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# MULTIPLE SOLUTIONS FOR QUASILINEAR ELLIPTIC EQUATIONS WITH SIGN-CHANGING POTENTIAL

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Abstract. In this article, we study the quasilinear elliptic equation

$$-\Delta_p u - (\Delta_p u^2)u + V(x)|u|^{p-2}u = g(x,u), \quad x \in \mathbb{R}^N,$$

where the potential $V(x)$ and the nonlinearity $g(x,u)$ are allowed to be sign-changing. Under some suitable assumptions on $V$ and $g$, we obtain the multiplicity of solutions by using minimax methods.

## 1. Introduction

In this article, we are concerned with the multiplicity of nontrivial solutions for the quasilinear elliptic equation

$$-\Delta_p u - (\Delta_p u^2)u + V(x)|u|^{p-2}u = g(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the $p$-Laplacian operator with $2 \le p < N$, $N \ge 3$, $V \in C(\mathbb{R}^N)$ and $g \in C(\mathbb{R}^N \times \mathbb{R})$ satisfy superlinear growth at infinity.

In recent years, there has been increasingly interest in the study of the quasilinear Schrödinger equation

$$-\Delta u - \Delta(u^2)u + V(x)u = g(x,u), \quad x \in \mathbb{R}^N. \tag{1.2}$$

Such equations are related to the existence of solitary wave solutions for quasilinear Schrödinger equations

$$i\partial_t \psi = -\Delta\psi + W(x)\psi - g(x,|\psi|^2)\psi - \kappa\Delta[\rho(|\psi|^2)]\rho'(|\psi|^2)\psi, \tag{1.3}$$

where $\psi : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$, $W(x)$ is a given potential, $\kappa$ is a real constant and $\rho, g$ are real functions. Quasilinear Schrödinger equations of the type (1.3) with $\kappa > 0$ arise in various branches of mathematical physics and have been derived as models of several physical phenomena, such as superfluid film equations in plasma physics [11] and the fluid mechanics in condensed matter theory [5, 12, 19, 23, 17] and so on. The related Schrödinger equations for $\kappa = 0$ have been extensively studied (see e.g. [4, 10, 9] and their references therein) in the last few decades. For $\kappa > 0$, the existence of a positive ground state solution has been proved in [18] by using a constrained minimization argument, which gives a solution of (1.2) with an unknown Lagrange multiplier $\lambda$ in front of nonlinear term. In [14], the authors

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establish the existence of ground states of soliton type solutions by a minimization argument. In [15], by a change of variables the quasilinear problem was transformed to a semilinear one and Orlicz space framework was used as the working space, and they were able to prove the existence of positive solutions of (1.2) by the mountain-pass theorem. The same method of change of variables was used recently also in [8], but the usual Sobolev space $H^1(\mathbb{R}^N)$ framework was used as the working space and they studied different class of nonlinearities. In [16], it was established the existence of both one-sign and nodal ground states of soliton type solutions by the Nehari method. In [27], where the potential $V(x)$ and $g$ is allowed to be sign-changing, $g$ is of superlinear growth at infinity in $u$, the author obtain the existence of infinitely many nontrivial solutions by using dual approach and symmetric mountain pass theorem.

Recently, there has been a lot of results on existence and multiplicity for problem (1.1). The existence of nontrivial weak solutions of (1.1) has been proved in [21] by using minimax methods and method of Changes of variable, where $V$ is a positive continuous potential bounded away from zero. In [2], the authors use variational method together with the Lusternick-Schnirelmann category theory to get the existence and multiplicity of nontrivial weak solutions, where $V$ is also a positive continuous potential bounded away from zero. In [1], the authors established the multiplicity of positive weak solutions through using minimax methods, where the potential $V$ is of form $V(x) = \lambda A(x) + 1$ and $A(x)$ is a nonnegative continuous function. The other related results can be seen in [3] and the references therein.

In the above mentioned paper, the potential $V$ is always assumed to be positive or vanish at infinity except [27]. In the present paper we shall consider problem (1.1) with non-constant and sign-changing potential. We will investigate the existence of at least two solutions and the existence of infinitely many nontrivial solutions of (1.1) through using the Ekeland's variational principle, variant mountain pass theorem and symmetric mountain pass theorem. Our main results improve the corresponding theorems in [27] in some sense.

For stating our main result, we make the following assumptions on the potential function $V(x)$

- (A1) $V \in C(\mathbb{R}^N)$ and $\inf_{x\in\mathbb{R}^N} V(x) > -\infty$, and there exists a constant $d_0 > 0$ such that
$$\lim_{|y|\to\infty} \operatorname{meas}(\{x \in \mathbb{R}^N : |x-y| \le d_0,\ V(x) \le M\}) = 0, \quad \forall M > 0.$$

Inspired by [13, 27], we can find a constant $V_0 > 0$ such that $\overline{V}(x) = V(x) + V_0 \ge 1$ for all $x \in \mathbb{R}^N$, and let $\overline{g}(x,u) = g(x,u) + V_0|u|^{p-2}u$, for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$. Then it is easy to show the following Lemma.

**Lemma 1.1.** *Equation* (1.1) *is equivalent to the problem*
$$-\Delta_p u - (\Delta_p u^2)u + \overline{V}(x)|u|^{p-2}u = \overline{g}(x,u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

In what follows, we impose some assumptions on $\overline{g}$ and its primitive $\overline{G}(x,t) = \int_0^t \overline{g}(x,s)\mathrm{d}s$ as follows:

- (A2) $\overline{g} \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist constant $C > 0$ and $2p < q < 2p^*$ such that
$$|\overline{g}(x,u)| \le C(|u|^{p-1} + |u|^{q-1}), \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R};$$

(A3) $\lim_{|u|\to\infty} \overline{G}(x,u)/|u|^{2p} = +\infty$ uniformly in $x \in \mathbb{R}^N$, and there exists $r_0 > 0$, $\tau < p$ and $C_0$ such that $\inf \overline{G}(x,u) \geq C_0|u|^\tau > 0$, for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$, $|u| \geq r_0$;

(A4) $\widetilde{\overline{G}}(x,u) = \frac{1}{2p}u\overline{g}(x,u) - \overline{G}(x,u) \geq 0$, There exist $C_1$ and $\sigma > \frac{2N}{N+p}$ such that

$$\Big(\overline{G}(x,u)\Big)^\sigma \leq C_1|u|^{p\sigma}\widetilde{\overline{G}}(x,u) \quad \text{for all } (x,u) \in \mathbb{R}^N \times \mathbb{R},\ |u| \geq r_0;$$

(A5) There exist $\mu > 2p$ and $C_2 > 0$ such that $\mu\overline{G}(x,u) \leq u\overline{g}(x,u) + C_2|u|^p$, for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$;

(A6) There exist $\mu > 2p$ and $r_1 > 0$ such that $\mu\overline{G}(x,u) \leq u\overline{g}(x,u)$, for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$ with $|u| \geq r_1$;

(A7) $\lim_{|u|\to 0} \frac{\overline{G}(x,u)}{|u|^p} = 0$ uniformly in $x \in \mathbb{R}^N$;

(A8) $\overline{g}(x,-u) = -\overline{g}(x,u)$ for all $(x,u) \in \mathbb{R}^N \times \mathbb{R}$.

**Remark 1.2.** It follows from (A3) and (A4) that

$$\widetilde{\overline{G}}(x,u) \geq \frac{1}{C_1}\Big(\frac{\overline{G}(x,u)}{|u|^p}\Big)^\sigma \to \infty \tag{1.5}$$

uniformly for $x \in \mathbb{R}^N$ as $|u| \to \infty$.

Now, we state our main results.

**Theorem 1.3.** *Suppose that conditions* (A1)–(A4) *are satisfied. Then* (1.1) *possesses at least two solutions.*

**Theorem 1.4.** *Suppose that conditions* (A1)–(A3), (A5) *are satisfied. Then* (1.1) *possesses at least two solutions.*

From (A2) and (A6), it is easy to verified that (A5) holds. Thus we have the following corollary.

**Corollary 1.5.** *Suppose that conditions* (A1)–(A3), (A6) *are satisfied. Then* (1.1) *possesses at least two solutions.*

If we add the hypothesis (A8), we can obtain the infinitely many solutions for problem (1.1).

**Theorem 1.6.** *Assume that* (A1)–(A4), (A8) *are satisfied. Then* (1.1) *possesses infinitely many nontrivial solutions.*

**Theorem 1.7.** *Assume that* (A1)–(A3), (A5), (A7), (A8) *are satisfied. Then* (1.1) *possesses infinitely many nontrivial solutions.*

**Corollary 1.8.** *Assume that* (A1)–(A3), (A6)–(A8) *are satisfied. Then* (1.1) *possesses infinitely many nontrivial solutions.*

**Remark 1.9.** If we use the following assumption instead of (A2):

(A2') $\overline{g} \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist constant $C_3 > 0$, $p < r \leq 2p$ and $2p < q < 2p^*$ such that

$$|\overline{g}(x,u)| \leq C_3(|u|^{r-1} + |u|^{q-1}),\ \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

Then the assumption (A7) is not needed. Thus we can get the similar results as Theorem 1.3–1.7. Here we omit their statements.

## 2. Variational setting and preliminary results

As usual, for $1 \le s \le +\infty$, we let

$$\|u\|_s = (\int_{\mathbb{R}^N} |u(x)|^s)^{1/s}, u \in L^s(\mathbb{R}^N).$$

We denote $C$, $C_i$ $(i = 0, 1, 2, \cdots)$ as the various positive constants throughout this paper. Throughout this section, we make the following assumption on $\overline{V}$ instead of (A1):

(A1') $\overline{V} \in C(\mathbb{R}^N, \mathbb{R})$, and $\inf_{x\in\mathbb{R}^N} \overline{V}(x) > 0$, and there exists a constant $d_0 > 0$ such that

$$\lim_{|y|\to\infty} \text{meas}(\{x \in \mathbb{R}^N : |x - y| \le d_0,\ \overline{V}(x) \le M\}) = 0, \quad \forall M > 0.$$

Let

$$E := \{u \in W^{1,p}(\mathbb{R}^N, \mathbb{R}) : \int_{\mathbb{R}^N} \overline{V}(x)|u|^p \mathrm{d}x < \infty\},$$

which is endowed with the norm

$$\|u\|_E = \Big(\int_{\mathbb{R}^N} (|\nabla u|^p + \overline{V}(x)|u|^p) \,\mathrm{d}x\Big)^{1/p}.$$

Under assumption (A1'), the embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ is continuous for $s \in [p, p^*)$, and $E \hookrightarrow L^s_{loc}(\mathbb{R}^N)$ is compact for $s \in [p, p^*)$, i.e., there are constants $a_s > 0$ such that

$$\|u\|_s \le a_s\|u\|_E, \quad \forall u \in E,\ s \in [p, p^*).$$

Furthermore, under assumption (A1'), we have the following compactness embedding lemma due to [7, 6, 26].

**Lemma 2.1.** *Under assumption* (A1')*, the embedding from $E$ into $L^s(\mathbb{R}^N)$ is compact for $p \le s < p^*$.*

The energy functional $J : E \to \mathbb{R}$ formally can be given by

$$\begin{aligned} J(u) &= \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p \mathrm{d}x + \frac{2^{p-1}}{p}\int_{\mathbb{R}^N} |\nabla u|^p|u|^p \mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} \overline{V}(x)|u|^p \mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N} \overline{G}(x, u)\mathrm{d}x \\ &= \frac{1}{p}\int_{\mathbb{R}^N} (1 + 2^{p-1}|u|^p)|\nabla u|^p \mathrm{d}x + \frac{1}{p}\int_{\mathbb{R}^N} \overline{V}(x)|u|^p \mathrm{d}x - \int_{\mathbb{R}^N} \overline{G}(x, u)\mathrm{d}x. \end{aligned}$$

Since the integral $\int_{\mathbb{R}^N} |\nabla u|^p|u|^p \mathrm{d}x$ may be infinity, $J$ is not well defined in general in $E$. To overcome this difficulty, we apply an argument developed by [15]. We make the change of variables by $v = f^{-1}(u)$, where $f$ is defined by

$$f'(t) = \frac{1}{[1 + 2^{p-1}|f(t)|^p]^{1/p}}, \quad t \in [0, \infty),$$

and

$$f(-t) = -f(t),\ \mathrm{t} \in (-\infty, 0].$$

Some properties of the function $f$ are listed as follows.

**Lemma 2.2.** *Concerning the function $f(t)$ and its derivative satisfy the following properties:*

(1) *$f$ is uniquely defined, $C^2$ and invertible;*

(2) $|f'(t)| \le 1$ *for all* $t \in \mathbb{R}$;
(3) $|f(t)| \le |t|$ *for all* $t \in \mathbb{R}$;
(4) $\frac{f(t)}{t} \to 1$ *as* $t \to 0$;
(5) $\frac{f(t)}{\sqrt{t}} \to a > 0$ *as* $t \to +\infty$;
(6) $\frac{f(t)}{2} \le tf'(t) \le f(t)$ *for all* $t > 0$;
(7) $\frac{f^2(t)}{2} \le tf'(t)f(t) \le f^2(t)$ *for all* $t \in \mathbb{R}$;
(8) $|f(t)| \le 2^{\frac{1}{2p}}|t|^{\frac{1}{2}}$ *for all* $t \in \mathbb{R}$;
(9) *there exists a positive constant* $C_4$ *such that*

$$|f(t)| \ge \begin{cases} C_4|t|, & |t| \le 1, \\ C_4|t|^{\frac{1}{2}}, & |t| \ge 1; \end{cases}$$

(10)

$$f^2(st) \le \begin{cases} sf^2(t), & 0 \le s \le 1, \\ s^2 f^2(t), & s \ge 1; \end{cases}$$

(11) $|f(t)f'(t)| \le \frac{1}{2^{\frac{p-1}{p}}}$.

*Proof.* We only prove properties (10). Since the function $(f^2)'' > 0$, in $[0, +\infty)$, and therefore item $f^2$ is strictly convex,

$$f^2((1-s)0 + st) \le (1-s)f^2(0) + sf^2(t) = sf^2(t).$$

In order to prove $f^2(st) \le s^2 f^2(t)$, when $s \ge 1$. We notice that, since $f'' \le 0$ in $[0, +\infty)$, we have that $f'$ is non-increasing in this interval. For any $t \ge 0$ fixed we consider the function $h(s) := f(st) - sf(t)$ defined for $s \ge 1$. We have that $h'(s) := tf'(st) - f(t) \le tf'(t) - f(t) \le 0$, by $(f_6)$. Since $h(1) = 0$ we consider that $h(s) \le 0$ for any $s \ge 1$; that is, $f(st) \le sf(t)$ for any $t \ge 0$ and $s \ge 1$. Thus the proof is complete. □

By the change of variables, from $J(u)$ we can define the following functional

$$I(v) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla v|^p + \overline{V}(x)|f(v)|^p)\mathrm{d}x - \int_{\mathbb{R}^N} \overline{G}(x, f(v))\mathrm{d}x, \tag{2.1}$$

which is well defined on the space $E$. From (A2), we have

$$\overline{G}(x, u) \le C(|u|^p + |u|^q), \quad \text{for all } (x, u) \in \mathbb{R}^N \times \mathbb{R}.$$

By standard arguments, it is easy to show that $I \in C^1(E, \mathbb{R})$, and

$$\begin{aligned} \langle I'(v), w\rangle = &\int_{\mathbb{R}^N} |\nabla v|^{p-2}\nabla v \nabla w \mathrm{d}x + \int_{\mathbb{R}^N} \overline{V}(x)|f(v)|^{p-2} f(v)f'(v)w\mathrm{d}x \\ &- \int_{\mathbb{R}^N} \overline{g}(x, f(v))f'(v)w\mathrm{d}x, \end{aligned} \tag{2.2}$$

for any $w \in E$. Moreover, the critical points of $I$ are the weak solutions of the following equation

$$-\Delta_p v + \overline{V}(x)|f(v)|^{p-2}f(v)f'(v) = \overline{g}(x, f(v))f'(v).$$

We also observe that if $v$ is a critical point of the functional $I$, then $u = f(v)$ is a critical point of the functional $J$, i.e. $u = f(v)$ is a solution of problem (1.4).

Next, we present the relationship between the norm $\|u\|_E$ in $E$ and $\int_{\mathbb{R}^N}(|\nabla u|^p + \overline{V}(x)|f(u)|^p)\mathrm{d}x$.

**Proposition 2.3.** *There exist two constants $C_5 > 0$ and $\rho > 0$ such that*

$$\int_{\mathbb{R}^N} (|\nabla u|^p + \overline{V}(x)|f(u)|^p)\mathrm{d}x \geq C_5\|u\|_E^p, \quad \forall\, u \in \{u \in E : \|u\|_E \leq \rho\}.$$

*Proof.* Suppose by contradiction, there exists a sequence $\{u_n\} \subset E$ verifying $u_n \neq 0$, for all $n \in \mathbb{N}$ and $\|u_n\|_E \to 0$, such that

$$\int_{\mathbb{R}^N} \Big(\frac{|\nabla u_n|^p}{\|u_n\|_E^p} + \overline{V}(x)\frac{|f(u_n)|^p}{\|u_n\|_E^p}\Big)\mathrm{d}x \to 0. \tag{2.3}$$

Set $v_n = u_n/\|u_n\|_E$, then $\|v_n\|_E = 1$, passing to a subsequence, by Lemma 2.1, we may assume that $v_n \rightharpoonup v$ in $E$, $v_n \to v$ in $L^s(\mathbb{R}^N)$ for $s \in [p, p^*)$, $v_n \to v$ a.e $\mathbb{R}^N$. Therefore, (2.3) implies that

$$\int_{\mathbb{R}^N} |\nabla v_n|^p\mathrm{d}x \to 0, \quad \int_{\mathbb{R}^N} \overline{V}(x)\frac{|f(u_n)|^p}{\|u_n\|_E^p}\mathrm{d}x \to 0, \quad \int_{\mathbb{R}^N} \overline{V}(x)|v_n|^p\mathrm{d}x \to 1. \tag{2.4}$$

Similar to the idea in [25], we assert that for each $\varepsilon > 0$, there exists $C_6 > 0$ independent of $n$ such that $\mathrm{meas}(\Omega_n) < \varepsilon$, where $\Omega_n := \{x \in \mathbb{R}^N : |u_n(x)| \geq C_6\}$. Otherwise, there is an $\varepsilon_0 > 0$ and a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that for any positive integer $k$,

$$\mathrm{meas}(\{x \in \mathbb{R}^N : |u_{n_k}(x)| \geq k\}) \geq \varepsilon_0 > 0.$$

Set $\Omega_{n_k} := \{x \in \mathbb{R}^N : |u_{n_k}(x)| \geq k\}$. By (3) and (9) of Lemma 2.2, we have

$$\begin{aligned} \|u_{n_k}\|_E^p &\geq \int_{\mathbb{R}^N} \overline{V}(x)|u_{n_k}|^p\mathrm{d}x \geq \int_{\mathbb{R}^N} \overline{V}(x)|f(u_{n_k})|^p\mathrm{d}x \\ &\geq \int_{\Omega_{n_k}} \overline{V}(x)|f(u_{n_k})|^p\mathrm{d}x \geq C_6 k^{\frac{p}{2}}\varepsilon_0, \end{aligned}$$

which implies a contradiction. Hence the assertion is true.

On the one hand, by the absolutely continuity of Lebesgue integral, there exists $\delta > 0$ such that when $A \subset \mathbb{R}^N$ with $\mathrm{meas}\,(A) < \delta$, we have

$$\int_A \overline{V}(x)|v_n(x)|^p\mathrm{d}x < \frac{1}{p}.$$

Hence, we can find a constant $C_7 > 0$ such that $\mathrm{meas}\,(\Omega_n) < \delta$. Thus we infer that

$$\int_{\Omega_n} \overline{V}(x)|v_n(x)|^p\,\mathrm{d}x \leq \frac{1}{p}. \tag{2.5}$$

On the other hand, when $|u_n(x)| \leq C_6$, by (9) and (10) of Lemma 2.2, we have

$$\int_{\mathbb{R}^N\setminus\Omega_n} \overline{V}(x)|v_n|^p\mathrm{d}x = \int_{\mathbb{R}^N\setminus\Omega_n} \overline{V}(x)\frac{|u_n|^p}{\|u_n\|_E^p}\mathrm{d}x \leq C_7\int_{\mathbb{R}^N\setminus\Omega_n} \overline{V}(x)\frac{|f(u_n)|^p}{\|u_n\|_E^p}\mathrm{d}x \to 0. \tag{2.6}$$

Combining (2.5) and (2.6), we have

$$\int_{\mathbb{R}^N} \overline{V}(x)|v_n(x)|^p = \int_{\mathbb{R}^N\setminus\Omega_n} \overline{V}(x)|v_n(x)|^p + \int_{\Omega_n} \overline{V}(x)|v_n(x)|^p \leq \frac{1}{p} + o(1),$$

which implies that $1 \leq \frac{1}{p}$, a contradiction. The proof is complete. $\square$

**Proposition 2.4.** *For any sequence* $\{u_n\} \subset E$ *satisfying*

$$\int_{\mathbb{R}^N} (|\nabla u_n|^p + \overline{V}(x)|f(u_n)|^p)\mathrm{d}x \le C_8,$$

*there exists a constant* $C_9 > 0$ *such that*

$$\int_{\mathbb{R}^N} (|\nabla u_n|^p + \overline{V}(x)|f(u_n)|^p)\mathrm{d}x \ge C_9\|u_n\|_E^p, \quad \forall\, n \in \mathbb{N}.$$

*Proof.* We argue by conradiction, so there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ such that

$$\int_{\mathbb{R}^N} \Big(\frac{|\nabla u_{n_k}|^p}{\|u_{n_k}\|_E^p} + \overline{V}(x)\frac{|f(u_{n_k})|^p}{\|u_{n_k}\|_E^p}\Big)\mathrm{d}x \to 0 \quad \text{as } k \to \infty.$$

The rest of the proof is similar to Proposition 2.3, we can deduce the conclusion. □

At the end of this section, we recall the variant mountain pass theorem and symmetric mountain pass theorem which are used to prove our main result.

**Theorem 2.5** ([22])**.** *Let* $E$ *be a real Banach space with its dual space* $E^*$*, and suppose that* $I \in C^1(E, \mathbb{R})$ *satisfies*

$$\max\{I(0), I(e)\} \le \mu < \eta \le \inf_{\|u\|=\rho} I(u),$$

*for some* $\rho > 0$ *and* $e \in E$ *with* $\|e\| > \rho$*. Let* $c \ge \eta$ *be characterized by*

$$c = \inf_{\gamma\in\Gamma} \max_{0\le\tau\le1} I(\gamma(\tau)),$$

*where* $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$ *is the set of continuous paths joining* $0$ *and* $e$*, then there exists a sequence* $\{u_n\} \subset E$ *such that*

$$I(u_n) \to c \ge \eta \quad \textit{and} \quad (1 + \|u_n\|)\|I'(u_n)\|_{E^*} \to 0, \quad \textit{as } n \to \infty. \tag{2.7}$$

A sequence $\{v_n\} \subset E$ is said to be a Cerami sequence (simply $(C)_c$) if $I(v_n) \to c$ and $(1 + \|v_n\|_E)I'(v_n) \to 0$, $I$ is said to satisfy the $(C)_c$ condition if any $(C)_c$ sequence has a convergent subsequence.

**Theorem 2.6** ([20])**.** *Let* $E$ *be an infinite dimensional Banach space,* $E = Y \oplus Z$*, where* $Y$ *is finite dimensional. If* $\varphi \in C^1(E, \mathbb{R})$ *satisfies* $(C)_c$*-condition for all* $c > 0$*, and*

(1) $\varphi(0) = 0$, $\varphi(-u) = \varphi(u)$ *for all* $u \in E$*;*
(2) *there exist constants* $\rho$*,* $\alpha$ *such that* $\varphi|\partial B_\rho \cap Z \ge \alpha$*;*
(3) *for any finite dimensional subspace* $\widetilde{E} \subset E$*, there is* $R = R(\widetilde{E}) > 0$ *such that* $\varphi(u) \le 0$ *on* $\widetilde{E} \setminus B_R$*.*

*Then* $\varphi$ *possesses an unbounded sequence of critical values.*

## 3. $(C)_c$ condition

In this section, we will prove the bondedness of $(C)_c$ sequence and then show that bounded $(C)_c$ sequence is strongly convergence in $E$.

**Lemma 3.1.** *Any bounded* $(C)_c$ *sequence of* $I$ *possesses a convergence subsequence in* $E$*.*

*Proof.* Assume that $\{v_n\} \subset E$ is a bounded sequence satisfying

$$I(v_n) \to c \quad \text{and} \quad (1 + \|v_n\|_E) I'(v_n) \to 0. \tag{3.1}$$

Going if necessary to a subsequence, we can assume that $v_n \rightharpoonup v$ in $E$. By Lemma 2.1, $v_n \to v$ in $L^s(\mathbb{R}^N)$ for all $p \le s < p^*$ and $v_n \to v$ a.e. on $\mathbb{R}^N$. First, we claim that there exists $C_{10} > 0$ such that

$$\begin{aligned} &\int_{\mathbb{R}^N} |\nabla(v_n - v)|^p + \overline{V}(x)\Big(|f(v_n)|^{p-2} f(v_n) f'(v_n) \\ &\quad - |f(v)|^{p-2} f(v) f'(v)\Big)(v_n - v) \mathrm{d}x \\ &\ge \int_{\mathbb{R}^N} (|\nabla v_n|^{p-1} - |\nabla v|^{p-1}) \nabla(v_n - v) + \overline{V}(x)\Big(|f(v_n)|^{p-2} f(v_n) f'(v_n) \\ &\quad - |f(v)|^{p-2} f(v) f'(v)\Big)(v_n - v) \mathrm{d}x \\ &\ge C_{10} \|v_n - v\|_E^p. \end{aligned} \tag{3.2}$$

Indeed, we may assume that $v_n \ne v$. Set

$$w_n = \frac{v_n - v}{\|v_n - v\|_E}, \quad h_n = \frac{|f(v_n)|^{p-2} f(v_n) f'(v_n) - |f(v)|^{p-2} f(v) f'(v)}{|v_n - v|^{p-1}}.$$

We argue by contradiction and assume that

$$\int_{\mathbb{R}^N} |\nabla w_n|^p + \overline{V}(x) h_n(x) w_n^p \mathrm{d}x \to 0.$$

Since

$$\frac{d}{dt}\Big(|f(t)|^{p-2} f(t) f'(t)\Big) = |f(t)|^{p-2} |f'(t)|^2 \Big[p - 1 - \frac{2^{p-1}|f(t)|^p}{1 + 2^{p-1}|f(t)|^p}\Big] > 0,$$

so, $|f(t)|^{p-2} f(t) f'(t)$ is strictly increasing and for each $C_{11} > 0$ there is $\delta_1 > 0$ such that

$$\frac{d}{dt}\Big(|f(t)|^{p-2} f(t) f'(t)\Big) \ge \delta_1 \quad \text{as } |t| \le C_{11}.$$

From this, we can see that $h_n(x)$ is positive. Hence

$$\int_{\mathbb{R}^N} |\nabla w_n|^p \mathrm{d}x \to 0, \quad \int_{\mathbb{R}^N} \overline{V}(x) h_n(x) w_n^p \mathrm{d}x \to 0, \quad \int_{\mathbb{R}^N} \overline{V}(x) |w_n|^p \mathrm{d}x \to 1.$$

By a similar argument as Proposition 2.3, we can conclude a contradiction.

On the other hand, by (2), (3), (8) and (11) of Lemma 2.6, (A2) and the definition of the $f'(t)$, we have

$$\begin{aligned} &\Big| \int_{\mathbb{R}^N} \Big(\overline{g}(x, f(v_n)) f'(v_n) - \overline{g}(x, f(v)) f'(v)\Big)(v_n - v) \mathrm{d}x \Big| \\ &\le \Big( \int_{\mathbb{R}^N} |\overline{g}(x, f(v_n)) f'(v_n)| + \int_{\mathbb{R}^N} |\overline{g}(x, f(v)) f'(v)| \Big) |v_n - v| \mathrm{d}x \\ &\le \int_{\mathbb{R}^N} C_{12}\big(|f(v_n)|^{p-1} + |f(v_n)|^{q-1}\big) |f'(v_n)| |v_n - v| \mathrm{d}x \\ &\quad + \int_{\mathbb{R}^N} C_{12}\Big(|f(v)|^{p-1} + |f(v)|^{q-1}\Big) |f'(v)| |v_n - v| \mathrm{d}x \\ &\le \int_{\mathbb{R}^N} C_{12}\Big(|f(v_n)|^{p-1} |f'(v_n)| + |f(v_n)|^{q-1} |f'(v_n)|\Big) |v_n - v| \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&+\int_{\mathbb{R}^N} C_{12}\Big(|f(v)|^{p-1}|f'(v)|+|f(v)|^{q-1}|f'(v)|\Big)|v_n-v|\mathrm{d}x\\
&\le \int_{\mathbb{R}^N} C_{12}\Big(|f(v_n)|^{p-1}+\frac{|f(v_n)|^{q-1}}{[1+2^{p-1}|f(v_n)|^p]^{1/p}}\Big)|v_n-v|\mathrm{d}x\\
&\quad+\int_{\mathbb{R}^N} C_{12}\Big(|f(v)|^{p-1}+\frac{|f(v)|^{q-1}}{[1+2^{p-1}|f(v)|^p]^{1/p}}\Big)|v_n-v|\mathrm{d}x\\
&\le \int_{\mathbb{R}^N} C_{12}\Big(|f(v_n)|^{p-1}+|f(v_n)|^{q-2}+|f(v)|^{p-1}+|f(v)|^{q-2}\Big)|v_n-v|\mathrm{d}x\\
&\le \int_{\mathbb{R}^N} C_{12}\Big(|v_n|^{p-1}+|v_n|^{\frac{q}{2}-1}+|v|^{p-1}+|v|^{\frac{q}{2}-1}\Big)|v_n-v|\mathrm{d}x\\
&\le C_{12}\Big((\|v_n\|_p^{p-1}+\|v\|_p^{p-1})\|v_n-v\|_p\Big)+C_{12}\Big((\|v_n\|_{\frac{q}{2}}^{\frac{q-2}{2}}+\|v\|_{\frac{q}{2}}^{\frac{q-2}{2}})\|v_n-v\|_{\frac{q}{2}}\Big)\\
&=o(1).
\end{aligned}
$$

Therefore, by (3.2) and the above inequality, we have

$$
\begin{aligned}
o(1)&=\langle I'(v_n)-I'(v),v_n-v\rangle\\
&=\int_{\mathbb{R}^N}\Big[|\nabla(v_n-v)|^p+\overline{V}(x)\Big(|f(v_n)|^{p-2}f(v_n)f'(v_n)\\
&\quad-|f(v)|^{p-2}f(v)f'(v)\Big)(v_n-v)\Big]\mathrm{d}x\\
&\quad-\int_{\mathbb{R}^N}\Big(\overline{g}(x,f(v_n))f'(v_n)-\overline{g}(x,f(v))f'(v)\Big)(v_n-v)\mathrm{d}x\\
&\ge C_{13}\|v_n-v\|_E^p+o(1),
\end{aligned}
$$

which implies that $\|v_n-v\|_E\to 0$ as $n\to\infty$. The proof is complete. $\square$

**Lemma 3.2.** *Suppose that* (A1'), (A2)-(A4) *are satisfied. Then any* $(C)_c$ *sequence of* $I$ *is bounded in* $E$.

*Proof.* Let $\{v_n\}\subset E$ be such that

$$
I(v_n)\to c \quad\text{and}\quad (1+\|v_n\|_E)I'(v_n)\to 0. \tag{3.3}
$$

Thus, there is a constant $C_{14}>0$ such that

$$
I(v_n)-\frac{1}{2p}\langle I'(v_n),v_n\rangle\le C_{14}. \tag{3.4}
$$

Firstly, we prove that there exists $C_{15}>0$ independent of $n$ such that

$$
\int_{\mathbb{R}^N}\Big(|\nabla v_n|^p+\overline{V}(x)|f(v_n)|^p\Big)\mathrm{d}x\le C_{15}. \tag{3.5}
$$

Suppose by contradiction that

$$
\|v_n\|_0^p:=\int_{\mathbb{R}^N}\Big(|\nabla v_n|^p+\overline{V}(x)|f(v_n)|^p\Big)\mathrm{d}x\to\infty \quad \text{as } n\to\infty.
$$

Setting $\widetilde{f}(v_n):=f(v_n)/\|v_n\|_0$, then $\|\widetilde{f}(v_n)\|_E\le 1$. Passing to a subsequence, we may assume that $\widetilde{f}(v_n)\rightharpoonup w$ in $E$, $\widetilde{f}(v_n)\to w$ in $L^s(\mathbb{R}^N)$, $p\le s<p^*$, and $\widetilde{f}(v_n)\to w$ a.e. $\mathbb{R}^N$. It follows from (3.3) that

$$
\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|\overline{G}(x,f(v_n))|}{\|v_n\|_0^p}\mathrm{d}x\ge\frac{1}{p}. \tag{3.6}
$$

Let $\varphi_n = f(v_n)/f'(v_n)$, by (3.4), we have

$$\begin{aligned}
C_{14} &\geq I(v_n) - \frac{1}{2p}\langle I'(v_n), \varphi_n\rangle \\
&= \frac{1}{2p}\int_{\mathbb{R}^N} |\nabla v_n|^p |f'(v_n)|^p \mathrm{d}x + \frac{1}{2p}\int_{\mathbb{R}^N} \overline{V}(x)|f(v_n)|^p \mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} \frac{1}{2p}\overline{g}(x, f(v_n))f(v_n)\mathrm{d}x - \int_{\mathbb{R}^N} \overline{G}(x, f(v_n))\mathrm{d}x,
\end{aligned}$$

which implies

$$C_{14} \geq \int_{\mathbb{R}^N} \widetilde{\overline{G}}(x, f(v_n))\mathrm{d}x. \tag{3.7}$$

Set

$$h(r) := \inf\{\widetilde{\overline{G}}(x, f(v_n)) : x \in \mathbb{R}^N, |f(v_n)| \geq r\} \quad r \geq 0.$$

By (1.5), $h(r) \to \infty$ as $r \to \infty$. For $0 \leq a < b$, let $\Omega_n(a,b) = \{x \in \mathbb{R}^N : a \leq |f(v_n(x))| < b\}$. Hence, it follows from (3.7) that

$$\begin{aligned}
C_{14} &\geq \int_{\Omega_n(0,r)} \widetilde{\overline{G}}(x, f(v_n)) + \int_{\Omega_n(r,+\infty)} \widetilde{\overline{G}}(x, f(v_n)) \\
&\geq \int_{\Omega_n(0,r)} \widetilde{\overline{G}}(x, f(v_n)) + h(r)\,\mathrm{meas}(\Omega_n(r,+\infty)),
\end{aligned}$$

which implies that $\mathrm{meas}(\Omega_n(r,+\infty)) \to 0$ as $r \to \infty$ uniformly in $n$. Thus, for any $s \in [p, 2p^*)$, by (8) of Lemma 2.2, Hölder inequality and Sobolev embedding, we have

$$\begin{aligned}
&\int_{\Omega_n(r,+\infty)} \widetilde{f}^s(v_n)\mathrm{d}x \\
&\leq \Big(\int_{\Omega_n(r,+\infty)} \widetilde{f}^{2p^*}(v_n)\mathrm{d}x\Big)^{\frac{s}{2p^*}} \Big(\mathrm{meas}(\Omega_n(r,+\infty))\Big)^{\frac{2p^*-s}{2p^*}} \\
&\leq \frac{C_{16}}{\|v_n\|_0^s}\Big(\int_{\Omega_n(r,+\infty)} |\nabla f^2(v_n)|^p\Big)^{\frac{s}{2p}} \Big(\mathrm{meas}(\Omega_n(r,+\infty))\Big)^{\frac{2p^*-s}{2p^*}} \\
&\leq \frac{C_{17}}{\|v_n\|_0^s}\Big(\int_{\Omega_n(r,+\infty)} |\nabla v_n|^p\Big)^{\frac{s}{2p}} \Big(\mathrm{meas}(\Omega_n(r,+\infty))\Big)^{\frac{2p^*-s}{2p^*}} \\
&\leq C_{17}\|v_n\|_0^{-\frac{s}{2}} \Big(\mathrm{meas}(\Omega_n(r,+\infty))\Big)^{\frac{2p^*-s}{2p^*}} \to 0,
\end{aligned} \tag{3.8}$$

as $r \to \infty$ uniformly in $n$.

If $w=0$, then $\widetilde{f}(v_n)=\frac{f(v_n)}{\|v_n\|_0}\to 0$ in $L^s(\mathbb{R}^N)$, $p\le s<p^*$. For any $0<\epsilon<\frac{1}{4p}$, there exist large $r_1$, $N_0\in\mathbb{N}$ such that

$$\begin{aligned}
&\int_{\Omega_n(0,r_1)}\frac{|\overline{G}(x,f(v_n))|}{|f(v_n)|^p}|\widetilde{f}(v_n)|^p\mathrm{d}x\\
&\le\int_{\Omega_n(0,r_1)}\frac{C_{18}|f(v_n)|^p+C_{19}|f(v_n)|^q}{|f(v_n)|^p}|\widetilde{f}(v_n)|^p\mathrm{d}x\\
&\le(C_{18}+C_{19}r_1^{q-p})\int_{\Omega_n(0,r_1)}|\widetilde{f}(v_n)|^p\mathrm{d}x\\
&\le(C_{18}+C_{19}r_1^{q-p})\int_{\mathbb{R}^N}|\widetilde{f}(v_n)|^p\mathrm{d}x<\epsilon,
\end{aligned}\tag{3.9}$$

for all $n>N_0$. Set $\sigma'=\frac{\sigma}{\sigma-1}$. Since $\sigma>\frac{2N}{N+p}$, so $p\sigma'\in(p,2p^*)$. Hence, it follows from (A4) and (3.7) that

$$\begin{aligned}
&\int_{\Omega_n(r_1,+\infty)}\frac{|\overline{G}(x,f(v_n))|}{|f(v_n)|^p}|\widetilde{f}(v_n)|^p\mathrm{d}x\\
&\le\Big(\int_{\Omega_n(r_1,+\infty)}(\frac{|\overline{G}(x,f(v_n))|}{|f(v_n)|^p})^\sigma\mathrm{d}x\Big)^{1/\sigma}\Big(\int_{\Omega_n(r_1,+\infty)}|\widetilde{f}(v_n)|^{p\sigma'}\mathrm{d}x\Big)^{1/\sigma'}\\
&\le C_{20}^{1/\sigma}\Big(\int_{\Omega_n(r_1,+\infty)}\widetilde{\overline{G}}(x,f(v_n)\mathrm{d}x\Big)^{1/\sigma}\Big(\int_{\Omega_n(r_1,+\infty)}|\widetilde{f}(v_n)|^{p\sigma'}\mathrm{d}x\Big)^{1/\sigma'}\\
&\le C_{21}\Big(\int_{\Omega_n(r_1,+\infty)}|\widetilde{f}(v_n)|^{p\sigma'}\mathrm{d}x\Big)^{1/\sigma'}<\epsilon,
\end{aligned}\tag{3.10}$$

for all $n$. Combining (3.9) with (3.10), we have

$$\int_{\mathbb{R}^N}\frac{\overline{G}(x,f(v_n))}{\|v_n\|_0^p}\mathrm{d}x=\Big(\int_{\Omega_n(0,r_1)}+\int_{\Omega_n(r_1,+\infty)}\Big)\frac{\overline{G}(x,f(v_n))}{|f(v_n)|^p}|\widetilde{f}(v_n)|^p\mathrm{d}x<2\epsilon<\frac{1}{p},$$

for all $n>N_0$, which contradicts (3.6).

If $w\neq 0$, then $\mathrm{meas}(\Omega)>0$, where $\Omega:=\{x\in\mathbb{R}^N:w\neq 0\}$. For $x\in\Omega$, $|f(v_n)|\to\infty$ as $n\to\infty$. Hence $\Omega\subset\Omega_n(r_0,\infty)$ for large $n\in N$, where $r_0$ is given in $(A3)$. By $(A3)$, we have

$$\frac{\overline{G}(x,f(v_n))}{|f(v_n)|^{2p}}\to+\infty\quad\text{as } n\to\infty.$$

Hence, using Fatou's lemma, we have

$$\int_{\mathbb{R}^N}\frac{\overline{G}(x,f(v_n))}{|f(v_n)|^{2p}}\to+\infty\quad\text{as }\ n\to\infty.\tag{3.11}$$

It follows from (3.3) and (3.11) that

$$\begin{aligned}
0&=\lim_{n\to\infty}\frac{c+o(1)}{\|v_n\|_0^p}=\lim_{n\to\infty}\frac{I(v_n)}{\|v_n\|_0^p}\\
&=\lim_{n\to\infty}\frac{1}{\|v_n\|_0^p}\Big(\frac{1}{p}\int_{\mathbb{R}^N}(|\nabla v_n|^p+\overline{V}(x)|f(v_n)|^p)\mathrm{d}x-\int_{\mathbb{R}^N}\overline{G}(x,f(v_n))\mathrm{d}x\Big)\\
&=\lim_{n\to\infty}\Big(\frac{1}{p}-\int_{\Omega_n(0,r_0)}\frac{\overline{G}(x,f(v_n))}{|f(v_n)|^p}|\widetilde{f}(v_n)|^p\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\quad - \int_{\Omega_n(r_0,+\infty)} \frac{\overline{G}(x,f(v_n))}{|f(v_n)|^p} |\widetilde{f}(v_n)|^p \mathrm{d}x\Big) \\
&\le \frac{1}{p} + \limsup_{n\to\infty} (C_{22} + C_{23} r_0^{q-p}) \int_{\mathbb{R}^N} |\widetilde{f}(v_n)|^p \mathrm{d}x \\
&\quad - \int_{\Omega_n(r_0,+\infty)} \frac{\overline{G}(x,f(v_n))}{|f(v_n)|^p} |\widetilde{f}(v_n)|^p \mathrm{d}x \\
&\le C_{24} - \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{\overline{G}(x,f(v_n))}{|f(v_n)|^{2p}} |f(v_n)\widetilde{f}(v_n)|^p \mathrm{d}x = -\infty,
\end{aligned}
$$

which is a contradiction. Thus, there exists $C_{15} > 0$ such that

$$
\int_{\mathbb{R}^N} (|\nabla v_n|^p + \overline{V}(x)|f(v_n)|^p) \mathrm{d}x \le C_{15}.
$$

Hence, from Proposition 2.4, we have that $\{v_n\}$ is bounded in $E$. □

**Lemma 3.3.** *Suppose that* (A1'), (A2), (A3), (A5) *are satisfied. Then any* $(C)_c$ *sequence of* $I$ *is bounded.*

*Proof.* Let $\{v_n\} \subset E$ be such that

$$
I(v_n) \to c \quad \text{and} \quad (1 + \|v_n\|_E) I'(v_n) \to 0. \tag{3.12}
$$

Thus, there is a constant $C_{25} > 0$ such that

$$
I(v_n) - \frac{1}{\mu} \langle I'(v_n), v_n \rangle \le C_{25}. \tag{3.13}
$$

Firstly, we prove that there exists $C_{26} > 0$ independent of $n$ such that

$$
\int_{\mathbb{R}^N} \Big(|\nabla v_n|^p + \overline{V}(x)|f(v_n)|^p\Big) \mathrm{d}x \le C_{26}.
$$

Suppose by contradiction, we assume that

$$
\|v_n\|_0^p := \int_{\mathbb{R}^N} \Big(|\nabla v_n|^p + \overline{V}(x)|f(v_n)|^p\Big) \mathrm{d}x \to \infty \quad \text{as } n \to \infty.
$$

As

$$
\nabla\big(\frac{f(v_n)}{f'(v_n)}\big) = \nabla\Big[f(v_n) \cdot (1 + 2^{p-1}|f(v_n)|^p)^{1/p}\Big] = \nabla v_n\big[1 + \frac{2^{p-1}|f(v_n)|^p}{1 + 2^{p-1}|f(v_n)|^p}\big].
$$

By (A5) and $\mu > 2p$ we can obtain

$$
\begin{aligned}
C_{25} &\geq I(v_n) - \frac{1}{\mu}\langle I'(v_n), \frac{f(v_n)}{f'(v_n)}\rangle \\
&= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla v_n|^p + \overline{V}(x)|f(v_n)|^p)\mathrm{d}x - \int_{\mathbb{R}^N} \overline{G}(x, f(v_n))\mathrm{d}x \\
&\quad - \frac{1}{\mu}\int_{\mathbb{R}^N} \Big(|\nabla v_n|^{p-2}\nabla(v_n)\nabla(\frac{f(v_n)}{f'(v_n)})\Big)\mathrm{d}x \\
&\quad + \frac{1}{\mu}\int_{\mathbb{R}^N} \Big(\overline{g}(x, f(v_n))f'(v_n)\frac{f(v_n)}{f'(v_n)}\Big)\mathrm{d}x \\
&\quad - \frac{1}{\mu}\int_{\mathbb{R}^N} \Big(\overline{V}(x)|f(v_n)|^{p-2}f(v_n)f'(v_n)\frac{f(v_n)}{f'(v_n)}\Big)\mathrm{d}x \\
&= \int_{\mathbb{R}^N} \Big[\frac{1}{p} - \frac{1}{\mu}\Big(1 + \frac{2^{p-1}|f(v_n)|^p}{1+2^{p-1}|f(v_n)|^p}\Big)\Big]|\nabla v_n|^p\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^N} (\frac{1}{p} - \frac{1}{\mu})(\overline{V}(x)|f(v_n)|^p)\mathrm{d}x \\
&\quad + \frac{1}{\mu}\int_{\mathbb{R}^N} \Big[\overline{g}(x, f(v_n))f(v_n) - \mu\overline{G}(x, f(v_n))\Big]\mathrm{d}x \\
&\geq \int_{\mathbb{R}^N} (\frac{1}{p} - \frac{2}{\mu})|\nabla v_n|^p\mathrm{d}x + \int_{\mathbb{R}^N} (\frac{1}{p} - \frac{1}{\mu})(\overline{V}(x)|f(v_n)|^p)\mathrm{d}x \\
&\quad - \frac{1}{\mu}\int_{\mathbb{R}^N} |f(v_n)|^p\mathrm{d}x \\
&\geq (\frac{1}{p} - \frac{2}{\mu})\int_{\mathbb{R}^N} \Big(|\nabla v_n|^p + \overline{V}(x)|f(v_n)|^p\Big)\mathrm{d}x - \frac{1}{\mu}\int_{\mathbb{R}^N} |f(v_n)|^p\mathrm{d}x \\
&\geq (\frac{1}{p} - \frac{2}{\mu})\|v_n\|_0^p - \frac{1}{\mu}\int_{\mathbb{R}^N} |f(v_n)|^p\mathrm{d}x.
\end{aligned} \tag{3.14}
$$

Setting $\widetilde{f}(v_n) := f(v_n)/\|v_n\|_0$, we have $\|\widetilde{f}(v_n)\|_E \leq 1$. Passing to a subsequence, we may assume that $\widetilde{f}(v_n) \rightharpoonup w$ in $E$, $\widetilde{f}(v_n) \to w$ in $L^s(\mathbb{R}^N)$, $p \leq s < p^*$, and $\widetilde{f}(v_n) \to w$ a.e. $\mathbb{R}^N$.

From (3.14),

$$
\frac{C_{25}}{\|v_n\|_0^p} \geq (\frac{1}{p} - \frac{2}{\mu}) - \frac{1}{\mu}\int_{\mathbb{R}^N} |\widetilde{f}(v_n)|^p\mathrm{d}x.
$$

Hence, we obtain

$$
\frac{1}{\mu}\int_{\mathbb{R}^N} |\widetilde{f}(v_n)|^p\mathrm{d}x \geq (\frac{1}{p} - \frac{2}{\mu})\mu + o(1).
$$

Then $\widetilde{f}(v_n) \to w$ and $w \neq 0$, so $|f(v_n)| \to \infty$ as $n \to \infty$.

Also by (A3), we have

$$
\frac{\overline{G}(x, f(v_n))}{|f(v_n)|^{2p}} \to +\infty.
$$

So

$$
\int_{\mathbb{R}^N} \frac{\overline{G}(x, f(v_n))}{|f(v_n)|^{2p}} \to +\infty, \tag{3.15}
$$

From (3.12) and (3.15) it follows that

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c+o(1)}{\|v_n\|_0^p} = \lim_{n\to\infty} \frac{I(v_n)}{\|v_n\|_0^p} \\
&= \lim_{n\to\infty} \frac{1}{\|v_n\|_0^p}\Big(\frac{1}{p}\int_{\mathbb{R}^N} (|\nabla v_n|^p + \overline{V}(x)|f(v_n)|^p)\mathrm{d}x - \int_{\mathbb{R}^N} \overline{G}(x, f(v_n))\mathrm{d}x\Big) \\
&= \lim_{n\to\infty} \Big(\frac{1}{p} - \int_{\mathbb{R}^N} \frac{\overline{G}(x, f(v_n))}{|f(v_n)|^p} |\widetilde{f}(v_n)|^p \mathrm{d}x\Big) \\
&\le C_{27} - \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{\overline{G}(x, f(v_n))}{|f(v_n)|^{2p}} |f(v_n)\widetilde{f}(v_n)|^p \mathrm{d}x = -\infty.
\end{aligned}
\tag{3.16}
$$

Which is a contradiction. Thus, there exists $C_{26} > 0$ such that

$$
\int_{\mathbb{R}^N} (|\nabla v_n|^p + \overline{V}(x)|f(v_n)|^p)\mathrm{d}x \le C_{26}.
$$

Hence, from Proposition 2.4, we obtain that $\{v_n\}$ is bounded in $E$. □

Since (A2) and (A6) imply ($A5$), we have the following corollary.

**Corollary 3.4.** *Suppose that* (A1'), (A2), (A3), (A6) *are satisfied. Then any* $(C)_c$ *sequence of* $I$ *is bounded.*

## 4. Proof of main results

**Proof of Theorems 1.3 and 1.4.**

**Lemma 4.1.** *The functional* $I$ *is bounded from below on a neighborhood of the origin. That is, there exist* $C_{28} \in \mathbb{R}$ *and* $\rho > 0$*, such that*

$$
I(u) \ge C_{28}, \quad \forall u \in B_\rho = \{u \in E : \|u\| \le \rho\}.
$$

*Proof.* If the conclusion is not true, there exists $\{u_n\} \subset E$, satisfying

$$
\|u_n\| \le \frac{1}{n}, \quad I(u_n) \to -\infty.
$$

So $u_n \to 0$ in $E$, and

$$
I(u_n) = \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u_n|^p + \overline{V}(x)|f(u_n)|^p)\mathrm{d}x - \int_{\mathbb{R}^N} \overline{G}(x, f(u_n))\mathrm{d}x.
$$

Obviously,

$$
\frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u_n|^p + \overline{V}(x)|f(u_n)|^p)\mathrm{d}x \to 0.
$$

From (A2), and (3) and (8) of Lemma 2.2, we have

$$
\begin{aligned}
\int_{\mathbb{R}^N} \overline{G}(x, f(u_n))\mathrm{d}x &\le C_{29} \int_{\mathbb{R}^N} \big(|f(u_n)|^p + |f(u_n)|^q\big)\mathrm{d}x \\
&\le C_{29} \int_{\mathbb{R}^N} (|u_n|^p + |u_n|^{\frac{q}{2}})\mathrm{d}x \to 0.
\end{aligned}
$$

Hence, $I(u_n) \to 0$, contradicts with $I(u_n) \to -\infty$, as $n \to +\infty$. □

**Lemma 4.2.** *There exists* $\vartheta \in E$*, such that* $I(t\vartheta) < 0$*, for* $t$ *small enough.*

*Proof.* Let $\vartheta \in C_0^\infty(\mathbb{R}^N,[0,1])\backslash\{0\}$, and $K=\operatorname{supp}\vartheta$. From (A3), we have
$$\overline{G}(x,u)\ge C_{30}|u|^\tau>0,$$
for all $(x,u)\in\mathbb{R}^N\times\mathbb{R}$, $|u|\ge r_0$. By $(A2)$, for a.e. $x\in\mathbb{R}^N$ and $0\le |u|\le 1$, there exists $M>0$ such that
$$\Big|\frac{\overline{g}(x,u)u}{|u|^p}\Big|\le\Big|\frac{C(|u|^{p-1}+|u|^{q-1})\cdot|u|}{|u|^p}\Big|\le M,$$
which implies that
$$\overline{g}(x,u)u\ge -M|u|^p.$$
We can use the equality $\overline{G}(x,u)=\int_0^1\overline{g}(x,tu)u\mathrm{d}t$, for a.e. $x\in\mathbb{R}^N$ and $0\le|u|\le 1$, to obtain
$$\overline{G}(x,u)\ge -\frac{M}{p}|u|^p\,.$$
Then
$$\overline{G}(x,u)\ge -\frac{M}{p}|u|^p+C_{30}|u|^\tau. \tag{4.1}$$
So from (4.1),
$$\begin{aligned}
&I(t\vartheta)\\
&=\frac{t^p}{p}\int_{\mathbb{R}^N}|\nabla\vartheta|^p\mathrm{d}x+\frac{1}{p}\int_{\mathbb{R}^N}\overline{V}(x)|f(t\vartheta)|^p\mathrm{d}x-\int_{\mathbb{R}^N}\overline{G}(x,f(t\vartheta))\mathrm{d}x\\
&\le\frac{t^p}{p}\int_{\mathbb{R}^N}\big(|\nabla\vartheta|^p+\overline{V}(x)|\vartheta|^p\big)\mathrm{d}x+\frac{M}{p}\int_{\mathbb{R}^N}|f(t\vartheta)|^p\mathrm{d}x-C_{31}\int_{\mathbb{R}^N}|f(t\vartheta)|^\tau\mathrm{d}x\\
&\le\frac{t^p}{p}\int_{\mathbb{R}^N}\big(|\nabla\vartheta|^p+\overline{V}(x)|\vartheta|^p+M|\vartheta|^p\big)\mathrm{d}x-C_{31}\int_{\mathbb{R}^N}|f(t\vartheta)|^\tau\mathrm{d}x.
\end{aligned}\tag{4.2}$$
Since $f(t)/t$ is decreasing and $0\le t\vartheta\le t$, for $t\ge 0$. We obtain $f(t\vartheta)\ge f(t)\vartheta$. By (9) of Lemma 2.2, we obtain $f(t\vartheta)\ge Ct\vartheta$, for $0\le t\le 1$. Hence
$$I(t\vartheta)\le\frac{t^p}{p}\int_{\mathbb{R}^N}\big(|\nabla\vartheta|^p+\overline{V}(x)|\vartheta|^p+M|\vartheta|^p\big)\mathrm{d}x-C_{32}t^\tau\int_{\mathbb{R}^N}|\vartheta|^\tau\mathrm{d}x,$$
and since $\tau<p$, we obtain $I(t\vartheta)<0$, for $t$ sufficiently small and the Lemma is proved. □

Thus, we obtain that
$$c_0=\inf\{I(u):u\in\overline{B_\rho}\}<0,$$
which $\rho>0$ is given in Lemma 4.1. Then we can apply the Ekeland's variational principle and [24, corollary 2.5], there exists a sequence $\{u_n\}\subset\overline{B_\rho}$ such that $C_{33}\le I(u_n)<C_{33}+\frac{1}{n}$. Hence
$$I(u)\ge I(u_n)-\frac{1}{n}\|w-u_n\|_E,\quad\forall w\in\overline{B_\rho}.$$
Then, following the idea in [24], we can show that $\{u_n\}$ is a bounded Cerami sequence of $I$. Therefore, Lemma 3.1 implies that there exists a function $u_0\in E$ such that $I'(u_0)=0$ and $I(u_0)=c_0<0$.

Next, we show that there exists a second solution for problem 1.1.

**Lemma 4.3.** *If the conditions* (A1)–(A3), (A7) *are satisfied, there exist two constants $\rho_1>0$, $\alpha>0$, such that*
$$I(u)\ge\alpha>0,\quad\forall u\in S_{\rho_1}=\{u\in E:\|u\|_E=\rho_1\}.$$

*Proof.* From (A2) and (A7), it follows that

$$|\overline{G}(x,u)| \le \varepsilon |u|^p + C_\varepsilon |u|^q, \quad \forall (x,u) \in \mathbb{R}^N \times \mathbb{R}.$$

Thus, by Proposition 2.3, we take $u \in E$ with $\|u\| \le \rho$, where $\rho$ is given in Proposition 2.3, we can deduce that

$$\begin{aligned} I(u) &= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + \overline{V}(x)|f(u)|^p)\mathrm{d}x - \int_{\mathbb{R}^N} \overline{G}(x,f(u))\mathrm{d}x \\ &\ge \frac{C_{34}}{p}\|u\|_E^p - C\varepsilon \|u\|_E^p - C_\varepsilon \|u\|_E^q \\ &\ge \frac{C_{35}}{2p}\|u\|_E^p - C_{36}\|u\|_E^q, \end{aligned} \tag{4.3}$$

and since $q > 2p$, there exists $\alpha, \rho_1 > 0$ such that $I(u) \ge \alpha > 0$ for $\|u\|_E = \rho_1$. □

**Lemma 4.4.** *There exist a $v \in E$ with $\|v\|_E > \rho_1$, such that $I(v) < 0$, which $\rho_1$ is defined in Lemma 4.3.*

*Proof.* Let $u_0 \in E$ and $u_0 > 0$. From $(A3)$, (9) of Lemma 2.2, and Fatou's Lemma, we have

$$\begin{aligned} \lim_{t\to\infty} \frac{I(tu_0)}{t^p} &= \lim_{t\to\infty}\Big(\frac{1}{pt^p}\int_{\mathbb{R}^N}(|\nabla tu_0|^p + \overline{V}(x)|f(tu_0)|^p)\mathrm{d}x - \int_{\mathbb{R}^N}\frac{\overline{G}(x,f(tu_0))}{t^p}\mathrm{d}x\Big) \\ &\le \lim_{t\to\infty}\Big(\int_{\mathbb{R}^N}\frac{|\nabla u_0|^p}{p}\mathrm{d}x + \int_{\mathbb{R}^N}\frac{\overline{V}(x)|tu_0|^p}{pt^p}\mathrm{d}x \\ &\quad - \int_{\mathbb{R}^N}\frac{\overline{G}(x,f(tu_0))}{(f(tu_0))^{2p}}\frac{(f(tu_0))^{2p}}{(tu_0)^p}(u_0)^p\mathrm{d}x\Big) \\ &= \frac{\|u_0\|_E^p}{p} - \lim_{t\to\infty}\int_{\mathbb{R}^N}\frac{\overline{G}(x,f(tu_0))}{(f(tu_0))^{2p}}\frac{(f(tu_0))^{2p}}{(tu_0)^p}(u_0)^p\mathrm{d}x \\ &\le \frac{\|u_0\|_E^p}{p} - \int_{\mathbb{R}^N}\liminf_{t\to\infty}\frac{\overline{G}(x,f(tu_0))}{(f(tu_0))^{2p}}\frac{(f(tu_0))^{2p}}{(tu_0)^p}(u_0)^p\mathrm{d}x = -\infty. \end{aligned}$$

Thus, this lemma is proved by taking $v = tu_0$ with $t > 0$ large enough. □

Based on Lemmas 4.3 and 4.4, Theorem 2.5 implies that there is a sequence $\{u_n\} \subset E$ such that

$$I(u_n) \to c \quad \text{and} \quad (1 + \|u_n\|_E)I'(u_n) \to 0.$$

From Lemma 3.2 and 3.1, it shows that this sequence $\{u_n\}$ has a convergent subsequence in $E$. Thus, there exists $u_1 \in E$ such that $I'(u_1) = 0$ and $I(u_1) = c_1 > 0$. Consequently, the proof of Theorem 1.3 is complete.

By the similar arguments as the proof of Theorem 1.3, Theorem 1.4 and Corollary 1.5 can be proved.

**Proof of Theorems 1.6 and 1.7.** Let $\{e_i\}_{i\in\mathbb{N}} \in E$ is a total orthonormal basis of $E$ and $\{e_j^*\}_{j\in\mathbb{N}} \in E^*$, so that

$$E = \overline{\operatorname{span}\{e_i : i = 1,2,\cdots\}}, \quad E^* = \overline{\operatorname{span}\{e_j^* : j = 1,2,\cdots\}},$$

$$\langle e_i, e_j^* \rangle = \begin{cases} 1, & i = j, \\ 0, & i \ne j; \end{cases}$$

So we define $X_j = \mathbb{R}e_j$,

$$Y_k = \oplus_{j=1}^{k} X_j, \quad Z_k = \overline{\oplus_{j=k+1}^{\infty} X_j}, \quad k \in \mathbb{Z}$$

and $Y_k$ is finite-dimensional. Similar to [24, Lemma 3.8], we have the following lemma.

**Lemma 4.5.** *Under assumption* (A1'), *for* $p \le s < p^*$,

$$\beta_k(s) := \sup_{v \in Z_k, \|v\|=1} \|v\|_s \to 0, \quad k \to \infty.$$

**Lemma 4.6.** *Suppose that* (A1'), (A2) *are satisfied. Then there exist constants* $\rho > 0$, $\alpha > 0$ *such that* $I\big|_{S_\rho \bigcap Z_m} \ge \alpha$.

*Proof.* For any $v \in Z_m$ with $\|v\|_E = \rho < 1$, by (3) and (8) of Lemma 2.2, and proposition 2.3, we have

$$\begin{aligned} I(v) &= \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla v|^p + \overline{V}(x)|f(v)|^p)\mathrm{d}x - \int_{\mathbb{R}^N} \overline{G}(x, f(v))\mathrm{d}x \\ &\ge \frac{C_{37}}{p}\|v\|_E^p - C_{38}\int_{\mathbb{R}^N} (|f(v)|^p + |f(v)|^q)\mathrm{d}x \\ &\ge \frac{C_{37}}{p}\|v\|_E^p - C_{39}\int_{\mathbb{R}^N} (|v|^p + |v|^{\frac{q}{2}})\mathrm{d}x. \end{aligned} \tag{4.4}$$

By Lemma 4.5, we can choose an integer $m \ge 1$ such that

$$C_{39}\|v\|_p^p \le \frac{C_{37}}{2p}\|v\|_E^p, \quad C_{39}\|v\|_{\frac{q}{2}}^{\frac{q}{2}} \le \frac{C_{37}}{2p}\|v\|_E^{\frac{q}{2}}, \quad \forall v \in Z_m.$$

Combining the above inequality with (4.4), we have

$$I(v) \ge \frac{C_{37}}{p}\|v\|_E^p - \frac{C_{37}}{2p}\|v\|_E^p - \frac{C_{37}}{2p}\|v\|_E^{\frac{q}{2}} = \frac{C_{37}}{2p}\|v\|_E^p(1 - \|v\|_E^{\frac{q-2p}{2}}) > 0,$$

since $q > 2p$. This completes the proof. $\square$

**Lemma 4.7.** *Suppose that* (A1'), (A2), (A3) *are satisfied. Then for any finite dimensional subspace* $\widetilde{E} \subset E$, *there is* $R = R(\widetilde{E}) > 0$ *such that*

$$I(v) \le 0, \quad \forall v \in \widetilde{E} \backslash B_R.$$

*Proof.* For any finite dimensional subspace $\widetilde{E} \subset E$, there exists a $m \in \mathbb{N}$ such that $\widetilde{E} \subset E_m$. Suppose by contradiction, we assume that there exists a sequence $\{v_n\} \subset \widetilde{E}$ such that $\|v_n\|_E \to \infty$ and $I(v_n) > 0$. Hence

$$\frac{1}{p}\int_{\mathbb{R}^N} (|\nabla v_n|^p + \overline{V}(x)|f(v_n)|^p)\mathrm{d}x > \int_{\mathbb{R}^N} \overline{G}(x, f(v_n))\mathrm{d}x. \tag{4.5}$$

Set $w_n = \frac{v_n}{\|v_n\|_E}$. Then, up to a subsequence, we can assume that $w_n \rightharpoonup w$ in $E$, $w_n \to w$ in $L^s(\mathbb{R}^N)$ for all $p \le s < p^*$, and $w_n \to w$ a.e.on $\mathbb{R}^N$. Set $\Omega_1 := \{x \in \mathbb{R}^N : w(x) \ne 0\}$ and $\Omega_2 := \{x \in \mathbb{R}^N : w(x) = 0\}$. If meas$(\Omega_1) > 0$, by (A3), (5) of Lemma 2.2, and Fatou's lemma, we have

$$\int_{\Omega_1} \frac{\overline{G}(x, f(v_n))}{\|v_n\|_E^p}\mathrm{d}x = \int_{\Omega_1} \frac{\overline{G}(x, f(v_n))}{|f(v_n)|^{2p}} \frac{|f(v_n)|^{2p}}{|v_n|^p}|w_n|^p\mathrm{d}x \to +\infty. \tag{4.6}$$

On the other hand, by (A2) and (A3), there exists $C_{40} > 0$ such that

$$\overline{G}(x, t) \ge -C_{40}|t|^p, \quad \text{for all } (x, t) \in \mathbb{R}^N \times \mathbb{R}.$$

Hence

$$\int_{\Omega_2} \frac{\overline{G}(x, f(v_n))}{\|v_n\|_E^p} \mathrm{d}x \geq -C_{40} \int_{\Omega_2} \frac{|f(v_n)|^p}{\|v_n\|_E^p} \mathrm{d}x \geq -C_{41} \int_{\Omega_2} |w_n|^p \mathrm{d}x.$$

Hence, by the fact that $w_n \to w$ in $L^p(\mathbb{R}^N)$, we obtain

$$\liminf_{n\to\infty} \int_{\Omega_2} \frac{\overline{G}(x, f(v_n))}{\|v_n\|_E^p} \mathrm{d}x \geq 0.$$

Combining this with (4.6), we have

$$\int_{\mathbb{R}^N} \frac{\overline{G}(x, f(v_n))}{\|v_n\|_E^p} \mathrm{d}x = +\infty,$$

which implies a contradiction with (4.5). Hence, meas$(\Omega_1) = 0$, i.e. $w(x) = 0$ a.e. on $\mathbb{R}^N$. By the fact that all norms are equivalent in $\widetilde{E}$, there exists $C_{42} > 0$ such that

$$\|v\|_p^p \geq C_{42} \|v\|_E^p, \quad \forall v \in \widetilde{E}.$$

Hence

$$0 = \lim_{n\to\infty} \|w_n\|_p^p \geq \lim_{n\to\infty} C_{42} \|w_n\|_E^p = C_{42},$$

this results in a contradiction. The proof is complete. □

*Proof of theorem 1.3.* Let $X = E$, $Y = Y_m$ and $Z = Z_m$. Obviously, $I(0) = 0$ and (A8) imply that $I$ is even. By Lemma 3.2, Lemma 4.2 and Lemma 4.3, all conditions of Theorem 2.6 are satisfied. Thus, problem (2.1) possesses infinitely many nontrivial solutions $\{v_n\}$ such that $I(v_n) \to \infty$ as $n \to \infty$. Namely, problem (1.1) also possesses infinitely many nontrivial solutions $\{u_n\}$ such that $J(u_n) \to \infty$ as $n \to \infty$. □

By the similar arguments as Theorem 1.6, we can give the proof of Theorem 1.7 and Corollary 1.8.

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## References

[1] C. O. Alves, G. M. Figueiredo. Multiple solutions for a quasilinear Schrödinger equation on $\mathbb{R}^{\mathbb{N}}$. *Acta Appl Math.* 136:91–117, 2015.

[2] C. O. Alves, G. M. Figueiredo, U. B. Severo. Multiplicity of positive solutions for a class of quasilinear problems. *Advances in Differential Equations.* 14:911–942, 2009.

[3] C. O. Alves, G. M. Figueiredo, U. B. Severo. A result of multiplicity of solutions for a class of quasilinear equations. *Proceedings of the Edinburgh Mathematical Society.* 54:1–19, 2011.

[4] H. Berestycki, P. L. Lions. Nonlinear scalar field equations, I: Existence of a ground state. *Arch. Rational Mech. Anal.*. 82:313–346, 1983.

[5] F. G. Bass, N. N. Nasanov. Nonlinear electromagnetic spin waves. *Physics Reports.* 189:165–223, 1990.

[6] T. Bartsch, Z. Q. Wang. Existence and multiplicity results for some superlinear elliptic problems on $\mathbb{R}^{\mathbb{N}}$. *Comm. Partial Differential Equations.* 20:1725–1741, 1995.

[7] T. Bartsch, Z. Q. Wang, M. Willem. The Dirichlet probllem for superlinear elliptic equations. *in: M. Chipot, P. Quittner (Eds), Handbook of Differential Equations- Sationary Partial Differental Equations.* vol. 2, Elsevier, 2005.

[8] M. Colin, L. Jeanjean. Solutions for a quasilinear Schrödinger equation: a dual approach. *Nonlinear Anal.*. 56:213–226, 2004.

[9] A. Floer, A. Weinstein. Nonspreading wave packets for the cubic Schödinger with a bounded potential. *J. Funct. Anal.*. 69:397–408, 1986.
[10] L. Jeanjean, K. Tanaka. A positive solution for a nonlinear Schrödinger equation on $\mathbb{R}^n$. *Indiana Univ. Math.*. 54:443–464, 2005.
[11] S. Kurihura. Large-amplitude quasi-solitons in superfluid films. *J. Phys. Soc. Japan*. 50:3262–3267, 1981.
[12] A. M. Kosevich, B. A. Ivanov, A. S. Kovalev. Magnetic solitons in superfluid films. *Physics Reports*. 194:117–238, 1990.
[13] H. L. Liu, H. B. Chen, X. X. Yang. Multiple solutions for superlinear Schrödinger-Poisson system with sign-changing potential and nonlinearity. *Computers and Mathematics with Applications*. 68:1982–1990, 2014.
[14] J. Q. Liu, Z. Q. Wang. Soliton solutions for quasilinear Schrödinger equations, I. *Proc. Amer. Math. Soc.*. 131:441–448, 2002..
[15] J. Q. Liu, Y. Q. Wang, Z. Q. Wang. Soliton solutions for quasilinear Schrödinger equations, II. *J. Differential Equations*. 187:473–493, 2003.
[16] J. Q. Liu, Y. Q. Wang, Z. Q. Wang. Solutions for quasilinear Schrödinger equations via the Nehari method. *Comm. Partial Differential Equations*. 29:879–901, 2004.
[17] V. G. Makhankov, V. K. Fedyanin. Non-linear effects in quasi-one-dimensional models of condensed matter theory. *Physics Reports*. 104:1–86, 1984.
[18] M. Poppenberg, K. Schmitt, Z. Q. Wang. On the existence of soliton solutions to quasilinear Schrödinger equations. *Calc. Var. Partial Differential Equations*. 14:329–344, 2002.
[19] G. R. W. Quispel, H. W. Capel. Equation of motion for the heisenberg spin chain. *Physica*. 110 A:41–80, 1982.
[20] P. H. Rabinowitz. Minimax Methods in Critical Point Theory with Applications to Differential Equations. *CBMS Reg. Conf. Ser. Math., vol. 65, Amer. Math. Soc., Providence,RI.*. 1986.
[21] U. Severo. Existence of weak solutions for quasilinear elliptic equations involving the p-Lapacian. *Electronic Journal of Differential Equations*. 56:1-16, 2008.
[22] J. Sun, H. Chen, L. Yang. Positive solutions of asymptotically linear Schrödinger-Poisson system with a radial potential vanishing at infinity. *Nonlinear Anal.*. 74:413-423, 2011.
[23] S. Takeno, S. Homma. Classical planar heisenberg ferromagnet, complex scalar fields and nonlinear excitation. *Progr. Theoret. Physics*. 65:172–189, 1981.
[24] M. Willem. Minimax Theorems. Birkhäuser, Boston, 1996.
[25] X. Wu. Multiple solutions for quasilinear Schrödinger equations with a parameter. *J. Differential Equations*. 256:2619–2632, 2014.
[26] W. M. Zou, M. Schechter. Critical Point Theory and Its Applications. *Springer, New York*. 2006.
[27] J. Zhang, X. H. Tang, W. Zhang. Infinitely many solutions of quasilinear Schrödinger equation with sign-changing potential. *J. Math. Anal. Appl.*. 420:1762–1775, 2014.

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