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LONG TIME DECAY FOR 3D NAVIER-STOKES EQUATIONS IN SOBOLEV-GEVREY SPACES

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Abstract. In this article, we study the long time decay of global solution to 3D incompressible Navier-Stokes equations. We prove that if $u \in \mathcal{C}([0,\infty), H^1_{a,\sigma}(\mathbb{R}^3))$ is a global solution, where $H^1_{a,\sigma}(\mathbb{R}^3)$ is the Sobolev-Gevrey spaces with parameters $a > 0$ and $\sigma > 1$, then $\|u(t)\|_{H^1_{a,\sigma}(\mathbb{R}^3)}$ decays to zero as time approaches infinity. Our technique is based on Fourier analysis.

1. Introduction

The 3D incompressible Navier-Stokes equations are

$$\begin{gathered}\partial_t u - \Delta u + u \cdot \nabla u = -\nabla p \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ u(0,x) = u^0(x) \quad \text{in } \mathbb{R}^3,\end{gathered} \tag{1.1}$$

where, we assume that the fluid viscosity $\nu = 1$, and $u = u(t,x) = (u_1, u_2, u_3)$ and $p = p(t,x)$ denote respectively the unknown velocity and the unknown pressure of the fluid at the point $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$, $(u \cdot \nabla u) := u_1\partial_1 u + u_2\partial_2 u + u_3\partial_3 u$, and $u^0 = (u_1^o(x), u_2^o(x), u_3^o(x))$ is a given initial velocity. If u^0 is quite regular, the divergence free condition determines the pressure p.

We define the Sobolev-Gevrey spaces as follows; for $a, s \geq 0$, $\sigma > 1$ and $|D| = (-\Delta)^{1/2}$,

$$H^s_{a,\sigma}(\mathbb{R}^3) = \{f \in L^2(\mathbb{R}^3) : e^{a|D|^{1/\sigma}} f \in H^s(\mathbb{R}^3)\}$$

which is equipped with the norm

$$\|f\|_{H^s_{a,\sigma}} = \|e^{a|D|^{1/\sigma}} f\|_{H^s}$$

and its associated inner product

$$\langle f \mid g\rangle_{H^s_{a,\sigma}} = \langle e^{a|D|^{1/\sigma}} f \mid e^{a|D|^{1/\sigma}} g\rangle_{H^s}.$$

There are several authors who have studied the behavior of the norm of the solution to infinity in the different Banach spaces. Wiegner [8] proved that the L^2 norm of the solutions vanishes for any square integrable initial data, as time approaches infinity, and gave a decay rate that seems to be optimal for a class of

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initial data. Schonbek and Wiegner [7, 9] derived some asymptotic properties of the solution and its higher derivatives under additional assumptions on the initial data. Benameur and Selmi [4] proved that if u is a Leray solution of the 2D Navier-Stokes equation, then $\lim_{t\to\infty}\|u(t)\|_{L^2(\mathbb{R}^2)}=0$. For the critical Sobolev spaces $\dot H^{1/2}$, Gallagher, Iftimie and Planchon [6] proved that $\|u(t)\|_{\dot H^{1/2}}$ approaches zero at infinity. Now, we state our main result.

Theorem 1.1. *Let $a>0$ and $\sigma>1$. Let $u\in\mathcal{C}([0,\infty),H^1_{a,\sigma}(\mathbb{R}^3))$ be a global solution to* (1.1)*. Then*
$$\limsup_{t\to\infty}\|u(t)\|_{H^1_{a,\sigma}}=0. \tag{1.2}$$

Note that the existence of local solutions to (1.1) was studied recently in [3].

This article is organized as follows: In section 2, we give some notations and important preliminary results. Section 3 is devoted to prove that if $u\in\mathcal{C}(\mathbb{R}^+,H^1(\mathbb{R}^3))$ is a global solution to (1.1) then $\|u(t)\|_{H^1}$ decays to zero as time approaches infinity. The proof is based on the fact that
$$\lim_{t\to\infty}\|u(t)\|_{\dot H^{1/2}}=0 \tag{1.3}$$
and the energy estimate
$$\|u(t)\|_{L^2}^2+\int_0^t\|\nabla u(\tau)\|_{L^2}^2d\tau\le\|u^0\|_{L^2}^2. \tag{1.4}$$
In section 4, we generalize the results of Foias-Temam [5] to $\mathbb{R}^3$ and in section 5, we prove the main theorem.

2. Notation and preliminary results

2.1. **Notation.** In this section, we collect notation and definitions that will be used later. First, the Fourier transformation is normalized as
$$\mathcal{F}(f)(\xi)=\widehat{f}(\xi)=\int_{\mathbb{R}^3}\exp(-ix\cdot\xi)f(x)dx,\quad \xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3,$$
the inverse Fourier formula is
$$\mathcal{F}^{-1}(g)(x)=(2\pi)^{-3}\int_{\mathbb{R}^3}\exp(i\xi\cdot x)g(\xi)d\xi,\quad x=(x_1,x_2,x_3)\in\mathbb{R}^3,$$
and the convolution product of a suitable pair of functions f and g on $\mathbb{R}^3$ is
$$(f*g)(x):=\int_{\mathbb{R}^3}f(y)g(x-y)dy.$$
For $s\in\mathbb{R}$, $H^s(\mathbb{R}^3)$ denotes the usual non-homogeneous Sobolev space on $\mathbb{R}^3$ and $\langle\cdot\mid\cdot\rangle_{H^s}$ denotes the usual scalar product on $H^s(\mathbb{R}^3)$. For $s\in\mathbb{R}$, $\dot H^s(\mathbb{R}^3)$ denotes the usual homogeneous Sobolev space on $\mathbb{R}^3$ and $\langle\cdot\mid\cdot\rangle_{\dot H^s}$ denotes the usual scalar product on $\dot H^s(\mathbb{R}^3)$. We denote by $\mathbb{P}$ the Leray projection operator defined by the formula
$$\mathcal{F}(\mathbb{P}f)(\xi)=\widehat{f}(\xi)-\frac{(f(\xi)\cdot\xi)}{|\xi|^2}\xi.$$
The fractional Laplacian operator $(-\Delta)^\alpha$ for a real number α is defined through the Fourier transform, namely
$$\widehat{(-\Delta)^\alpha f}(\xi)=|\xi|^{2\alpha}\hat f(\xi).$$

Finally, If $f=(f_1,f_2,f_3)$ and $g=(g_1,g_2,g_3)$ are two vector fields, we set

$$f\otimes g := (g_1 f, g_2 f, g_3 f),$$

and

$$\operatorname{div}(f\otimes g) := (\operatorname{div}(g_1 f), \operatorname{div}(g_2 f), \operatorname{div}(g_3 f)).$$

2.2. **Preliminary results.** In this section, we recall some classical results and we give a new technical lemma.

Lemma 2.1 ([1]). *Let $(s,t)\in\mathbb{R}^2$ be such that $s<3/2$ and $s+t>0$. Then, there exists a constant $C:=C(s,t)>0$, such that for all $u,v\in\dot H^s(\mathbb{R}^3)\cap\dot H^t(\mathbb{R}^3)$, we have*

$$\|uv\|_{\dot H^{s+t-\frac32}(\mathbb{R}^3)} \le C(\|u\|_{\dot H^s(\mathbb{R}^3)}\|v\|_{\dot H^t(\mathbb{R}^3)} + \|u\|_{\dot H^t(\mathbb{R}^3)}\|v\|_{\dot H^s(\mathbb{R}^3)}).$$

If $s<3/2$, $t<3/2$ and $s+t>0$, then there exists a constant $c:=c(s,t)>0$, such that

$$\|uv\|_{\dot H^{s+t-\frac32}(\mathbb{R}^3)} \le c\|u\|_{\dot H^s(\mathbb{R}^3)}\|v\|_{\dot H^t(\mathbb{R}^3)}.$$

Lemma 2.2. *Let $f\in\dot H^{s_1}(\mathbb{R}^3)\cap\dot H^{s_2}(\mathbb{R}^3)$, where $s_1<\frac32<s_2$. Then, there is a constant $c=c(s_1,s_2)$ such that*

$$\|f\|_{L^\infty(\mathbb{R}^3)} \le \|\hat f\|_{L^1(\mathbb{R}^3)} \le c\|f\|_{\dot H^{s_1}(\mathbb{R}^3)}^{\frac{s_2-\frac32}{s_2-s_1}}\|f\|_{\dot H^{s_2}(\mathbb{R}^3)}^{\frac{\frac32-s_1}{s_2-s_1}}.$$

Proof. We have

$$\begin{aligned}
\|f\|_{L^\infty(\mathbb{R}^3)} &\le \|\widehat{f}\|_{L^1(\mathbb{R}^3)}\\
&\le \int_{\mathbb{R}^3}|\widehat{f(\xi)}|d\xi\\
&\le \int_{|\xi|<\lambda}|\widehat{f(\xi)}|d\xi + \int_{|\xi|>\lambda}|\widehat{f(\xi)}|d\xi.
\end{aligned}$$

We take

$$I_1 = \int_{|\xi|<\lambda}\frac{1}{|\xi|^{s_1}}|\xi|^{s_1}|\widehat{f(\xi)}|d\xi.$$

Using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
I_1 &\le \Big(\int_{|\xi|<\lambda}\frac{1}{|\xi|^{2s_1}}d\xi\Big)^{1/2}\|f\|_{\dot H^{s_1}}\\
&\le 2\sqrt{\pi}\Big(\int_0^\lambda \frac{1}{r^{2s_1-2}}dr\Big)^{1/2}\|f\|_{\dot H^{s_1}}\\
&\le c_{s_1}\lambda^{\frac32-s_1}\|f\|_{\dot H^{s_1}}.
\end{aligned}$$

Similarly, take

$$I_2 = \int_{|\xi|>\lambda}\frac{1}{|\xi|^{s_2}}|\xi|^{s_2}|\widehat{f(\xi)}|d\xi.$$

Then we have

$$\begin{aligned}
I_2 &\le \Big(\int_{|\xi|>\lambda}\frac{1}{|\xi|^{2s_2}}d\xi\Big)^{1/2}\|f\|_{\dot H^{s_2}}\\
&\le 2\sqrt{\pi}\Big(\int_\lambda^\infty \frac{1}{r^{2s_2-2}}dr\Big)^{1/2}\|f\|_{\dot H^{s_2}}
\end{aligned}$$

$$\leq c_{s_2}\lambda^{\frac{3}{2}-s_2}\|f\|_{\dot{H}^{s_2}}.$$

Therefore,

$$\|f\|_{L^\infty} \leq A\lambda^{\frac{3}{2}-s_1} + B\lambda^{\frac{3}{2}-s_2},$$

with $A = c_{s_1}\|f\|_{\dot{H}^{s_1}}$ and $B = c_{s_2}\|f\|_{\dot{H}^{s_2}}$.

Since the function

$$\lambda \mapsto \varphi(\lambda) = A\lambda^{\frac{3}{2}-s_1} + B\lambda^{\frac{3}{2}-s_2}$$

attains its minimum at $\lambda = \lambda^* = c(s_1, s_2)(B/A)^{\frac{1}{s_2-s_1}}$. Then

$$\|f\|_{L^\infty(\mathbb{R}^3)} \leq c' A^{\frac{s_2-\frac{3}{2}}{s_2-s_1}} B^{\frac{\frac{3}{2}-s_1}{s_2-s_1}}.$$

□

We remark that, for $s_1 = 1$ and $s_2 = 2$, where $f \in \dot{H}^1(\mathbb{R}^3) \cap \dot{H}^2(\mathbb{R}^3)$, we obtain

$$\|f\|_{L^\infty(\mathbb{R}^3)} \leq \|\hat{f}\|_{L^1(\mathbb{R}^3)} \leq c\|f\|^{1/2}_{\dot{H}^1(\mathbb{R}^3)}\|f\|^{1/2}_{\dot{H}^2(\mathbb{R}^3)}. \tag{2.1}$$

3. Long time decay of (1.1) in $H^1(\mathbb{R}^3)$

In this section, we prove that if $u \in \mathcal{C}(\mathbb{R}^+, H^1(\mathbb{R}^3))$ is a global solution of (1.1), then

$$\limsup_{t\to\infty} \|u(t)\|_{H^1} = 0. \tag{3.1}$$

This proof is done in two steps.

Step 1: We shall prove that

$$\limsup_{t\to\infty} \|u(t)\|_{\dot{H}^1} = 0. \tag{3.2}$$

We have

$$\partial_t u - \Delta u + u \cdot \nabla u = -\nabla p.$$

Taking the $\dot{H}^{1/2}(\mathbb{R}^3)$ inner product of the above equality with u, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{\dot{H}^{1/2}} + \|\nabla u\|^2_{\dot{H}^{1/2}} \leq |\langle(u\cdot\nabla u) \mid u\rangle_{\dot{H}^{1/2}}|.$$

Using the fundamental property $u \cdot \nabla v = \operatorname{div}(u \otimes v)$ if $\operatorname{div} v = 0$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|^2_{\dot{H}^{1/2}} + \|\nabla u\|^2_{\dot{H}^{1/2}} &\leq |\langle(u\cdot\nabla u) \mid u\rangle_{\dot{H}^{1/2}}| \\
&\leq |\langle \operatorname{div}(u\otimes u) \mid u\rangle_{\dot{H}^{1/2}}| \\
&\leq |\langle u\otimes u \mid \nabla u\rangle_{\dot{H}^{1/2}}| \\
&\leq \|u\otimes u\|_{\dot{H}^{1/2}}\|\nabla u\|_{\dot{H}^{1/2}} \\
&\leq \|u\otimes u\|_{\dot{H}^{1/2}}\|u\|_{\dot{H}^{3/2}}.
\end{aligned}$$

Hence, from Lemma (2.1) there would exist a constant $c > 0$ such that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{\dot{H}^{1/2}} + \|u\|^2_{\dot{H}^{3/2}} \leq c\|u\|_{\dot{H}^{1/2}}\|u\|^2_{\dot{H}^{3/2}}.$$

From the equality (1.3) there would exist $t_0 > 0$ such that, for all $t \geq t_0$,

$$\|u(t)\|_{\dot{H}^{1/2}} < \frac{1}{2c}.$$

Then

$$\frac{1}{2}\frac{d}{dt}\|u\|^2_{\dot{H}^{1/2}} + \frac{1}{2}\|u\|^2_{\dot{H}^{3/2}} \leq 0, \quad \forall t \geq t_0.$$

Integrating with respect to time, we obtain

$$\|u(t)\|_{\dot H^{1/2}}^2 + \int_{t_0}^t \|u(\tau)\|_{\dot H^{3/2}}^2 d\tau \le \|u(t_0)\|_{\dot H^{1/2}}^2, \quad \forall t \ge t_0.$$

Let $s > 0$ and $c = c_s$. There exists $T_0 = T_0(s, u^0) > 0$, such that

$$\|u(T_0)\|_{\dot H^{1/2}} < \frac{1}{2c_s}.$$

Then

$$\|u(t)\|_{\dot H^{1/2}} < \frac{1}{2c_s}, \quad \forall t \ge T_0.$$

Now, for $s > 0$ we have

$$\partial_t u - \Delta u + u \cdot \nabla u = -\nabla p.$$

Taking the $\dot H^s(\mathbb{R}^3)$ inner product of the above equality with u, we obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{\dot H^s}^2 + \|\nabla u\|_{\dot H^s}^2 \le |\langle (u\cdot\nabla u) \mid u\rangle_{\dot H^s}|.$$

Using the fundamental property $u \cdot \nabla v = \operatorname{div}(u \otimes v)$ if $\operatorname{div} v = 0$, we obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|u\|_{\dot H^s}^2 + \|u\|_{\dot H^{s+1}}^2 &\le |\langle (u\cdot\nabla u) \mid u\rangle_{\dot H^s}| \\
&\le |\langle \operatorname{div}(u\otimes u)/u\rangle_{\dot H^s}| \\
&\le |\langle u\otimes u \mid \nabla u\rangle_{\dot H^s}| \\
&\le \|u\otimes u\|_{\dot H^s}\|\nabla u\|_{\dot H^s} \\
&\le \|u\otimes u\|_{\dot H^s}\|u\|_{\dot H^{s+1}} \\
&\le c_s\|u\|_{\dot H^{1/2}}\|u\|_{\dot H^{s+1}}^2.
\end{aligned}$$

Thus

$$\frac{1}{2}\frac{d}{dt}\|u\|_{\dot H^s}^2 + \frac{1}{2}\|u(t)\|_{\dot H^{s+1}}^2 \le 0, \quad \forall t \ge T_0.$$

So, for $T_0 \le t' \le t$,

$$\|u(t)\|_{\dot H^s}^2 + \int_{t'}^t \|u(\tau)\|_{\dot H^{s+1}}^2 d\tau \le \|u(t')\|_{\dot H^s}^2.$$

In particular, for $s = 1$,

$$\|u(t)\|_{\dot H^1}^2 + \int_{t'}^t \|u(\tau)\|_{\dot H^2}^2 d\tau \le \|u(t')\|_{\dot H^1}^2.$$

Then, the map $t \to \|u(t)\|_{\dot H^1}$ is decreasing on $[T_0, \infty)$ and $u \in L^2([0,\infty), \dot H^2(\mathbb{R}^3))$. Now, let $\varepsilon > 0$ be small enough. Then the L^2-energy estimate

$$\|u(t)\|_{L^2}^2 + 2\int_{T_0}^t \|\nabla u(\tau)\|_{L^2}^2 d\tau \le \|u(T_0)\|_{L^2}^2, \quad \forall t \ge T_0$$

implies that $u \in L^2([T_0,\infty), \dot H^1(\mathbb{R}^3))$ and there is a time $t_\varepsilon \ge T_0$ such that

$$\|u(t_\varepsilon)\|_{\dot H^1} < \varepsilon.$$

Since the map $t \mapsto \|u(t)\|_{\dot H^1}$ is decreasing on $[T_0, \infty)$, it follows that

$$\|u(t)\|_{\dot H^1} < \varepsilon, \quad \forall t \ge t_\varepsilon.$$

Therefore (3.2) is proved.

Step 2: In this step, we prove that

$$\limsup_{t\to\infty} \|u(t)\|_{L^2} = 0. \tag{3.3}$$

This proof is inspired by [2] and [4]. For $\delta > 0$ and a given distribution f, we define the operators $A_\delta(D)$ and $B_\delta(D)$ as follows

$$A_\delta(D)f = \mathcal{F}^{-1}(\mathbf{1}_{\{|\xi|<\delta\}}\mathcal{F}(f)), \quad B_\delta(D)f = \mathcal{F}^{-1}(\mathbf{1}_{\{|\xi|\geq\delta\}}\mathcal{F}(f)).$$

It is clear that when applying $A_\delta(D)$ (respectively, $B_\delta(D)$) to any distribution, we are dealing with its low-frequency part (respectively, high-frequency part).

Let u be a solution to (1.1). Denote by ω_δ and υ_δ, respectively, the low-frequency part and the high-frequency part of u and so on ${\omega_\delta}^0$ and ${\upsilon_\delta}^0$ for the initial data u^0. We have

$$\partial_t u - \Delta u + u\cdot\nabla u = -\nabla p.$$

Then

$$\partial_t u - \Delta u + \mathbb{P}(u\cdot\nabla u) = 0.$$

Applying the pseudo-differential operators $A_\delta(D)$ to the above equality, we obtain

$$\begin{gathered}\partial_t A_\delta(D)u - \Delta A_\delta(D)u + A_\delta(D)\mathbb{P}(u\cdot\nabla u) = 0,\\ \partial_t\omega_\delta - \Delta\omega_\delta + A_\delta(D)\mathbb{P}(u\cdot\nabla u) = 0.\end{gathered}$$

Taking the $L^2(\mathbb{R}^3)$ inner product of the above equality with $\omega_\delta(t)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega_\delta(t)\|_{L^2}^2 + \|\nabla\omega_\delta(t)\|_{L^2}^2 &\leq |\langle A_\delta(D)\mathbb{P}(u(t)\cdot\nabla u(t)) \mid \omega_\delta(t)\rangle_{L^2}|\\ &\leq |\langle A_\delta(D)\operatorname{div}(u\otimes u)(t) \mid \omega_\delta(t)\rangle_{L^2}|\\ &\leq |\langle A_\delta(D)(u\otimes u)(t) \mid \nabla\omega_\delta(t)\rangle_{L^2}|\\ &\leq |\langle (u\otimes u)(t) \mid \nabla\omega_\delta(t)\rangle_{L^2}|\\ &\leq \|u\otimes u(t)\|_{L^2}\|\nabla\omega_\delta(t)\|_{L^2}\\ &\leq \|u\otimes u(t)\|_{L^2}\|\nabla\omega_\delta(t)\|_{L^2}.\end{aligned}$$

Lemma 2.1 gives

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega_\delta(t)\|_{L^2}^2 + \|\nabla\omega_\delta(t)\|_{L^2}^2 &\leq C\|u(t)\|_{\dot H^{1/2}}\|\nabla u(t)\|_{L^2}\|\nabla\omega_\delta(t)\|_{L^2}\\ &\leq CM\|\nabla u(t)\|_{L^2}\|\nabla\omega_\delta(t)\|_{L^2}.\end{aligned}$$

with $M = \sup_{t\geq 0}\|u(t)\|_{\dot H^{1/2}}$). Integrating with respect to t, we obtain

$$\|\omega_\delta(t)\|_{L^2}^2 \leq \|{\omega_\delta}^0\|_{L^2}^2 + CM\int_0^t \|\nabla u(\tau)\|_{L^2}\|\nabla\omega_\delta(\tau)\|_{L^2}d\tau.$$

Hence, we have $\|\omega_\delta(t)\|_{L^2}^2 \leq M_\delta$ for all $t\geq 0$, where

$$M_\delta = \|{\omega_\delta}^0\|_{L^2}^2 + CM\int_0^\infty \|\nabla u(\tau)\|_{L^2}\|\nabla\omega_\delta(\tau)\|_{L^2}d\tau.$$

Using the fact that $\lim_{\delta\to 0}\|{\omega_\delta}^0\|_{L^2(\mathbb{R}^3)}^2 = 0$ and thanks to the Lebesgue-dominated convergence theorem we deduce that

$$\lim_{\delta\to 0}\int_0^\infty \|\nabla u(\tau)\|_{L^2}\|\nabla\omega_\delta(\tau)\|_{L^2}d\tau = 0. \tag{3.4}$$

Hence $\lim_{\delta\to 0} M_\delta = 0$, and thus

$$\lim_{\delta\to 0}\sup_{t\geq 0}\|\omega_\delta(t)\|_{L^2} = 0. \tag{3.5}$$

We can take time equal to ∞ in the integral (3.4) because by definition of ω_δ we have

$$\begin{aligned}\|\nabla\omega_\delta\|_{L^2} &= \|\mathcal{F}(\nabla\omega_\delta)\|_{L^2}\\ &= \||\xi|\mathbf{1}_{\{|\xi|<\delta\}}\mathcal{F}(u)\|_{L^2}\\ &\leq \||\xi|\mathcal{F}(u)\|_{L^2}\\ &\leq \|\nabla u\|_{L^2}.\end{aligned}$$

Now, using the fact that $\lim_{\delta\to 0}\|\nabla\omega_\delta(t)\|_{L^2} = 0$ almost everywhere. Then, the sequence

$$\|\nabla u(t)\|_{L^2}\|\nabla\omega_\delta(t)\|_{L^2}$$

converges point-wise to zero. Moreover, using the above computations and the energy estimate (1.4), we obtain

$$\|\nabla u(t)\|_{L^2}\|\nabla\omega_\delta(t)\|_{L^2} \leq \|\nabla u(t)\|_{L^2}^2 \in L^1(\mathbb{R}^+).$$

Thus, the integral sequence is dominated. Hence, the limiting function is integrable and one can take the time $T = \infty$ in (3.4).

Now, let us investigate the high-frequency part. For this, we apply the pseudo-differential operators $B_\delta(D)$ to the (1.1) to obtain

$$\partial_t \upsilon_\delta - \Delta\upsilon_\delta + B_\delta(D)\mathbb{P}(u\cdot\nabla u) = 0.$$

Taking the Fourier transform with respect to the space variable, we obtain

$$\begin{aligned}\partial_t|\widehat{\upsilon_\delta}(t,\xi)|^2 + 2|\xi|^2|\widehat{\upsilon_\delta}(t,\xi)|^2 &\leq 2|\mathcal{F}(B_\delta(D)\mathbb{P}(u\cdot\nabla u))(t,\xi)||\widehat{\upsilon_\delta}(t,\xi)|\\ &\leq 2|\mathcal{F}(B_\delta(D)\mathbb{P}(\operatorname{div}(u\otimes u)))(t,\xi)||\widehat{\upsilon_\delta}(t,\xi)|\\ &\leq 2|\xi||\mathcal{F}(B_\delta(D)\mathbb{P}(u\otimes u))(t,\xi)||\widehat{\upsilon_\delta}(t,\xi)|\\ &\leq 2|\xi||\mathcal{F}(u\otimes u)(t,\xi)||\widehat{\upsilon_\delta}(t,\xi)|\\ &\leq 2|\mathcal{F}(u\otimes u)(t,\xi)||\widehat{\nabla\upsilon_\delta}(t,\xi)|.\end{aligned}$$

Multiplying the obtained equation by $\exp(2t|\xi|^2)$ and integrating with respect to time, we obtain

$$|\widehat{\upsilon_\delta}(t,\xi)|^2 \leq e^{-2t|\xi|^2}|\widehat{\upsilon_\delta^0}(\xi)|^2 + 2\int_0^t e^{-2(t-\tau)|\xi|^2}|\mathcal{F}(u\otimes u)(\tau,\xi)||\widehat{\nabla\upsilon_\delta}(\tau,\xi)|d\tau.$$

Since $|\xi| > \delta$, we have

$$|\widehat{\upsilon_\delta}(t,\xi)|^2 \leq e^{-2t\delta^2}|\widehat{\upsilon_\delta^0}(\xi)|^2 + 2\int_0^t e^{-2(t-\tau)\delta^2}|\mathcal{F}(u\otimes u)(\tau,\xi)||\widehat{\nabla\upsilon_\delta}(\tau,\xi)|d\tau.$$

Integrating with respect to the frequency variable ξ and using Cauchy-Schwarz inequality, we obtain

$$\|\upsilon_\delta(t)\|_{L^2}^2 \leq e^{-2t\delta^2}\|\upsilon_{\delta^0}\|_{L^2}^2 + 2\int_0^t e^{-2(t-\tau)\delta^2}\|u\otimes u(\tau)\|_{L^2}\|\nabla\upsilon_\delta(\tau)\|_{L^2}d\tau.$$

By the definition of υ_δ, we have

$$\|\upsilon_\delta(t)\|_{L^2}^2 \leq e^{-2t\delta^2}\|u^0\|_{L^2}^2 + 2\int_0^t e^{-2(t-\tau)\delta^2}\|u\otimes u(\tau)\|_{L^2}\|\nabla u(\tau)\|_{L^2}d\tau.$$

Lemma 2.1 and the equality (1.3) yield

$$\begin{aligned}\|\upsilon_\delta(t)\|^2_{L^2(\mathbb{R}^3)} &\le e^{-2t\delta^2}\|u^0\|^2_{L^2(\mathbb{R}^3)} + C\int_0^t e^{-2(t-\tau)\delta^2}\|u(\tau)\|_{\dot H^{1/2}}\|\nabla u(\tau)\|^2_{L^2}d\tau\\ &\le e^{-2t\delta^2}\|u^0\|^2_{L^2} + CM\int_0^t e^{-2(t-\tau)\delta^2}\|\nabla u(\tau)\|^2_{L^2}d\tau,\end{aligned}$$

where $M = \sup_{t\ge 0}\|u\|_{\dot H^{1/2}}$. Hence, $\|\upsilon_\delta(t)\|^2_{L^2} \le N_\delta(t)$, where

$$N_\delta(t) = e^{-2t\delta^2}\|u^0\|^2_{L^2} + CM\int_0^t e^{-2(t-\tau)\delta^2}\|\nabla u(\tau)\|^2_{L^2}d\tau.$$

Using the energy estimate (1.4), we obtain $N_\delta \in L^1(\mathbb{R}^+)$ and

$$\int_0^\infty N_\delta(t)dt \le \frac{\|u^0\|^2_{L^2}}{2\delta^2} + \frac{CM\|u^0\|^2_{L^2}}{4\delta^2}.$$

This leads to the fact that the function $t \to \|\upsilon_\delta(t)\|^2_{L^2}$ is both continuous and Lebesgue integrable over $\mathbb{R}^+$.

Now, let $\varepsilon > 0$. At first, the inequality (3.5) implies that there exists some $\delta_0 > 0$ such that

$$\|\omega_{\delta_0}(t)\|_{L^2} \le \varepsilon/2,\ \forall\, t \ge 0.$$

Let us consider the set R_{δ_0} defined by $\mathrm{R}_{\delta_0} := \{t \ge 0,\ \|\upsilon_\delta(t)\|_{L^2(\mathbb{R}^3)} > \varepsilon/2\}$. If we denote by $\lambda_1(\mathrm{R}_{\delta_0})$ the Lebesgue measure of R_{δ_0}, we have

$$\int_0^\infty \|\upsilon_{\delta_0}(t)\|^2_{L^2(\mathbb{R}^3)}dt \ge \int_{\mathrm{R}_{\delta_0}} \|\upsilon_\delta(t)\|^2_{L^2(\mathbb{R}^3)}dt \ge (\varepsilon/2)^2\lambda_1(\mathrm{R}_{\delta_0}).$$

By doing this, we can deduce that $\lambda_1(\mathrm{R}_{\delta_0}) = T^\varepsilon_{\delta^0} < \infty$, and there exists $t^\varepsilon_{\delta^0} > T^\varepsilon_{\delta^0}$ such that

$$\|\upsilon_{\delta_0}(t^\varepsilon_{\delta^0})\|^2_{L^2} \le (\varepsilon/2)^2.$$

So, $\|u(t^\varepsilon_{\delta^0})\|_{L^2} \le \varepsilon$ and from the energy estimate (1.4) we have

$$\|u(t)\|_{L^2} \le \varepsilon,\ \ \forall t \ge t^\varepsilon_{\delta^0}.$$

This completes the proof of (3.3).

4. Generalization of Foias-Temam result in $H^1(\mathbb{R}^3)$

Fioas and Temam [5] proved an analytic property for the Navier-Stokes equations on the torus $\mathbb{T}^3 = \mathbb{R}^3/\mathbb{Z}^3$. Here, we give a similar result on the whole space $\mathbb{R}^3$.

Theorem 4.1. *We assume that $u^0 \in H^1(\mathbb{R}^3)$. Then, there exists a time T that depends only on the $\|u^0\|_{H^1(\mathbb{R}^3)}$, such that*

- (1.1) *possesses on $(0,T)$ a unique regular solution u such that the function $t \mapsto e^{t|D|}u(t)$ is continuous from $[0,T]$ into $H^1(\mathbb{R}^3)$.*
- *If $u \in \mathcal{C}(\mathbb{R}^+, H^1(\mathbb{R}^3))$ is a global and bounded solution to* (1.1)*, then there are $M \ge 0$ and $t_0 > 0$ such that*

$$\|e^{t_0|D|}u(t)\|_{H^1(\mathbb{R}^3)} \le M, \quad \forall t \ge t_0.$$

Before proving this Theorem, we need the following Lemmas.

Lemma 4.2. *Let $t \mapsto e^{t|D|}u$ belong to $\dot H^1(\mathbb{R}^3) \cap \dot H^2(\mathbb{R}^3)$. Then*

$$\|e^{t|D|}(u\cdot\nabla v)\|_{L^2(\mathbb{R}^3)} \le \|e^{t|D|}u\|^{1/2}_{H^1(\mathbb{R}^3)}\|e^{t|D|}u\|^{1/2}_{H^2(\mathbb{R}^3)}\|e^{t|D|}v\|_{H^1(\mathbb{R}^3)}.$$

Proof. We have

$$\begin{aligned}
\|e^{t|D|}(u\cdot\nabla v)\|_{L^2}^2 &= \int_{\mathbb{R}^3} e^{2t|\xi|}|\widehat{u\cdot\nabla v}(\xi)|^2 d\xi\\
&\le \int_{\mathbb{R}^3} e^{2t|\xi|}\Big(\int_{\mathbb{R}^3}|\hat{u}(\xi-\eta)||\widehat{\nabla v}(\eta)|d\eta\Big)^2 d\xi\\
&\le \int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3} e^{t|\xi|}|\hat{u}(\xi-\eta)||\widehat{\nabla v}(\eta)|d\eta\Big)^2 d\xi.
\end{aligned}$$

Using the inequality $e^{|\xi|}\le e^{|\xi-\eta|}e^{|\eta|}$, we obtain

$$\begin{aligned}
\|e^{t|D|}(u\cdot\nabla v)\|_{L^2}^2 &\le \int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3} e^{t|\xi-\eta|}|\hat{u}(\xi-\eta)|e^{t|\eta|}|\widehat{\nabla v}(\eta)|d\eta\Big)^2 d\xi\\
&\le \int_{\mathbb{R}^3}\Big(\int_{\mathbb{R}^3}\big(e^{t|\xi-\eta|}|\hat{u}(\xi-\eta)|\big)\big(e^{t|\eta|}|\eta||\hat{v}(\eta)|\big)d\eta\Big)^2 d\xi\\
&\le \Big(\int_{\mathbb{R}^3} e^{t|\xi|}|\hat{u}(\xi)|d\xi\Big)^2\|e^{t|D|}\nabla v\|_{L^2}^2.
\end{aligned}$$

Hence, for $f=\mathcal{F}^{-1}(e^{t|\xi|}|\hat{u}(\xi)|)\in\dot{H}^1(\mathbb{R}^3)\cap\dot{H}^2(\mathbb{R}^3)$, inequality (2.1) gives

$$\begin{aligned}
\|e^{t|D|}(u\cdot\nabla v)\|_{L^2} &\le \|e^{t|D|}u\|_{\dot{H}^1}^{1/2}\|e^{t|D|}u\|_{\dot{H}^2}^{1/2}\|e^{t|D|}\nabla v\|_{L^2}\\
&\le \|e^{t|D|}u\|_{\dot{H}^1}^{1/2}\|e^{t|D|}u\|_{\dot{H}^2}^{1/2}\|e^{t|D|}v\|_{\dot{H}^1}\\
&\le \|e^{t|D|}u\|_{H^1}^{1/2}\|e^{t|D|}u\|_{H^2}^{1/2}\|e^{t|D|}v\|_{H^1}.
\end{aligned}$$

□

Lemma 4.3. *Let $t\mapsto e^{t|D|}u\in\dot{H}^1(\mathbb{R}^3)\cap\dot{H}^2(\mathbb{R}^3)$. Then*

$$\big|\langle e^{t|D|}(u\cdot\nabla v)\mid e^{t|D|}w\rangle_{H^1}\big|\le\|e^{t|D|}u\|_{H^1}^{1/2}\|e^{t|D|}u\|_{H^2}^{1/2}\|e^{t|D|}v\|_{H^1}\|e^{t|D|}w\|_{H^2}.$$

Proof. We have

$$\begin{aligned}
\langle u\cdot\nabla v\mid w\rangle_{H^1} &= \sum_{|j|=1}\langle\partial_j(u\cdot\nabla v)\mid\partial_j w\rangle_{L^2}\\
&= -\sum_{|j|=1}\langle u\cdot\nabla v\mid\partial_j^2 w\rangle_{L^2}\\
&= -\langle u\cdot\nabla v\mid\Delta w\rangle_{L^2}.
\end{aligned}$$

Then

$$\begin{aligned}
\big|\langle e^{t|D|}(u\cdot\nabla v)\mid e^{t|D|}w\rangle_{H^1}\big| &= \big|\langle e^{t|D|}(u\cdot\nabla v)\mid e^{t|D|}\Delta w\rangle_{L^2}\big|\\
&\le \|e^{t|D|}(u\cdot\nabla v)\|_{L^2}\|e^{t|D|}\Delta w\|_{L^2}\\
&\le \|e^{t|D|}(u\cdot\nabla v)\|_{L^2}\|e^{t|D|}w\|_{\dot{H}^2}\\
&\le \|e^{t|D|}(u\cdot\nabla v)\|_{L^2}\|e^{t|D|}w\|_{H^2}.
\end{aligned}$$

Finally, using Lemma 4.2, we obtain the desired result. □

Proof of Theorem 4.1. We have

$$\partial_t u-\Delta u+u\cdot\nabla u=-\nabla p.$$

Applying the fourier transform to the last equation and multiplying by $\overline{\widehat{u}}$, we obtain

$$\partial_t\widehat{u}\cdot\overline{\widehat{u}}+|\xi|^2|\widehat{u}|^2=-(\widehat{u\cdot\nabla u})\cdot\overline{\widehat{u}}.$$

Then

$$\partial_t|\widehat{u}|^2 + 2|\xi|^2|\widehat{u}|^2 = -2\,\mathrm{Re}((\widehat{u\cdot\nabla u})\cdot\widehat{u}).$$

Multiplying the above equation by $(1+|\xi|^2)e^{2t|\xi|}$, we obtain

$$(1+|\xi|^2)e^{2t|\xi|}\partial_t|\widehat{u}|^2 + 2(1+|\xi|^2)|\xi|^2e^{2t|\xi|}|\widehat{u}|^2 = -2\,\mathrm{Re}((\widehat{u\cdot\nabla u})\cdot\widehat{u})(1+|\xi|^2)e^{2t|\xi|}.$$

Integrating with respect to ξ, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^3}(1+|\xi|^2)e^{2t|\xi|}\partial_t|\widehat{u}(\xi)|^2d\xi + 2\int_{\mathbb{R}^3}(1+|\xi|^2)|\xi|^2e^{2t|\xi|}|\widehat{u}(\xi)|^2d\xi\\ &= -2\,\mathrm{Re}\int_{\mathbb{R}^3}((\widehat{u\cdot\nabla u})\cdot\widehat{u})(1+|\xi|^2)e^{2t|\xi|}d\xi.\end{aligned}$$

Thus

$$\langle e^{t|D|}\partial_t u/e^{t|D|}u\rangle_{H^1} + 2\|e^{t|D|}\nabla u\|^2_{H^1(\mathbb{R}^3)} = -2Re\langle e^{t|D|}(u\cdot\nabla u)\mid e^{t|D|}u\rangle_{H^1}. \tag{4.1}$$

Therefore,

$$\begin{aligned}\langle e^{t|D|}u'(t)\mid e^{t|D|}u(t)\rangle_{H^1} &= \langle (e^{t|D|}u(t))' - |D|e^{t|D|}u(t)\mid e^{t|D|}u(t)\rangle_{H^1}\\ &= \frac{1}{2}\frac{d}{dt}\|e^{t|D|}u\|^2_{H^1} - \langle e^{t|D|}|D|u(t)\mid e^{t|D|}u(t)\rangle_{H^1}\\ &\geq \frac{1}{2}\frac{d}{dt}\|e^{t|D|}u\|^2_{H^1} - \|e^{t|D|}u\|_{H^1}\|e^{t|D|}u\|_{H^2}.\end{aligned}$$

Using the Young inequality, we obtain

$$\frac{d}{dt}\|e^{t|D|}u\|^2_{H^1} - 2\|e^{t|D|}u\|^2_{H^1} - \frac{1}{2}\|e^{t|D|}u\|^2_{H^2} \leq 2\langle e^{t|D|}u'(t)\mid e^{t|D|}u(t)\rangle_{H^1}. \tag{4.2}$$

Hence, using Lemma 4.3 and Young inequality the right hand of (4.1) satisfies

$$\begin{aligned}|-2\,\mathrm{Re}\langle e^{t|D|}(u\cdot\nabla u)\mid e^{t|D|}u\rangle_{H^1}| &\leq 2\|e^{t|D|}u\|^{3/2}_{H^1}\|e^{t|D|}u\|^{3/2}_{H^2}\\ &\leq \frac{3}{4}\|e^{t|D|}u\|^2_{H^2} + \frac{c_1}{2}\|e^{t|D|}u\|^6_{H^1},\end{aligned}$$

where c_1 is a positive constant. Then, (4.1) yields

$$\langle e^{t|D|}u'(t)\mid e^{t|D|}u(t)\rangle_{H^1} + 2\|e^{t|D|}\nabla u\|^2_{H^1} \leq \frac{3}{4}\|e^{t|D|}u\|^2_{H^2} + \frac{c_1}{2}\|e^{t|D|}u\|^6_{H^1}. \tag{4.3}$$

Hence, using (4.2)–(4.3), we obtain

$$\frac{d}{dt}\|e^{t|D|}u\|^2_{H^1} - 2\|e^{t|D|}u\|^2_{H^1} - 2\|e^{t|D|}u\|^2_{H^2} + 4\|e^{t|D|}\nabla u\|^2_{H^1} \leq c_1\|e^{t|D|}u\|^6_{H^1}.$$

The equality $\|e^{t|D|}u\|^2_{H^2} = \|e^{t|D|}u\|^2_{H^1} + \|e^{t|D|}\nabla u\|^2_{H^1}$ yields

$$\begin{aligned}\frac{d}{dt}\|e^{t|D|}u\|^2_{H^1} + 2\|e^{t|D|}\nabla u\|^2_{H^1} &\leq 4\|e^{t|D|}u\|^2_{H^1} + c_1\|e^{t|D|}u\|^6_{H^1}\\ &\leq c_2 + 2c_1\|e^{t|D|}u\|^6_{H^1}.\end{aligned}$$

where c_2 is a positive constant. Finally, we obtain

$$y(t) \leq y(0) + K_1\int_0^t y^3(s)ds.$$

where

$$y(t) = 1 + \|e^{t|D|}u(t)\|^2_{H^1} \quad \text{and} \quad K_1 = 2c_1 + c_2.$$

Let

$$T_1 = \frac{2}{K_1 y^2(0)}$$

and $0 < T \leq T^*$ be such that $T = \sup\{t \in [0, T^*) \mid \sup_{0 \leq s \leq t} y(s) \leq 2y(0)\}$. Hence for $0 \leq t \leq \min(T_1, T)$, we have

$$\begin{aligned} y(t) &\leq y(0) + K_1 \int_0^t y^3(s) ds \\ &\leq y(0) + K_1 \int_0^t 8y^3(0) ds \\ &\leq \big(1 + K_1 8 T_1 y^2(0)\big)\, y(0). \end{aligned}$$

Taking $1 + K_1 8 T_1 y^2(0) < 2$, we obtain $T > T_1$. Then $y(t) \leq 2y(0)$ for all $t \in [0, T_1]$. This shows that $t \mapsto e^{t|D|} u(t) \in H^1(\mathbb{R}^3)$ for all $t \in [0, T_1]$. In particular

$$\|e^{T_1|D|} u(T_1)\|_{H^1}^2 \leq 2 + 2\|u_0\|_{H^1}^2.$$

Now, from the hypothesis, we assume that there exists $M_1 > 0$ such that

$$\|u(t)\|_{H^1} \leq M_1 \quad \text{for all } t \geq 0.$$

Define the system

$$\begin{gathered} \partial_t w - \Delta w + w.\nabla w = -\nabla p \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ \operatorname{div} w = 0 \quad \text{in } \mathbb{R}^+ \times \mathbb{R}^3 \\ w(0) = u(T) \quad \text{in } \mathbb{R}^3, \end{gathered}$$

where $w(t) = u(T + t)$. Using a similar technique, we can prove that there exists $T_2 = \frac{2}{K_1}(1 + M_1^2)^{-2}$ such that

$$y(t) = 1 + \|e^{t|D|} w(t)\|_{H^1}^2 \leq 2(1 + M_1^2), \quad \forall t \in [0, T_2].$$

This implies $1 + \|e^{t|D|} u(T + t)\|_{H^1}^2 \leq 2(1 + M_1^2)$. Hence, for $t = T_2$ we have

$$\|e^{T_2|D|} u(T + T_2)\|_{H^1}^2 \leq 2(1 + M_1^2).$$

Since $t = T + T_2 \geq T_2$ for all $T \geq 0$, we obtain

$$\|e^{T_2|D|} u(t)\|_{H^1}^2 \leq 2(1 + M_1^2), \quad \forall t \geq T_2.$$

Then

$$\|e^{T_2|D|} u(t)\|_{H^1}^2 \leq 2(1 + M_1^2), \quad \forall t \geq T_2,$$

where

$$T_2 = T_2(M_1) = \frac{2}{K_1}(1 + M_1^2)^{-2}.$$

$\square$

5. Proof of the main result

In this section, we prove Theorem 1.1. This proof uses the results of sections 3 and 4.

Let $u \in \mathcal{C}(\mathbb{R}^+, H^1_{a,\sigma}(\mathbb{R}^3))$. As $H^1_{a,\sigma}(\mathbb{R}^3) \hookrightarrow H^1(\mathbb{R}^3)$, then $u \in \mathcal{C}(\mathbb{R}^+, H^1(\mathbb{R}^3))$. Applying Theorem 4.1, there exist $t_0 >$ such that

$$\|e^{t_0|D|}u(t)\|_{H^1} \leq c_0 = 2 + M_1^2, \quad \forall t \geq t_0, \tag{5.1}$$

where $t_0 = \frac{2}{K_1}(1+M_1^2)^{-2}$. Let $a > 0$, $\beta > 0$. Then there exists $c_3 \geq 0$ such that

$$ax^{1/\sigma} \leq c_3 + \beta x, \quad \forall x \geq 0.$$

Indeed, $\frac{1}{\sigma} + \frac{\sigma-1}{\sigma} = \frac{1}{p} + \frac{1}{q} = 1$. Using the Young inequality, we obtain

$$\begin{aligned} ax^{1/\sigma} &= a\beta^{\frac{-1}{\sigma}}(\beta^{1/\sigma}x^{1/\sigma}) \\ &\leq \frac{(a\beta^{\frac{-1}{\sigma}})^q}{q} + \frac{(\beta^{1/\sigma}x^{1/\sigma})^p}{p} \\ &\leq c_3 + \frac{\beta x}{\sigma} \leq c_3 + \beta x, \end{aligned}$$

where $c_3 = \frac{\sigma-1}{\sigma} a^{\frac{\sigma}{\sigma-1}} \beta^{\frac{1}{1-\sigma}}$.

Take $\beta = \frac{t_0}{2}$, using (5.1) and the Cauchy Schwarz inequality, we have

$$\begin{aligned} \|u(t)\|^2_{H^1_{a,\sigma}} &= \|e^{a|D|^{1/\sigma}}u(t)\|^2_{H^1} \\ &= \int (1+|\xi|^2)e^{2a|\xi|^{1/\sigma}}|\widehat{u}(t,\xi)|^2 d\xi \\ &\leq \int (1+|\xi|^2)e^{2(c_3+\beta|\xi|)}|\widehat{u}(t,\xi)|^2 d\xi \\ &\leq \int (1+|\xi|^2)e^{2c_3}e^{t_0|\xi|}|\widehat{u}(t,\xi)|^2 d\xi \\ &\leq e^{2c_3}\Big(\int (1+|\xi|^2)|\widehat{u}(t,\xi)|^2 d\xi\Big)^{1/2}\Big(\int (1+|\xi|^2)e^{2t_0|\xi|}|\widehat{u}(t,\xi)|^2 d\xi\Big)^{1/2} \\ &\leq e^{2c_3}\|u\|^{1/2}_{H^1}\|e^{t_0|D|}u(t)\|^{1/2}_{H^1} \\ &\leq c\|u\|^{1/2}_{H^1}, \end{aligned}$$

where $c = e^{2c_3}c_0^{1/2}$. Using the inequality (3.1), we obtain

$$\limsup_{t\to\infty} \|e^{a|D|^{1/\sigma}}u(t)\|_{H^1} = 0.$$

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