\dots}$ is called the *spectral data* of L.

Lemma 2.2. *The equality*

$$\dot\Delta(\lambda_n)=-2\lambda_n\beta_n\gamma_n \tag{2.4}$$

holds. Here $\dot\Delta(\lambda)=\frac{d}{d\lambda}\Delta(\lambda)$.

Proof. Since

$$-\varphi''(x,\lambda_n)+q(x)\varphi(x,\lambda_n)=\lambda_n^2\varphi(x,\lambda_n),\quad -\psi''(x,\lambda)+q(x)\psi(x,\lambda)=\lambda^2\psi(x,\lambda),$$

we obtain

$$\frac{d}{dx}\langle\varphi(x,\lambda_n),\psi(x,\lambda)\rangle=(\lambda^2-\lambda_n^2)\varphi(x,\lambda_n)\psi(x,\lambda).$$

Integrating from 0 to π and using the conditions (1.2), (1.3), we obtain

$$\int_0^\pi\varphi(x,\lambda_n)\psi(x,\lambda)dx-\frac{2}{\lambda_n+\lambda}\sum_{s=1}^m\alpha_s\varphi(a_s,\lambda_n)\psi(a_s,\lambda)=\frac{\Delta(\lambda_n)-\Delta(\lambda)}{\lambda^2-\lambda_n^2}.$$

Because $\lambda\to\lambda_n$, by Lemma 2.1, this yields

$$\dot\Delta(\lambda_n)=-2\lambda_n\beta_n\gamma_n.$$

□

Corollary 2.3. *For BVP L, $\lambda_n \neq 0$, $\gamma_n \neq 0$ and $\lambda_n \neq \lambda_k$ $(n \neq k)$ hold.*

Now, consider the solutions $\varphi(x,\lambda)$ and $\psi(x,\lambda)$. Let $\tau := \operatorname{Im}\lambda$. The following asymptotic estimates hold uniformly with respect to $x \in (a_s, a_{s+1})$, $0 \le s \le m$, as $|\lambda| \to \infty$:

$$
\begin{aligned}
&\varphi(x,\lambda)\\
&= \alpha_1'\alpha_2'\dots\alpha_s'\cos(\lambda x - \theta_1 - \theta_2 - \dots - \theta_s)\\
&\quad + \sum_{1\le i\le s} \alpha_1'\dots\alpha_i\dots\alpha_s' \sin[\lambda(x-2a_i) + \theta_1 + \dots + \theta_{i-1} - \theta_{i+1} - \dots - \theta_s]\\
&\quad + \sum_{1\le i<j\le s} \alpha_1'\dots\alpha_i\dots\alpha_j\dots\alpha_s' \cos\big[\lambda(x+2a_i-2a_j) + \theta_1 + \dots\\
&\quad + \theta_{i-1} - \theta_{i+1} - \dots - \theta_{j-1} + \theta_{j+1} + \dots + \theta_s\big]\\
&\quad + \dots + \alpha_1\alpha_2\dots\alpha_s\{\begin{cases} \cos & \text{if } s \text{ is even}\\ \sin & \text{if } s \text{ is odd}\end{cases}\\
&\quad \big[\lambda(x + 2(-1)^s a_1 + 2(-1)^{s-1}a_2 + \dots - 2\alpha_s)\big] + O\Big(\frac{1}{|\lambda|}\exp(|\tau|x)\Big),
\end{aligned}
\tag{2.5}
$$

$$
\begin{aligned}
&\varphi'(x,\lambda)\\
&= \lambda\Big[-\alpha_1'\alpha_2'\dots\alpha_s'\sin(\lambda x - \theta_1 - \theta_2 - \dots - \theta_s)\\
&\quad + \sum_{1\le i\le s} \alpha_1'\dots\alpha_i\dots\alpha_s' \cos\big[\lambda(x-2a_i) + \theta_1 + \dots + \theta_{i-1} - \theta_{i+1} - \dots - \theta_s\big]\\
&\quad - \sum_{1\le i<j\le s} \alpha_1'\dots\alpha_i\dots\alpha_j\dots\alpha_s' \sin\big[\lambda(x+2a_i-2a_j) + \theta_1 + \dots + \theta_{i-1}\\
&\quad - \theta_{i+1} - \dots - \theta_{j-1} + \theta_{j+1} + \dots + \theta_s\big] + \dots + \alpha_1\alpha_2\dots\alpha_s \begin{cases} -\sin & \text{if } s \text{ is even}\\ \cos & \text{if } s \text{ is odd}\end{cases}\\
&\quad \big[\lambda\big(x + 2(-1)^s a_1 + 2(-1)^{s-1}a_2 + \dots - 2\alpha_s)\big] + O\Big(\exp(|\tau|x)\Big),
\end{aligned}
\tag{2.6}
$$

and

$$
\begin{aligned}
&\psi(x,\lambda)\\
&= \alpha_{s+1}'\alpha_{s+2}'\dots\alpha_m'\cos[\lambda(\pi - x) - \theta_{s+1} - \theta_{s+2} - \theta_m]\\
&\quad - \sum_{s+1\le i\le m} \alpha_{s+1}'\dots\alpha_i\dots\alpha_m' \sin[\lambda(\pi + x - 2a_m) - \theta_{s+1} - \theta_{s+2} - \theta_{m-1}]\\
&\quad + \sum_{s+1\le i<j\le m} \alpha_{s+1}'\dots\alpha_i\dots\alpha_j\dots\alpha_m' \cos\big[\lambda(\pi + x + 2a_i - 2a_j) + \theta_{s+1} +\\
&\quad \dots + \theta_{i-1} - \theta_{i+1} - \dots - \theta_{j-1} + \theta_{j+1} + \dots + \theta_m\big] + \dots\\
&\quad + \alpha_{s+1}\alpha_{s+2}\dots\alpha_m \begin{cases} \cos & \text{if } s \text{ is even}\\ \sin & \text{if } s \text{ is odd}\end{cases}\\
&\quad [\lambda(\pi + x - 2a_{s+1} + \dots + 2(-1)^{s-1}a_{m-1} + 2(-1)^s a_m)]\\
&\quad + O\Big(\frac{1}{|\lambda|}\exp(|\tau|(\pi - x))\Big),
\end{aligned}
\tag{2.7}
$$

$$
\begin{aligned}
&\psi'(x,\lambda)\\
&=\lambda[\alpha'_{s+1}\alpha'_{s+2}\dots\alpha'_m\sin[\lambda(\pi-x)-\theta_{s+1}-\theta_{s+2}-\theta_m]\\
&\quad-\sum_{s+1\le i\le m}\alpha'_{s+1}\dots\alpha_i\dots\alpha'_m\cos[\lambda(\pi+x-2a_m)-\theta_{s+1}-\theta_{s+2}-\theta_{m-1}]\\
&\quad+\sum_{s+1\le i<j\le m}\alpha'_{s+1}\dots\alpha_i\dots\alpha_j\dots\alpha'_m\sin\big[\lambda(\pi+x+2a_i-2a_j)+\theta_{s+1}\\
&\quad+\dots+\theta_{i-1}-\theta_{i+1}-\dots-\theta_{j-1}+\theta_{j+1}+\dots+\theta_m\big]+\dots\\
&\quad+\alpha_{s+1}\alpha_{s+2}\dots\alpha_m\begin{cases}-\sin & \text{if } s \text{ is even}\\ \cos & \text{if } s \text{ is odd}\end{cases}\\
&\quad\big[\lambda(\pi+x-2a_{s+1}+\dots+2(-1)^{s-1}a_{m-1}+2(-1)^s a_m)\big]\\
&\quad+O\Big(\exp(|\tau|(\pi-x))\Big),
\end{aligned}
\tag{2.8}
$$

where $\alpha'_i=\sqrt{1+\alpha_i^2}$, $\theta_i=\tan^{-1}\alpha_i$, $i=\overline{1,m}$.

It is easy to have the following asymptotic formulae for $\varphi(x,\lambda)$ and $\psi(x,\lambda)$,

$$|\varphi^{(\nu)}(x,\lambda)|=O\Big(|\lambda|^\nu e^{|\tau|x}\Big),\quad 0\le x\le\pi,\ \nu=0,1, \tag{2.9}$$

$$|\psi^{(\nu)}(x,\lambda)|=O(|\lambda|^\nu e^{|\tau|(\pi-x)}). \tag{2.10}$$

It follows from (2.3), (2.5) and (2.6) that

$$\Delta(\lambda)=\Delta_0(\lambda)+O\Big(\exp(|\tau|\pi)\Big), \tag{2.11}$$

where

$$
\begin{aligned}
&\Delta_0(\lambda)\\
&=\lambda\Big[-\alpha'_1\alpha'_2\dots\alpha'_s\sin(\lambda\pi-\theta_1-\theta_2-\dots-\theta_s)\\
&\quad+\sum_{1\le i\le s}\alpha'_1\dots\alpha_i\dots\alpha'_s\cos\big[\lambda(\pi-2a_i)+\theta_1+\dots+\theta_{i-1}-\theta_{i+1}-\dots-\theta_s\big]\\
&\quad-\sum_{1\le i<j\le s}\alpha'_1\dots\alpha_i\dots\alpha_j\dots\alpha'_s\sin\big[\lambda(\pi+2a_i-2a_j)+\theta_1+\dots+\theta_{i-1}\\
&\quad-\theta_{i+1}-\dots-\theta_{j-1}+\theta_{j+1}+\dots+\theta_s\big]+\dots+\alpha_1\alpha_2\dots\alpha_s\begin{cases}-\sin & \text{if } s \text{ is even}\\ \cos & \text{if } s \text{ is odd}\end{cases}\\
&\quad\big[\lambda(\pi+2(-1)^s a_1+2(-1)^{s-1}a_2+\dots-2\alpha_s)\big]+O\Big(\exp(|\tau|\pi)\Big).
\end{aligned}
\tag{2.12}
$$

Let $\{\lambda_n^0\}$ be zeros of $\Delta_0(\lambda)$. Using (2.11), by the well known methods (see, for example, [3]) one can obtain the following properties of the characteristic function $\Delta(\lambda)$ of the BVP L:

1. For $|\lambda|\to\infty$, $\Delta(\lambda)=O(|\lambda|\exp(|\tau|\pi))$.
2. Denote $G_\delta:=\{\lambda:|\lambda-\lambda_n^0|\ge\delta\}$. Then exist $C_\delta>0$ such that

$$|\Delta(\lambda)|\ge C_\delta|\lambda|\exp(|\tau|\pi)\quad\text{for all } \lambda\in G_\delta\quad(\delta>0). \tag{2.13}$$

(3) For sufficiently large values of n, one has

$$|\Delta(\lambda)-\Delta_0(\lambda)|<\frac{C_\delta}{2}\exp(|\tau|\pi),$$
$$\lambda\in\Gamma_n:=\{\lambda:|\lambda|=|\lambda_n^0|+\frac{1}{2}\inf_{n\neq m}|\lambda_n^0-\lambda_m^0|\}.$$

Then

$$\lambda_n=\lambda_n^0+o(1),\ n\to\infty. \tag{2.14}$$

Substituting (2.5), (2.6) and (2.13) into (2.3) we obtain

$$\gamma_n=\gamma_n^0+o(1),\ n\to\infty, \tag{2.15}$$

where

$$\gamma_n^0=\int_0^\pi\varphi^2(x,\lambda_n^0)dx-\frac{1}{\lambda_n^0}\sum_{s=1}^m\alpha_s\varphi^2(a_s,\lambda_n^0).$$

3. Formulation of the inverse problem. Uniqueness theorems

In this section, we study three inverse problems of recovering L from its spectral characteristics, namely

i from the Weyl function,
ii from the so-called spectral data,
iii from two spectra.

For each class of inverse problems we prove the corresponding uniqueness theorems and show the connection between the different spectral characteristics.

Let $\Phi(x,\lambda)$ be the solution of (1.4) under the conditions $U(\Phi)=1$ and $V(\Phi)=0$. We set $M(\lambda):=\Phi(\lambda)$. The functions $\Phi(x,\lambda)$ and $M(\lambda)$ are called the Weyl solution and the Weyl function for BVP L, respectively. Clearly

$$\Phi(x,\lambda)=\frac{\psi(x,\lambda)}{\Delta(\lambda)}=S(x,\lambda)+M(\lambda)\varphi(x,\lambda), \tag{3.1}$$

$$M(\lambda)=\frac{\psi(0,\lambda)}{\Delta(\lambda)}, \tag{3.2}$$

where $\psi(0,\lambda)$ is the characteristic function of BVP L_1 which is equation (1.4) with the boundary conditions $U(y)=0$, $y(\pi)=0$, and $S(x,\lambda)$ is defined from the equality

$$\psi(x,\lambda)=\psi(0,\lambda)\varphi(x,\lambda)+\Delta(\lambda)S(x,\lambda).$$

Note that, by equalities $\langle\varphi(x,\lambda),S(x,\lambda)\rangle\equiv 1$ and (3.1), one has

$$\langle\Phi(x,\lambda),\varphi(x,\lambda)\rangle\equiv 1. \tag{3.3}$$

Let $\{\mu_n^2\}_{n\geq 0}$ be zeros of $\psi(0,\lambda)$, i.e. the eigenvalues of L_1.

First, let us prove the uniqueness theorems for the solutions of the problems (i)-(iii). For this purpose we agree that together with L we consider a BVP $\widetilde{L}$ of the same form but with different coefficients $\widetilde{q}(x)$, $\widetilde{h}$, $\widetilde{H}$, $\widetilde{a_s}$ and $\widetilde{\alpha_s}$, $s=\overline{1,m}$. Everywhere below if a certain symbol e denotes an object related to L, then the corresponding symbol $\widetilde{e}$ with tilde denotes the analogous object related to $\widetilde{L}$.

Theorem 3.1. *If $M(\lambda)=\widetilde{M}(\lambda)$, then $L=\widetilde{L}$. Thus, the specification of the Weyl function $M(\lambda)$ uniquely determines L.*

Proof. Let us define the matrix $P(x,\lambda)=[P_{jk}(x,\lambda)]_{j,k=1,2}$ by the formula

$$P(x,\lambda)\begin{bmatrix}\widetilde{\varphi}(x,\lambda) & \widetilde{\Phi}(x,\lambda)\\ \widetilde{\varphi}'(x,\lambda) & \widetilde{\Phi}'(x,\lambda)\end{bmatrix}=\begin{bmatrix}\varphi(x,\lambda) & \Phi(x,\lambda)\\ \varphi'(x,\lambda) & \Phi'(x,\lambda)\end{bmatrix}. \tag{3.4}$$

By (3.1), we calculate

$$\begin{gathered}P_{j1}(x,\lambda)=\varphi^{(j-1)}(x,\lambda)\widetilde{\Phi}'(x,\lambda)-\Phi^{(j-1)}(x,\lambda)\widetilde{\varphi}'(x,\lambda),\\ P_{j2}(x,\lambda)=\Phi^{(j-1)}(x,\lambda)\widetilde{\varphi}(x,\lambda)-\varphi^{(j-1)}(x,\lambda)\widetilde{\Phi}(x,\lambda),\end{gathered} \tag{3.5}$$

and

$$\begin{gathered}\varphi(x,\lambda)=P_{11}(x,\lambda)\widetilde{\varphi}(x,\lambda)+P_{12}(x,\lambda)\widetilde{\varphi}'(x,\lambda),\\ \Phi(x,\lambda)=P_{11}(x,\lambda)\widetilde{\Phi}(x,\lambda)+P_{12}(x,\lambda)\widetilde{\Phi}'(x,\lambda).\end{gathered} \tag{3.6}$$

Taking (2.5), (2.6) and (2.13) into account, we infer that

$$|P_{11}(x,\lambda)-1|\le C_\delta|\lambda|^{-1},\quad |P_{12}(x,\lambda)|\le C_\delta|\lambda|^{-1},\quad \lambda\in G_\delta, \tag{3.7}$$

where G_δ is defined in (2.13) and C_δ is a constant.

On the other hand according to (3.2) and (3.5),

$$\begin{gathered}P_{11}(x,\lambda)=\varphi(x,\lambda)\widetilde{S}'(x,\lambda)-S(x,\lambda)\widetilde{\varphi}'(x,\lambda)+(\widetilde{M}(\lambda)-M(\lambda))\varphi(x,\lambda)\widetilde{\varphi}'(x,\lambda),\\ P_{12}(x,\lambda)=S(x,\lambda)\widetilde{\varphi}(x,\lambda)-\varphi(x,\lambda)\widetilde{S}(x,\lambda)+(M(\lambda)-\widetilde{M}(\lambda))\varphi(x,\lambda)\widetilde{\varphi}(x,\lambda).\end{gathered}$$

Since $M(\lambda)=\widetilde{M}(\lambda)$, it follows that for each fixed x the functions $P_{1k}(x,\lambda)$, $k=1,2$ are entire in λ. With the help of (3.7) and well-known Liouville's theorem, this yields $P_{11}(x,\lambda)\equiv 1$, $P_{12}(x,\lambda)\equiv 0$. Substituting into (3.6), we obtain $\varphi(x,\lambda)\equiv\widetilde{\varphi}(x,\lambda)$, $\Phi(x,\lambda)=\widetilde{\Phi}(x,\lambda)$ for all $x\in\cup_{s=0}^{m}(x_s,x_{s+1})$ and λ. Taking this into account, from (1.1) we obtain $q(x)=\widetilde{q}(x)$ a.e. on $(0,\pi)$, from (2.2) we obtain $h=\widetilde{h}$, $H=\widetilde{H}$, and from (1.3) we conclude that $a_s=\widetilde{a_s}$, $\alpha_s=\widetilde{\alpha_s}$ $(s=\overline{1,m})$. Consequently, $L=\widetilde{L}$. □

Theorem 3.2. *If $\lambda_n=\widetilde{\lambda_n}$, $\gamma_n=\widetilde{\gamma_n}$, $n=0,\pm1,\pm2,\dots$, then $L=\widetilde{L}$. Thus, the specification of the spectral data $\{\lambda_n,\gamma_n\}_{n=0,\pm1,\pm2,\dots}$ uniquely determines the operator.*

Proof. It follows from (3.2) that the Weyl function $M(\lambda)$ is meromorphic with simple poles at points λ_n^2. Using (3.2), Lemma 2.1 and equality $\dot{\Delta}(\lambda_n)=-2\lambda_n\beta_n\gamma_n$, we have

$$\operatorname{Re}s_{\lambda-\lambda_n}M(\lambda)=\frac{\psi(0,\lambda)}{\dot{\Delta}(\lambda_n)}=\frac{\beta_n}{\dot{\Delta}(\lambda_n)}=\frac{1}{-2\lambda_n\gamma_n}. \tag{3.8}$$

Since the Weyl function $M(\lambda)$ is regular for $\lambda\in\Gamma_n$, applying the Rouche theorem [7, p.112], we conclude that

$$M(\lambda)=\frac{1}{2\pi i}\int_{\Gamma_n}\frac{M(\mu)}{\lambda-\mu}d\mu,\ \lambda\in int\Gamma_n,$$

where the contour Γ_n is assumed to have the counterclockwise circuit.

Taking (2.13) and (3.2) into account, we arrive at $|M(\lambda)|\le C_\delta|\lambda|^{-1}$, $\lambda\in G_\delta$. Hence, by the residue theorem, we have

$$M(\lambda)=\sum_{n=-\infty}^{\infty}\frac{1}{2\lambda_n\gamma_n(\lambda-\lambda_n)}. \tag{3.9}$$

Under the hypothesis of the theorem we obtain, in view of (3.9), that $M(\lambda) = \widetilde{M}(\lambda)$ and consequently by Theorem 3.1, $L = \widetilde{L}$. □

Theorem 3.3. *If $\lambda_n = \widetilde{\lambda_n}$ and $\mu_n = \widetilde{\mu_n}$, $n = 0, \pm 1, \pm 2, \dots$, then $L = \widetilde{L}$. Thus the specification of two spectra $\{\lambda_n, \mu_n\}_{n=0,\pm 1,\pm 2,\dots}$ uniquely determines L.*

Proof. It is obvious that characteristic functions $\Delta(\lambda)$ and $\psi(0,\lambda)$ are uniquely determined by the sequences $\{\lambda_n^2\}$ and $\{\mu_n^2\}$ $(n = 0, \pm 1, \pm 2, \dots)$, respectively. If $\lambda_n = \widetilde{\lambda_n}$, $\mu_n = \widetilde{\mu_n}$, $n = 0, \pm 1, \pm 2, \dots$, then $\Delta(\lambda) \equiv \widetilde{\Delta}(\lambda)$, $\psi(0,\lambda) = \widetilde{\psi}(0,\lambda)$. Together with (3.2) this yields $M(\lambda) = \widetilde{M}(\lambda)$. By Theorem 3.1 we obtain $L = \widetilde{L}$. □

Remark 3.4. By (3.2), the specification of two spectra $\{\lambda_n, \mu_n\}_{n=0,\pm 1,\pm 2,\dots}$ is equivalent to the specification of the Weyl function $M(\lambda)$. On the other hand, it follows from (3.9) that the specification of the Weyl function $M(\lambda)$ is equivalent to the specification of the spectral data $\{\lambda_n, \gamma_n\}_{n=0,\pm 1,\pm 2,\dots}$. Consequently, three statements of the inverse problem of reconstruction of the problem L from the Weyl function, from the spectral data and from two spectra are equivalent.

4. Solution of the inverse problem

In this section, we will solve the inverse problems of recovering the BVP L by the method of spectral mappings with the help of Cauchy's integral formula and Residue theorem. Finally, we provide the algorithms for the solution of the inverse problems by using the solution of the main equation.

For definiteness we consider the inverse problem of recovering L from the spectral data $\{\lambda_n, \gamma_n\}_{n=0,\pm 1,\pm 2,\dots}$. Let BVPs L and $\widetilde{L}$ be such that

$$a_s = \widetilde{a}_s, \quad s = \overline{1,m}, \quad \sum_{n=-\infty}^{\infty} \zeta_n |\lambda_n| < \infty, \tag{4.1}$$

where $\zeta_n := |\lambda_n - \widetilde{\lambda}_n| + |\gamma_n - \widetilde{\gamma}_n|$. Denote

$$\lambda_{n0} = \lambda_n,\ \lambda_{n1} = \widetilde{\lambda}_n, \quad \gamma_{n_0} = \gamma_n, \gamma_{n1} = \widetilde{\gamma}_n,$$
$$\varphi_{ni}(x) = \varphi(x, \lambda_{ni}), \quad \widetilde{\varphi}_{ni}(x) = \widetilde{\varphi}(x, \lambda_{ni}),$$
$$Q_{kj}(x,\lambda) = \frac{\langle \varphi(x,\lambda), \varphi_{kj}(x)\rangle}{2\lambda_{kj}\gamma_{kj}(\lambda - \lambda_{kj})} = \frac{1}{2\lambda_{kj}\gamma_{kj}} \int_0^x \varphi(t,\lambda)\varphi_{kj}(t)dt,$$
$$Q_{ni,kj}(x) = Q_{kj}(x, \lambda_{ni})$$

for $i, j = 0, 1$ and $n, k = 0, \pm 1, \pm 2, \dots$, where $\widetilde{\varphi}(x,\lambda)$ is the solution of (1.4) with the potential $\widetilde{q}$ under the initial conditions $\widetilde{\varphi}(0,\lambda) = 1$, $\widetilde{\varphi}'(0,\lambda) = \widetilde{h}$. Analogously, we can define $\widetilde{Q}_{kj}(x,\lambda)$ by replacing φ with $\widetilde{\varphi}$ in the above definition.

Using Schwarz's lemma [7, p.130] and (2.5)-(2.8), (2.14) we obtain the following asymptotic estimates:

$$|\varphi_{nj}^{(\nu)}(x)| \le C(|\lambda_n^0| + 1)^{\nu}, \tag{4.2}$$

$$|Q_{ni,kj}(x)| \le \frac{C}{|\lambda_n^0 - \lambda_k^0| + 1}, \quad |Q_{ni,kj}^{(\nu+1)}(x)| \le C(|\lambda_n^0| + |\lambda_k^0| + 1)^{(\nu)}, \tag{4.3}$$

where $n, k = 0, \pm 1, \pm 2, \dots$, $i, j, \nu = 0, 1$ and C is a positive constant. Analogous estimates are also valid for $\widetilde{\varphi}_{ni}(x)$, $\widetilde{Q}_{ni,kj}(x)$.

Lemma 4.1. *Let $\varphi_{nj}(x)$ and $Q_{ni,kj}(x)$ be defined as above. Then the following representations hold for $i,j=0,1$ and $n,k=0,\pm1,\pm2,\dots$:*

$$\widetilde{\varphi}_{ni}(x)=\varphi_{n_i}(x)+\sum_{l=-\infty}^{\infty}\Big(\widetilde{Q}_{ni,l0}(x)\varphi_{k0}(x)-\widetilde{Q}_{ni,l1}(x)\varphi_{k1}(x)\Big). \tag{4.4}$$

Series (4.4) converge absolutely and uniformly with respect to $x\in[0,\pi]\backslash\{a_s\}_{s=1}^m$.

Proof. By (4.1) we have $a_s=\widetilde{a_s}$, $\alpha_s=\widetilde{\alpha_s}$ for $s=\overline{1,m}$. Then it follows from (2.5)-(2.6) that

$$|\varphi^{(\nu)}(x,\lambda)-\widetilde{\varphi}^{(\nu)}(x,\lambda)|\le C|\lambda|^{\nu-1}\exp(|\tau|x). \tag{4.5}$$

Similarly,

$$|\psi^{(\nu)}(x,\lambda)-\widetilde{\psi}^{(\nu)}(x,\lambda)|\le C|\lambda|^{\nu-1}\exp(|\tau|(\pi-x)). \tag{4.6}$$

Denote $G_\delta^0=G_\delta\cap\widetilde{G}_\delta$. In view of (2.10), (2.13), (3.1) and (4.6) we obtain

$$|\Phi^{(\nu)}(x,\lambda)-\widetilde{\Phi}^{(\nu)}(x,\lambda)|\le C_\delta|\lambda|^{\nu-2}\exp(-|\tau|x),\ \lambda\in G_\delta^0. \tag{4.7}$$

Let $P(x,\lambda)$ be the matrix defined in Section 3. Since for each fixed x, the functions $P_{1k}(x,\lambda)$ are meromorphic in λ with simple poles λ_n and $\widetilde{\lambda}_n$, we obtain by Cauchy's theorem [7, p.84]

$$P_{1k}(x,\lambda)-\delta_{1k}=\frac{1}{2\pi i}\int_{\Gamma_n}\frac{P_{1k}(x,\zeta)-\delta_{1k}}{\lambda-\zeta},\ k=1,2, \tag{4.8}$$

where $\lambda\in int\Gamma_n$, and δ_{jk} is the Kronecker delta.

Further, (3.3) and (3.5) imply

$$P_{11}(x,\lambda)=1+(\varphi(x,\lambda)-\widetilde{\varphi}(x,\lambda))\widetilde{\Phi}'(x,\lambda)-(\Phi(x,\lambda)-\widetilde{\Phi}(x,\lambda))\widetilde{\varphi}'(x,\lambda). \tag{4.9}$$

Using (2.9), (2.10), (3.5), (4.5), (4.7) and (4.9) we infer

$$|P_{1k}(x,\lambda)-\delta_{1k}|\le C_\delta|\lambda|^{-1},\quad \lambda\in G_\delta^0. \tag{4.10}$$

By (4.8) and (4.10)

$$P_{1k}(x,\lambda)-\delta_{1k}=\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{n1}}\frac{P_{1k}(x,\zeta)}{\zeta-\lambda}d\zeta,$$

where $\Gamma_{n1}=\{\lambda:|\lambda|=|\lambda_n^0|,\ n=0,\pm1,\pm2,\dots\}$. Substituting this into (3.6), we obtain

$$\varphi(x,\lambda)=\widetilde{\varphi}(x,\lambda)+\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{n1}}\frac{\widetilde{\varphi}(x,\lambda)P_{11}(x,\zeta)+\widetilde{\varphi}'(x,\lambda)P_{12}(x,\zeta)}{\lambda-\zeta}d\zeta.$$

Taking (3.1) and (3.5) into account we calculate

$$\widetilde{\varphi}(x,\lambda)=\varphi(x,\lambda)+\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Gamma_{n1}}\frac{\langle\widetilde{\varphi}(x,\lambda),\widetilde{\varphi}(x,\zeta)\rangle}{\lambda-\zeta}\widetilde{M}(\zeta)\varphi(x,\zeta)d\zeta. \tag{4.11}$$

It follows from (3.8) that

$$\operatorname{Re}s_{\zeta=\lambda_{kj}}\frac{\langle\widetilde{\varphi}(x,\lambda),\widetilde{\varphi}(x,\zeta)\rangle}{\lambda-\zeta}\widetilde{M}(\zeta)\varphi(x,\zeta)=\widetilde{Q}_{kj}(x,\lambda)\varphi_{kj}(x).$$

Consequently, calculating the integral in (4.11) by the residue theorem [7, p.112] and then taking $\lambda=\lambda_{ni}$, we arrive at (4.4) as $n\to\infty$. Furthermore, according to asymptotic formulaes (4.2) and (4.3), we derive for $x\in(a_s,a_{s+1})$, $0\le s\le m$ that the series converges absolutely and uniformly on $x\in[0,\pi]\backslash\{a_s\}_{s=1}^m$. $\square$

From the above arguments, it is seen that, for each fixed $x \in [0,\pi]\backslash\{a_s\}_{s=1}^m$, the relation (4.4) can be considered as a system of linear equations with respect to $\varphi_{ni}(x)$ for $n = 0, \pm1, \pm2, \dots$ and $i = 0, 1$. But the series in (4.4) converges only "with brackets". Therefore, it is not convenient to use (4.4) as a main equation of the inverse problem.

Let V be a set of indices $u = (n, i)$, $n = 0, \pm1, \pm2, \dots$ and $i = 0, 1$. For each fixed $x \in [0,\pi]\backslash\{a_s\}_{s=1}^m$, we define the vectors

$$\phi(x) = [\phi_u(x)]_{u\in V} = [\begin{bmatrix}\phi_{n0}(x)\\ \phi_{n1}(x)\end{bmatrix}, \quad \widetilde{\phi}(x) = [\widetilde{\phi}_u(x)]_{u\in V} = \begin{bmatrix}\widetilde{\phi}_{n0}(x)\\ \widetilde{\phi}_{n1}(x)\end{bmatrix}_{n=0,\pm1,\pm2,\dots}$$

by the formulas

$$\begin{bmatrix}\phi_{n0}(x)\\ \phi_{n1}(x)\end{bmatrix} = \begin{bmatrix}\zeta_n^{-1} & -\zeta_n^{-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\varphi_{n0}(x)\\ \varphi_{n1}(x)\end{bmatrix}, \quad \begin{bmatrix}\widetilde{\phi}_{n0}(x)\\ \widetilde{\phi}_{n1}(x)\end{bmatrix} = \begin{bmatrix}\zeta_n^{-1} & -\zeta_n^{-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}\widetilde{\varphi}_{n0}(x)\\ \widetilde{\varphi}_{n1}(x)\end{bmatrix}. \tag{4.12}$$

If $\zeta_n = 0$ for a certain n, then we put $\phi_{n0}(x) = \widetilde{\phi}_{n0}(x) = 0$.

Further, we define the block matrix

$$H(x) = [H_{u,v}(x)]_{u,v\in V} = \begin{bmatrix}H_{n0,k0}(x) & H_{n0,k1}(x)\\ H_{n1,k0}(x) & H_{n1,k1}(x)\end{bmatrix}_{n,k=0,\pm1,\pm2,\dots},$$

where $u = (n, i)$, $v = (k, j)$ and

$$\begin{bmatrix}H_{n0,k0}(x) & H_{n0,k1}(x)\\ H_{n1,k0}(x) & H_{n1,k1}(x)\end{bmatrix} = \begin{bmatrix}\zeta_n^{-1} & -\zeta_n^{-1}\\ 0 & 1\end{bmatrix}\begin{bmatrix}Q_{n0,k0}(x) & Q_{n0,k1}(x)\\ Q_{n1,k0}(x) & Q_{n1,k1}(x)\end{bmatrix}\begin{bmatrix}\zeta_k & \zeta_k\\ 0 & -1\end{bmatrix}.$$

Analogously we define $\widetilde{\phi}(x)$, $\widetilde{H}(x)$ by replacing in the previous definitions $\varphi_{ni}(x)$, $Q_{ni,kj}(x)$ by $\widetilde{\varphi}_{ni}(x)$, $\widetilde{Q}_{ni,kj}(x)$, respectively. It follows from (2.5)-(2.8), (2.14), (2.15), (4.2), (4.3) and Schwarz's lemma that

$$\begin{gathered}|\phi_{nj}^{(\nu)}(x)|,\ |\widetilde{\phi}_{nj}^{(\nu)}(x)| \le C(|\lambda_n^0| + 1)^\nu, \quad \nu = 0, 1,\\ |H_{ni,kj}(x)|,\ |\widetilde{H}_{ni,kj}(x)| \le \frac{C\zeta_k}{|\lambda_n^0 - \lambda_k^0| + 1},\end{gathered} \tag{4.13}$$

$$|H_{ni,kj}^{(\nu+1)}(x)|,\ |\widetilde{H}_{ni,kj}^{(\nu+1)}(x)| \le C(|\lambda_n^0| + |\lambda_k^0| + 1)^\nu \zeta_k, \quad \nu = 0, 1. \tag{4.14}$$

Let us consider the Banach space B of bounded sequences $\alpha = [\alpha_u]_{u\in V}$ with the norm $\|\alpha\|_B = \sup_{u\in V}|\alpha_u|$. It follows from (4.13), (4.14) that for each fixed $x \in (a_s, a_{s+1})$, $0 \le s \le m$, the operator $E + \widetilde{H}(x)$ and $E - H(x)$ (here E is the identity operator), acting from B to B, are linear bounded operator, and

$$\|H(x)\|, \|\widetilde{H}(x)\| \le C \sup_n \sum_{k=-\infty}^{\infty} \frac{\zeta_k}{|\lambda_n^0 - \lambda_k^0| + 1} < \infty.$$

Here, we give the main result of the section.

Theorem 4.2. *For each fixed $x \in [0,\pi]\backslash\{a_s\}_{s=1}^m$, the vector $\phi(x) \in B$ satisfies the equation*

$$\widetilde{\phi}(x) = (E + \widetilde{H}(x))\phi(x) \tag{4.15}$$

in the Banach space B. Moreover, the operator $E + \widetilde{H}(x)$ has a bounded inverse operator, i.e. the equation (4.15) is uniquely solvable.

Proof. Using the notation $\widetilde{\phi}(x)$ we rewrite (4.4) as

$$\widetilde{\phi}_{ni}(x) = \phi_{ni}(x) + \sum_{k,j} \widetilde{H}_{ni,kj}(x)\phi_{kj}(x), \quad (n,i) \in V, \quad (k,j) \in V,$$

which is equivalent to (4.4). Similarly, using the notation $H(x)$:

$$H_{ni,lj}(x) - \widetilde{H}_{ni,lj}(x) + \sum_{k,t} \widetilde{H}_{ni,kt}(x)H_{kt,lj}(x) = 0$$

for (n,i), (l,j), $(k,t) \in V$ or in the other form

$$(E + \widetilde{H}(x))(E - H(x)) = E.$$

Interchanging places for L and $\widetilde{L}$, we obtain analogously

$$\phi(x) = (E - H(x))\widetilde{\varphi}(x), \quad (E - H(x))(E + \widetilde{H}(x)) = E.$$

Hence the operator $(E + \widetilde{H}(x))^{-1}$ exists, and it is a linear bounded operator. $\square$

Equation (4.15) is called the *basic* equation of the inverse problem. Solving (4.15) we find the vector $\phi(x)$, and consequently, the functions $\varphi_{ni}(x)$. Thus, we obtain the following algorithms for the solution of our inverse problems.

Algorithm 4.3. Given the spectral data $\{\lambda_n, \gamma_n\}_{n=0,\pm1,\pm2,\dots}$, construct $q(x)$ and $h, H; a_s, \alpha_s$, $s = \overline{1,m}$.

(1) Choose $\widetilde{L}$ and calculate $\widetilde{\phi}(x)$ and $\widetilde{H}(x)$;
(2) Find $\phi(x)$ by solving the equation (4.15) and calculate $\varphi_{no}(x)$ via (4.12);
(3) Choose some n (e.g., $n = 0$) and construct $q(x)$ and $h, H; a_s, \alpha_s$, $s = \overline{1,m}$ by formulae

$$q(x) = \frac{\varphi''_{n0}(x)}{\varphi_{n0}(x)} + \lambda_n, \quad h = \varphi'_{n0}(0), \quad H = -\frac{\varphi'_{n0}(\pi)}{\varphi_{n0}(\pi)};$$

$$\varphi_{n0}(a_s + 0) = \varphi_{n0}(a_s - 0); \quad \alpha_s = \frac{\varphi'_{n0}(a_s + 0) - \varphi'_{n0}(a_s - 0)}{2\lambda_n \varphi_{n0}(a_s - 0)}; \quad s = \overline{1,m}.$$

Algorithm 4.4. Given $M(\lambda)$, we construct $q(x)$ and $h, H; a_s, \alpha_s$, $s = \overline{1,m}$.

(1) According to (3.9) construct the spectral data $\{\lambda_n, \gamma_n\}_{n=0,\pm1,\pm2,\dots}$.
(2) By Algorithm 4.3, construct $q(x)$ and $h, H; a_s, \alpha_s$, $s = \overline{1,m}$.

Algorithm 4.5. Given two spectra $\{\lambda_n, \mu_n\}_{n=0,\pm1,\pm2,\dots}$, we construct $q(x)$ and $h, H; a_s, \alpha_s$, $s = \overline{1,m}$.

(1) By (3.2) calculate $M(\lambda)$;
(2) By considering Algorithm 4.4, we construct $q(x)$ and $h, H; a_s, \alpha_s$, $s = \overline{1,m}$.

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