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NONHOMOGENEOUS ELLIPTIC EQUATIONS INVOLVING CRITICAL SOBOLEV EXPONENT AND WEIGHT

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Abstract. In this article we consider the problem

$$\begin{gathered} -\operatorname{div}\big(p(x)\nabla u\big) = |u|^{2^*-2}u + \lambda f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

where Ω is a bounded domain in $\mathbb{R}^N$, We study the relationship between the behavior of p near its minima on the existence of solutions.

1. Introduction and statement of main results

In this article we study the existence of solutions to the problem

$$\begin{gathered} -\operatorname{div}\big(p(x)\nabla u\big) = |u|^{2^*-2}u + \lambda f \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega, \end{gathered} \tag{1.1}$$

where Ω is a smooth bounded domain of $\mathbb{R}^N$, $N \geq 3$, f belongs to $H^{-1} = W^{-1,2}(\Omega) \setminus \{0\}$, $p \in H^1(\Omega) \cap C(\bar{\Omega})$ is a positive function, λ is a real parameter and $2^* = \frac{2N}{N-2}$ is the critical Sobolev exponent for the embedding of $H_0^1(\Omega)$ into $L^{2^*}(\Omega)$.

For a constant function p, problem (1.1) has been studied by many authors, in particular by Tarantello [8]. Using Ekeland's variational principle and minimax principles, she proved the existence of at least one solution of (1.1) with $\lambda = 1$ when $f \in H^{-1}$ and satisfies

$$\int_\Omega fu\,dx \leq K_N\Big(\int_\Omega |\nabla u|^2\Big)^{(N+2)/4} \quad \text{for } \int_\Omega |u|^{2^*} = 1,$$

with

$$K_N = \frac{4}{N-2}\Big(\frac{N-2}{N+2}\Big)^{(N+2)/4}.$$

Moreover when the above inequality is strict, she showed the existence of at least a second solution. These solutions are nonnegative when f is nonnegative.

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The following problem has been considered by several authors,

$$\begin{gathered} -\operatorname{div}(p(x)\nabla u) = |u|^{2^*-2}u + \lambda u \quad \text{in } \Omega \\ u > 0 \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega\,. \end{gathered} \tag{1.2}$$

We quote in particular the celebrate paper by Brezis and Nirenberg [4], and that of Hadiji and Yazidi [6]. In [4], the authors studied the case when p is constant.

To our knowledge, the case where p is not constant has been considered in [6] and [7]. The authors in [6] showed that the existence of solutions depending on a parameter λ, N, and the behavior of p near its minima. More explicitly: when $p \in H^1(\Omega) \cap C(\bar{\Omega})$ satisfies

$$p(x) = p_0 + \beta_k |x-a|^k + |x-a|^k \theta(x) \text{ in } B(a,\tau), \tag{1.3}$$

where k, β_k, τ are positive constants, and θ tends to 0 when x approaches a, with $a \in p^{-1}(\{p_0\}) \cap \Omega$, $p_0 = \min_{x\in\overline{\Omega}} p(x)$, and $B(a,\tau)$ denotes the ball with center 0 and radius τ, when $0 < k \le 2$, and p satisfies the condition

$$k\beta_k \le \frac{\nabla p(x).(x-a)}{|x-a|^k} \quad \text{a.e } x \in \Omega. \tag{1.4}$$

On the one hand, they obtained the existence of solutions to (1.2) if one of the following conditions is satisfied:

(i) $N \ge 4$, $k > 2$ and $\lambda \in\,]0, \lambda_1(p)[$;
(ii) $N \ge 4$, $k = 2$ and $\lambda \in\,]\tilde{\gamma}(N), \lambda_1(p)[$;
(iii) $N = 3$, $k \ge 2$ and $\lambda \in\,]\gamma(k), \lambda_1(p)[$;
(iv) $N \ge 3$, $0 < k < 2$ and p satisfies (1.4), $\lambda \in]\lambda^*, \lambda_1(p)[$;

where

$$\tilde{\gamma}(N) = \frac{(N-2)N(N+2)}{4(N-1)}\beta_2,$$

$\gamma(k)$ is a positive constant depending on k, and $\lambda^* \in [\tilde{\beta}_k \frac{N^2}{4}, \lambda_1(p)[$, with $\tilde{\beta}_k = \beta_k \min[(\operatorname{diam}\Omega)^{k-2}, 1]$.

On the other hand, non-existence results are given in the following cases:

(a) $N \ge 3$, $k > 0$ and $\lambda \le \delta(p)$.
(b) $N \ge 3$, $k > 0$ and $\lambda \ge \lambda_1(p)$.

We denote by $\lambda_1(p)$ the first eigenvalue of $(-\operatorname{div}(p\nabla.), H)$ and

$$\delta(p) = \frac{1}{2} \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega \nabla p(x)(x-a)|\nabla u|^2 dx}{\int_\Omega |u|^2 dx}.$$

Then we formulate the question: What happens in (1.1) when p is not necessarily a constant function? A response to this question is given in Theorem 1.5 below.

Notation. S is the best Sobolev constant for the embedding from $H_0^1(\Omega)$ to $L^{2^*}(\Omega)$. $\|\cdot\|$ is the norm of $H_0^1(\Omega)$ induced by the product $(u,v) = \int_\Omega \nabla u \nabla v\, dx$. $\|\cdot\|_{-1}$ and $|\cdot|_p = (\int_\Omega |.|^p dx)^{1/p}$ are the norms in H^{-1} and $L^p(\Omega)$ for $1 \le p < \infty$ respectively. We denote the space $H_0^1(\Omega)$ by H and the integral $\int_\Omega u\, dx$ by $\int u$. ω_N is the area of the sphere $\mathbb{S}^{N-1}$ in $\mathbb{R}^N$.

Let $E = \{u \in H : \int_\Omega \tilde{f}(x)u(x)dx > 0\}$ and

$$\alpha(p) := \frac{1}{2} \inf_{u\in E} \frac{\int_\Omega \hat{p}(x)|\nabla u(x)|^2 dx}{\int_\Omega \tilde{f}(x)u(x)dx},$$

with

$$\tilde{f}(x) := \nabla f(x).(x-a) + \frac{N+2}{2} f(x), \quad \hat{p}(x) = \nabla p(x).(x-a).$$

Put

$$\begin{gathered} \Lambda_0 := K_N \frac{p_0^{1/2}}{\|f\|_{-1}} (S(p))^{N/4}, \quad A_l = (N-2)^2 \int_{\mathbb{R}^N} \frac{|x|^{l+2}}{(1+|x|^2)^N}, \\ B = \int_{\mathbb{R}^N} \frac{1}{(1+|x|^2)^N}, \quad D := w_0(a) \int_{\mathbb{R}^N} (1+|x|^2)^{(N+2)/2}, \end{gathered} \tag{1.5}$$

where $l \geq 0$ and

$$S(p) := \inf_{u\in H\setminus\{0\}} \frac{\int_\Omega p(x)|\nabla u|^2}{|u|_{2^*}^2}.$$

Definition 1.1. We say that u is a ground state solution of (1.1) if $J_\lambda(u) = \min\{J_\lambda(v) : v \text{ is a solution of (1.1)}\}$. Here J_λ is the energy functional associate with (1.1).

Remark 1.2. By the Ekeland variational principle [5] we can prove that for $\lambda \in (0, \Lambda_0)$ there exists a ground state solution to (1.1) which will be denoted by w_0. The proof is similar to that in [8].

Remark 1.3. Noting that if u is a solution of the problem (1.1), then $-u$ is also a solution of the problem (1.1) with $-\lambda$ instead of λ. Without loss of generality, we restrict our study to the case $\lambda \geq 0$.

Our main results read as follows.

Theorem 1.4. *Suppose that Ω is a star shaped domain with respect to a and p satisfies* (1.3). *Then there is no solution of problem* (1.1) *in E for all $0 \leq \lambda \leq \alpha(p)$.*

Theorem 1.5. *Let $p \in H^1(\Omega) \cap C(\bar{\Omega})$ such that $p_0 > 0$ and p satisfies* (1.3) *then, for $0 < \lambda < \frac{\Lambda_0}{2}$, problem* (1.1) *admits at least two solutions in one of the following condition:*

(i) $k > \frac{N-2}{2}$,

(ii) $\beta_{(N-2)/2} > \frac{2D}{A_{(N-2)/2}} (\frac{A_0}{B})^{(6-N)/4}$.

This article is organized as follows: in the forthcoming section, we give some preliminaries. Section 3 and 4 present the proofs of our main results.

2. Preliminaries

A function u in H is said to be a weak solution of (1.1) if u satisfies

$$\int (p\nabla u \nabla v - |u|^{2^*-2}uv - \lambda f v) = 0 \quad \text{for all } v \in H.$$

It is well known that the nontrivial solutions of (1.1) are equivalent to the non zero critical points of the energy functional

$$J_\lambda(u) = \frac{1}{2}\int p|\nabla u|^2 - \frac{1}{2^*}\int |u|^{2^*} - \lambda \int fu\,. \tag{2.1}$$

We know that J_λ is not bounded from below on H, but it is on a natural manifold called Nehari manifold, which is defined by

$$\mathcal{N}_\lambda = \{u \in H \setminus \{0\} : \langle J'_\lambda(u), u\rangle = 0\}.$$

Therefore, for $u \in \mathcal{N}_\lambda$, we obtain

$$J_\lambda(u) = \frac{1}{N}\int p|\nabla u|^2 - \lambda\frac{N+2}{2N}\int fu, \tag{2.2}$$

or

$$J_\lambda(u) = -\frac{1}{2}\int p|\nabla u|^2 + \frac{N+2}{2N}\int |u|^{2^*}. \tag{2.3}$$

It is known that the constant S is achieved by the family of functions

$$U_\varepsilon(x) = \frac{\varepsilon^{(N-2)/2}}{(\varepsilon^2+|x|^2)^{(N-2)/2}} \quad \varepsilon > 0, \quad x \in \mathbb{R}^N, \tag{2.4}$$

For $a \in \Omega$, we define $U_{\varepsilon,a}(x) = U_\varepsilon(x-a)$ and $u_{\varepsilon,a}(x) = \xi_a(x)U_{\varepsilon,a}(x)$, where

$$\xi_a \in C_0^\infty(\Omega) \quad \text{with } \xi_a \geq 0 \text{ and } \xi_a = 1 \text{ in a neighborhood of } a. \tag{2.5}$$

We start with the following lemmas given without proofs and based essentially on [8].

Lemma 2.1. *The functional J_λ is coercive and bounded from below on $\mathcal{N}_\lambda$.*

Set

$$\Psi_\lambda(u) = \langle J'_\lambda(u), u\rangle. \tag{2.6}$$

For $u \in \mathcal{N}_\lambda$, we obtain

$$\langle \Psi'_\lambda(u), u\rangle = \int p|\nabla u|^2 - (2^*-1)\int |u|^{2^*} \tag{2.7}$$

$$= (2-2^*)\int p|\nabla u|^2 - \lambda(1-2^*)\int fu. \tag{2.8}$$

So it is natural to split $\mathcal{N}_\lambda$ into three subsets corresponding to local maxima, local minima and points of inflection defined respectively by

$$\mathcal{N}_\lambda^+ = \{u \in \mathcal{N}_\lambda : \langle \Psi'_\lambda(u), u\rangle > 0\}, \quad \mathcal{N}_\lambda^- = \{u \in \mathcal{N}_\lambda : \langle \Psi'_\lambda(u), u\rangle < 0\},$$
$$\mathcal{N}_\lambda^0 = \{u \in \mathcal{N}_\lambda : \langle \Psi'_\lambda(u), u\rangle = 0\}.$$

Lemma 2.2. *Suppose that u_0 is a local minimizer of J_λ on $\mathcal{N}_\lambda$. Then if $u_0 \notin \mathcal{N}_\lambda^0$, we have $J'_\lambda(u_0) = 0$ in H^{-1}.*

Lemma 2.3. *For each $\lambda \in (0, \Lambda_0)$ we have $\mathcal{N}_\lambda^0 = \emptyset$.*

By Lemma 2.3, we have $\mathcal{N}_\lambda = \mathcal{N}_\lambda^+ \cup \mathcal{N}_\lambda^-$ for all $\lambda \in (0, \Lambda_0)$. For $u \in H \setminus \{0\}$, let

$$t_m = t_{\max}(u) := \Big(\frac{\int p|\nabla u|^2}{(2^*-1)\int |u|^{2^*}}\Big)^{(N-2)/4}.$$

Lemma 2.4. *Suppose that $\lambda \in (0, \Lambda_0)$ and $u \in H \setminus \{0\}$, then*

(i) If $\int fu \leq 0$, then there exists an unique $t^+ = t^+(u) > t_m$ such that $t^+u \in \mathcal{N}_\lambda^-$ and

$$J_\lambda(t^+u) = \sup_{t \geq t_m} J_\lambda(tu).$$

(ii) If $\int fu > 0$, then there exist unique $t^- = t^-(u)$, $t^+ = t^+(u)$ such that $0 < t^- < t_m < t^+$, $t^-u \in \mathcal{N}_\lambda^+$, $t^+u \in \mathcal{N}_\lambda^-$ and

$$J_\lambda(t^+u) = \sup_{t\geq t_m} J_\lambda(tu); \quad J_\lambda(t^-u) = \inf_{0\leq t\leq t^+} J_\lambda(tu).$$

Thus we put

$$c = \inf_{u\in\mathcal{N}_\lambda} J_\lambda(u), \quad c^+ = \inf_{u\in\mathcal{N}_\lambda^+} J_\lambda(u), \quad c^- = \inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u).$$

Lemma 2.5. *(i) If $\lambda \in (0, \Lambda_0)$, then $c \leq c^+ < 0$.*
(ii) If $\lambda \in (0, \frac{\Lambda_0}{2})$, then $c^- > 0$.

3. Nonexistence result

Some properties of $\alpha(p)$.

Proposition 3.1. *(1) Assume that $p \in C^1(\Omega)$ and there exists $b \in \Omega$ such that $\nabla p(b)(b-a) < 0$ and $f \in C^1$ in a neighborhood of b. Then $\alpha(p) = -\infty$.*
(2) If $p \in C^1(\Omega)$ satisfying (1.3) with $k > 2$ and $\nabla p(x)(x-a) \geq 0$ for all $x \in \Omega$ and $f \in C^1$ in a neighborhood of a and $f(a) \neq 0$, then $\alpha(p) = 0$ for all $N \geq 3$.
(3) If $p \in H^1(\Omega) \cap C(\bar{\Omega})$ and $\nabla p(x)(x-a) \geq 0$ a.e $x \in \Omega$, then $\alpha(p) \geq 0$.

Proof. (1) Set $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that

$$0 \leq \varphi \leq 1, \quad \varphi(x) = \begin{cases} 1 & \text{if } x \in B(0;r) \\ 0 & \text{if } x \notin B(0;2r), \end{cases} \tag{3.1}$$

where $0 < r < 1$.

Set $\varphi_j(x) = \operatorname{sgn}[\tilde{f}(x)]\varphi(j(x-b))$ for $j \in \mathbb{N}^*$. We have

$$\alpha(p) \leq \frac{1}{2}\frac{\int_{B(b,\frac{2r}{j})} \hat{p}(x)|\nabla\varphi_j(x)|^2}{\int_{B(b,\frac{2r}{j})} \tilde{f}(x)\varphi_j(x)}.$$

Using the change of variable $y = j(x-b)$ and applying the dominated convergence theorem, we obtain

$$\alpha(p) \leq \frac{j^2}{2}\Big[\frac{\hat{p}(b)\int_{B(0,2r)} |\nabla\varphi(y)|^2}{|\tilde{f}(b)|\int_{B(0,2r)} \varphi(y)} + o(1)\Big],$$

letting $j \to \infty$, we obtain the desired result.

(2) Since $p \in C^1(\Omega)$ in a neighborhood V of a, we write

$$p(x) = p_0 + \beta_k|x-a|^k + \theta_1(x), \tag{3.2}$$

where $\theta_1 \in C^1(V)$ such that

$$\lim_{x\to a}\frac{\theta_1(x)}{|x-a|^k} = 0. \tag{3.3}$$

Thus, we deduce that there exists $0 < r < 1$, such that

$$\theta_1(x) \leq |x-a|^k, \quad \text{for all } x \in B(a,2r). \tag{3.4}$$

Let $\psi_j(x) = \operatorname{sgn}[\tilde{f}(x)]\varphi(j(x-a))$, $\varphi \in C_0^\infty(\mathbb{R}^N)$ defined as in (3.1), we have

$$0 \leq \alpha(p) \leq \frac{1}{2}\frac{\int \nabla p(x).(x-a)|\nabla\psi_j(x)|^2}{\int \tilde{f}(x)\psi_j(x)}.$$

Using (3.2), we obtain

$$0 \le \alpha(p) \le \frac{k\beta_k}{2} \frac{\int_{B(a,\frac{2r}{j})} |x-a|^k |\nabla\psi_j(x)|^2}{\int_{B(a,\frac{2r}{j})} \tilde{f}(x)\psi_j(x)} + \frac{1}{2} \frac{\int_{B(a,\frac{2r}{j})} \nabla\theta_1(x).(x-a)|\nabla\psi_j(x)|^2}{\int_{B(a,\frac{2r}{j})} \tilde{f}(x)\psi_j(x)}.$$

Using the change of variable $y = j(x-a)$, and integrating by parts the second term of the right hand side, we obtain

$$0 \le \alpha(p) \le \frac{k\beta_k}{2j^{k-2}} \frac{\int_{B(0,2r)} |y|^k |\nabla\varphi(y)|^2}{\int_{B(0,2r)} |\tilde{f}(\frac{y}{j}+a)|\varphi(y)} + \frac{j}{2} \frac{\int_{B(0,2r)} \theta_1(\frac{y}{j}+a) div(y|\nabla\varphi(y)|^2)}{\int_{B(0,2r)} |\tilde{f}(\frac{y}{j}+a)|\varphi(y)}.$$

Using (3.4) and applying the dominated convergence theorem, we obtain

$$\begin{aligned} 0 \le \alpha(p) \le & \frac{k\beta_k}{(N+2)j^{k-2}} \frac{\int_{B(0,2r)} |y|^k |\nabla\varphi(y)|^2}{|f(a)| \int_{B(0,2r)} \varphi(y)} \\ & + \frac{1}{(N+2)j^{k-1}} \frac{\int_{B0,2r)} |y|^k div(y|\nabla\varphi(y)|^2)}{|f(a)| \int_{B(0,2r)} \varphi(y)} + o(1). \end{aligned}$$

Therefore, for $k > 2$ we deduce that $\alpha(p) = 0$, which completes the proof. □

Proof of Theorem 1.4. Suppose that u is a solution of (1.1). We multiply (1.1) by $\nabla u(x).(x-a)$ and integrate over Ω, we obtain

$$\int |u|^{2^*-1} \nabla u(x).(x-a) = -\frac{N-2}{2} \int |u(x)|^{2^*}, \tag{3.5}$$

$$\lambda \int f(x) \nabla u(x).(x-a) = -\lambda \int (\nabla f(x).(x-a) + N f(x)) u(x), \tag{3.6}$$

$$\begin{aligned} & -\int \mathrm{div}(p(x)\nabla u(x)) \nabla u(x).(x-a) \\ & = -\frac{N-2}{2} \int p(x)|\nabla u(x)|^2 - \frac{1}{2} \int \nabla p(x).(x-a)|\nabla u(x)|^2 \\ & \quad - \frac{1}{2} \int_{\partial\Omega} p(x)(x-a).\nu |\frac{\partial u}{\partial \nu}|^2. \end{aligned} \tag{3.7}$$

Combining (3.5), (3.6) and (3.7), we obtain

$$\begin{aligned} & -\frac{N-2}{2} \int p(x)|\nabla u(x)|^2 - \frac{1}{2} \int \nabla p(x).(x-a)|\nabla u(x)|^2 \\ & - \frac{1}{2} \int_{\partial\Omega} p(x)(x-a).\nu |\frac{\partial u}{\partial \nu}|^2 \\ & = -\frac{N-2}{2} \int |u(x)|^{2^*} - \lambda \int (\nabla f(x).(x-a) + N f(x)) u(x). \end{aligned} \tag{3.8}$$

Multiplying (1.1) by $\frac{N-2}{2} u$ and integrating by parts, we obtain

$$\frac{N-2}{2} \int p(x)|\nabla u(x)|^2 = \frac{N-2}{2} \int |u(x)|^{2^*} + \lambda \frac{N-2}{2} \int f(x) u(x). \tag{3.9}$$

From (3.8) and (3.9), we obtain

$$-\frac{1}{2} \int \nabla p(x).(x-a)|\nabla u(x)|^2 - \frac{1}{2} \int_{\partial\Omega} p(x)(x-a).\nu |\frac{\partial u}{\partial \nu}|^2 + \lambda \int \tilde{f}(x) u(x) = 0.$$

Then

$$\lambda > \frac{1}{2}\frac{\int \nabla p(x).(x-a)|\nabla u(x)|^2}{\int \tilde{f}(x)u(x)} \geq \alpha(p). \tag{3.10}$$

Hence the desired result is obtained.

4. Existence of solutions

We begin by proving that

$$\inf_{u\in\mathcal{N}_\lambda^-} J_\lambda(u) = c^- < c + \frac{1}{N}(p_0 S)^{N/2}. \tag{4.1}$$

By some estimates in Brezis and Nirenberg [3], we have

$$\begin{aligned} |w_0 + Ru_{\varepsilon,a}|_{2^*}^{2^*} &= |w_0|_{2^*}^{2^*} + R^{2^*}|u_{\varepsilon,a}|_{2^*}^{2^*} + 2^*R\int |w_0|^{2^*-2}w_0u_{\varepsilon,a} \\ &\quad + 2^*R^{2^*-1}\int u_{\varepsilon,a}^{2^*-1}w_0 + o(\varepsilon^{(N-2)/2}), \end{aligned} \tag{4.2}$$

Put

$$|\nabla u_{\varepsilon,a}|_2^2 = A_0 + O(\varepsilon^{N-2}), \quad |u_{\varepsilon,a}|_{2^*}^{2^*} = B + O(\varepsilon^N), \tag{4.3}$$

$$S = S(1) = A_0 B^{-2/2^*}. \tag{4.4}$$

Lemma 4.1. *Let $p \in H^1(\Omega)\cap C(\bar{\Omega})$ satisfying (1.3) Then we have estimate*

$$\begin{aligned} &\int p(x)|\nabla u_{\varepsilon,a}(x)|^2 \\ &\leq \begin{cases} p_0A_0 + O(\varepsilon^{N-2}) & \text{if } N-2<k, \\ p_0A_0 + A_k\varepsilon^k + o(\varepsilon^k) & \text{if } N-2>k, \\ p_0A_0 + \frac{(N-2)^2}{2}(\beta_{N-2}+M)\omega_N\varepsilon^{N-2}|\ln\varepsilon| + o(\varepsilon^{N-2}|\ln\varepsilon|) & \text{if } N-2=k, \end{cases} \end{aligned}$$

where M is a positive constant.

Proof. by calculations,

$$\begin{aligned} &\varepsilon^{2-N}\int p(x)|\nabla u_{\varepsilon,a}(x)|^2 \\ &= \int \frac{p(x)|\nabla\xi_a(x)|^2}{(\varepsilon^2+|x-a|^2)^{N-2}} + (N-2)^2\int \frac{p(x)|\xi_a(x)|^2|x-a|^2}{(\varepsilon^2+|x-a|^2)^N} \\ &\quad - (N-2)\int \frac{p(x)\nabla\xi_a^2(x)(x-a)}{(\varepsilon^2+|x-a|^2)^{N-1}}. \end{aligned}$$

Suppose that $\xi_a \equiv 1$ in $B(a,r)$ with $r>0$ small enough. So, we obtain

$$\begin{aligned} &\varepsilon^{2-N}\int p(x)|\nabla u_{\varepsilon,a}(x)|^2 \\ &= \int_{\Omega\setminus B(a,r)} \frac{p(x)|\nabla\xi_a(x)|^2}{(\varepsilon^2+|x-a|^2)^{N-2}} + (N-2)^2\int \frac{p(x)|\xi_a(x)|^2|x-a|^2}{(\varepsilon^2+|x-a|^2)^N} \\ &\quad - 2(N-2)\int_{\Omega\setminus B(a,r)} \frac{p(x)\xi_a(x)\nabla\xi_a(x)(x-a)}{(\varepsilon^2+|x-a|^2)^{N-1}}. \end{aligned}$$

Applying the dominated convergence theorem,

$$\int p(x)|\nabla u_{\varepsilon,a}(x)|^2 = (N-2)^2\varepsilon^{N-2}\int \frac{p(x)|\xi_a(x)|^2|x-a|^2}{(\varepsilon^2+|x-a|^2)^N} + O(\varepsilon^{N-2}).$$

Using expression (1.3), we obtain

$$\begin{aligned}
&\varepsilon^{2-N}(N-2)^{-2}\int p(x)|\nabla u_{\varepsilon,a}(x)|^2\\
&= \int_{B(a,\tau)} \frac{p_0|x-a|^2+\beta_k|x-a|^{k+2}+\theta(x)|x-a|^{k+2}}{(\varepsilon^2+|x-a|^2)^N}\\
&\quad + \int_{\Omega\setminus B(a,\tau)} \frac{p(x)|\xi_a(x)|^2|x-a|^2}{(\varepsilon^2+|x-a|^2)^N} + O(\varepsilon^{N-2}).
\end{aligned}$$

Using again the definition of ξ_a, and applying the dominated convergence theorem, we obtain

$$\begin{aligned}
&\varepsilon^{2-N}(N-2)^{-2}\int p(x)|\nabla u_{\varepsilon,a}(x)|^2\\
&= p_0\int_{\mathbb{R}^N} \frac{|x-a|^2}{(\varepsilon^2+|x-a|^2)^N} + \beta_k\int_{B(a,\tau)} \frac{|x-a|^{k+2}}{(\varepsilon^2+|x-a|^2)^N}\\
&\quad + \int_{B(a,\tau)} \frac{\theta(x)|x-a|^{k+2}}{(\varepsilon^2+|x-a|^2)^N} + O(\varepsilon^{N-2}).
\end{aligned}$$

We distinguish three cases:

Case 1. If $k<N-2$,

$$\begin{aligned}
&\varepsilon^{2-N}(N-2)^{-2}\int p(x)|\nabla u_{\varepsilon,a}(x)|^2\\
&= p_0\int_{\mathbb{R}^N} \frac{|x-a|^2}{(\varepsilon^2+|x-a|^2)^N} + \int_{B(a,\tau)} \frac{\theta(x)|x-a|^{k+2}}{(\varepsilon^2+|x-a|^2)^N}\\
&\quad + \Big[\int_{\mathbb{R}^N} \frac{\beta_k|x-a|^{k+2}}{(\varepsilon^2+|x-a|^2)^N} - \int_{\mathbb{R}^N\setminus B(a,\tau)} \frac{\beta_k|x-a|^{k+2}}{(\varepsilon^2+|x-a|^2)^N}\Big] + O(\varepsilon^{N-2})
\end{aligned}$$

Using the change of variable $y=\varepsilon^{-1}(x-a)$ and applying the dominated convergence theorem, we obtain

$$\begin{aligned}
&(N-2)^{-2}\int p(x)|\nabla u_{\varepsilon,a}(x)|^2\\
&= p_0B_0 + \varepsilon^k\int_{\mathbb{R}^N} \frac{\beta_k|y|^{k+2}}{(1+|y|^2)^N} + \varepsilon^k\int_{\mathbb{R}^N} \frac{\theta(a+\varepsilon y)|y|^{k+2}}{(1+|y|^2)^N}\chi_{B(0,\frac{\tau}{\varepsilon})} + o(\varepsilon^k).
\end{aligned}$$

Since $\theta(x)$ tends to 0 when x tends to a, this gives us

$$\int p(x)|\nabla u_{\varepsilon,a}(x)|^2 = p_0A_0 + \beta_kA_k\varepsilon^k + o(\varepsilon^k).$$

Case 2. If $k>N-2$,

$$\begin{aligned}
\int p(x)|\nabla u_{\varepsilon,a}(x)|^2 = p_0A_0 + (N-2)^2\varepsilon^{N-2}\Big[&\int_{B(a,\tau)} \frac{(\beta_k+\theta(x))|x-a|^{k+2}}{(\varepsilon^2+|x-a|^2)^N}\\
&- \int_{B(a,\tau)\setminus\Omega} \frac{(\beta_k+\theta(x))|x-a|^{k+2}}{(\varepsilon^2+|x-a|^2)^N}\Big]
\end{aligned}$$

$$+ (N-2)^2\varepsilon^{N-2}\int_{B(a,\tau)}\frac{\theta(x)|x-a|^{k+2}}{(\varepsilon^2+|x-a|^2)^N} + O(\varepsilon^{N-2}).$$

By the change of variable $y = x - a$, we obtain

$$\begin{aligned}\int p(x)|\nabla u_{\varepsilon,a}(x)|^2 &= p_0A_0 + (N-2)^2\varepsilon^{N-2}\int_{B(0,\tau)}\frac{(\beta_k+\theta(a+y))|y|^{k+2}}{(\varepsilon^2+|y|^2)^N}\\ &\quad + (N-2)^2\varepsilon^{N-2}\int_{B(a,\tau)}\frac{\theta(a+y)|y|^{k+2}}{(\varepsilon+|y|^2)^N} + O(\varepsilon^{N-2}).\end{aligned}$$

Put $M := \max_{x\in\bar\Omega}\theta(x)$ where $\theta(x)$ is given by (1.3). Then

$$\begin{aligned}&\int p(x)|\nabla u_{\varepsilon,a}(x)|^2\\ &= p_0A_0 + \varepsilon^{N-2}(N-2)^2(\beta_k+M)\int_{B(0,\tau)}\frac{|y|^{k+2}}{(\varepsilon^2+|y|^2)^N}dy + O(\varepsilon^{N-2}).\end{aligned}$$

Applying the dominated convergence theorem,

$$\int p(x)|\nabla u_{\varepsilon,a}(x)|^2 = p_0A_0 + O(\varepsilon^{N-2}).$$

Case 3. If $k = N-2$, following the same previous steps, we obtain

$$\begin{aligned}&\int p(x)|\nabla u_{\varepsilon,a}(x)|^2\\ &= p_0A_0 + (N-2)^2\varepsilon^{N-2}\int_{B(a,\tau)}\frac{\theta(x)|x-a|^{k+2}}{(\varepsilon^2+|x-a|^2)^N}\\ &\quad + (N-2)^2\varepsilon^{N-2}\Big[\int_{B(a,\tau)}\frac{\beta_{N-2}|x-a|^N}{(\varepsilon^2+|x-a|^2)^N} - \int_{B(a,\tau)\setminus\Omega}\frac{\beta_{N-2}|x-a|^N}{(\varepsilon^2+|x-a|^2)^N}\Big]\\ &\quad + O(\varepsilon^{N-2}).\end{aligned}$$

Therefore,

$$\begin{aligned}&\int p(x)|\nabla u_{\varepsilon,a}(x)|^2\\ &= p_0A_0 + (N-2)^2\varepsilon^{N-2}\int_{B(a,\tau)}\frac{(\beta_{N-2}+\theta(x))|x-a|^N}{(\varepsilon^2+|x-a|^2)^N}\\ &\quad + (N-2)^2\varepsilon^{N-2}\int_{B(a,\tau)}\frac{\theta(x)|x-a|^{k+2}}{(\varepsilon^2+|x-a|^2)^N} + O(\varepsilon^{N-2}).\end{aligned}$$

Then

$$\begin{aligned}&\int p(x)|\nabla u_{\varepsilon,a}(x)|^2\\ &\le p_0A_0 + (N-2)^2\varepsilon^{N-2}(\beta_{N-2}+M)\int_{B(a,\tau)}\frac{|x-a|^N}{(\varepsilon^2+|x-a|^2)^N} + O(\varepsilon^{N-2}).\end{aligned}$$

On the other hand

$$\begin{aligned}\varepsilon^{N-2}\int_{B(a,\tau)}\frac{|x-a|^N}{(\varepsilon^2+|x-a|^2)^N} &= \omega_N\varepsilon^{N-2}\int_0^\tau\frac{r^{2N-1}}{(\varepsilon^2+r^2)^N}dr + O(\varepsilon^{N-2})\\ &= \frac{1}{2N}\omega_N\varepsilon^{N-2}\int_0^\tau\frac{((\varepsilon^2+r^2)^N)'}{(\varepsilon^2+r^2)^N}dr + O(\varepsilon^{N-2}),\end{aligned}$$

and

$$\varepsilon^{N-2}\int_{B(a,\tau)}\frac{|x-a|^N}{(\varepsilon^2+|x-a|^2)^N}=\frac{1}{2}\omega_N\varepsilon^{N-2}|\ln\varepsilon|+o(\varepsilon^{N-2}|\ln\varepsilon|), \tag{4.5}$$

Therefore,

$$\int p(x)|\nabla u_{\varepsilon,a}(x)|^2\le p_0A_0+\frac{(N-2)^2}{2}(\beta_{N-2}+M)\omega_N\varepsilon^{N-2}|\ln\varepsilon|+o(\varepsilon^{N-2}|\ln\varepsilon|).$$

□

Knowing that $w_0\neq 0$, we set $\Omega'\subset\Omega$ as a set of positive measure such that $w_0>0$ on Ω'. Suppose that $a\in\Omega'$ (otherwise replace w_0 by $-w_0$ and f by $-f$).

Lemma 4.2. *For each $R>0$ and $2k>N-2$, there exists $\varepsilon_0=\varepsilon_0(R,a)>0$ such that*

$$J_\lambda(w_0+Ru_{\varepsilon,a})<c+\frac{1}{N}(p_0S)^{N/2},\quad \text{for all } 0<\varepsilon<\varepsilon_0.$$

Proof. We have

$$\begin{aligned}J_\lambda(w_0+Ru_{\varepsilon,a})&=\frac12\int p|\nabla w_0|^2+R\int p\nabla w_0\nabla u_{\varepsilon,a}+\frac{R^2}{2}\int p|\nabla u_{\varepsilon,a}|^2\\&\quad-\frac{1}{2^*}\int|w_0+Ru_{\varepsilon,a}|^{2^*}-\lambda\int fw_0-\lambda R\int fu_{\varepsilon,a}.\end{aligned}$$

Using (4.2), (4.3) and the fact that w_0 satisfies (1.1), we obtain

$$\begin{aligned}&J_\lambda(w_0+Ru_{\varepsilon,a})\\&\le c+\frac{R^2}{2}\int p|\nabla u_{\varepsilon,a}|^2-\frac{R^{2^*}}{2^*}A-R^{2^*-1}\int u_{\varepsilon,a}^{2^*-1}w_0+o\big(\varepsilon^{(N-2)/2}\big).\end{aligned}$$

Taking $w=0$ the extension of w_0 by 0 outside of Ω, it follows that

$$\begin{aligned}\int u_{\varepsilon,a}^{2^*-1}w_0&=\int_{\mathbb{R}^N}w(x)\xi_a(x)\frac{\varepsilon^{(N+2)/2}}{(\varepsilon^2+|x-a|^2)^{(N+2)/2}}\\&=\varepsilon^{(N-2)/2}\int_{\mathbb{R}^N}w(x)\xi_a(x)\frac{1}{\varepsilon^N}\psi(\frac{x}{\varepsilon})\end{aligned}$$

where $\psi(x)=(1+|x|^2)^{(N+2)/2}\in L^1(\mathbb{R}^N)$. We deduce that

$$\int_{\mathbb{R}^N}w(x)\xi_a(x)\frac{1}{\varepsilon^N}\psi(\frac{x}{\varepsilon})\to D\quad\text{as } \varepsilon\to 0\,.$$

Then

$$\int u_{\varepsilon,a}^{2^*-1}w_0=\varepsilon^{(N-2)/2}D+o(\varepsilon^{(N-2)/2}).$$

Consequently

$$\begin{aligned}&J_\lambda(w_0+Ru_{\varepsilon,a})\\&\le c+\frac{R^2}{2}\int p|\nabla u_{\varepsilon,a}|^2-\frac{R^{2^*}}{2^*}B-R^{2^*-1}\varepsilon^{(N-2)/2}D+o\big(\varepsilon^{(N-2)/2}\big).\end{aligned}\tag{4.6}$$

Replacing $\int p|\nabla u_{\varepsilon,a}|^2$ by its value in (4.6), we obtain

$$J_\lambda(w_0+Ru_{\varepsilon,a})$$

$$\leq \begin{cases} c+\frac{R^2}{2}p_0A_0-\frac{R^{2^*}}{2^*}B-\varepsilon^{(N-2)/2}DR^{2^*-1}+o(\varepsilon^{(N-2)/2}) & \text{if } k>\frac{N-2}{2},\\ c+\frac{R^2}{2}p_0A_0-\frac{R^{2^*}}{2^*}B+\beta_kA_k\varepsilon^k+o(\varepsilon^k) & \text{if } k<\frac{N-2}{2},\\ c+\frac{R^2}{2}p_0A_0-\frac{R^{2^*}}{2^*}B-\varepsilon^{(N-2)/2}\Big(\frac{R^2}{2}\beta_{(N-2)/2}A_{(N-2)/2} & \\ -DR^{2^*-1}\Big)+o(\varepsilon^{(N-2)/2}) & \text{if } k=\frac{N-2}{2}. \end{cases}$$

Using that the function $R\mapsto\Phi(R)=\frac{R^2}{2}B-\frac{R^{2^*}}{2^*}A_0$ attains its maximum $\frac{1}{N}(p_0S)^{N/2}$ at the point $R_1:=(\frac{A_0}{B})^{(N-2)/4}$, we obtain

$$\begin{aligned} &J_\lambda(w_0+Ru_{\varepsilon,a})\\ &\leq \begin{cases} c+\frac{1}{N}(p_0S)^{N/2}-\varepsilon^{(N-2)/2}DR_1^{2^*-1}+o(\varepsilon^{(N-2)/2}) & \text{if } k>\frac{N-2}{2},\\ c+\frac{1}{N}(p_0S)^{N/2}+A_k\varepsilon^k+o(\varepsilon^k) & \text{if } k<\frac{N-2}{2},\\ c+\frac{1}{N}(p_0S)^{N/2}-\varepsilon^{(N-2)/2}\Big(\frac{R_1^2}{2}\beta_{(N-2)/2}A_{(N-2)/2} & \\ -DR_1^{2^*-1}\Big)+o(\varepsilon^{(N-2)/2}) & \text{if } k=\frac{N-2}{2}. \end{cases} \end{aligned}$$

So for $\varepsilon_0=\varepsilon_0(R,a)>0$ small enough, $k>\frac{N-2}{2}$ or $k=\frac{N-2}{2}$ and

$$\beta_{(N-2)/2}>\frac{2DR_1^{2^*-3}}{B_{(N-2)/2}},$$

we conclude that

$$J_\lambda(w_0+Ru_{\varepsilon,a})<c+\frac{1}{N}(p_0S)^{N/2}, \tag{4.7}$$

for all $0<\varepsilon<\varepsilon_0$. □

Proposition 4.3. *Let $\{u_n\}\subset\mathcal{N}_\lambda^-$ be a minimizing sequence such that:*

(a) $J_\lambda(u_n)\to c^-$ *and*
(b) $\|J'_\lambda(u_n)\|_{-1}\to 0$.

Then for all $\lambda\in(0,\Lambda_0/2)$, $\{u_n\}$ admits a subsequence that converges strongly to a point w_1 in H such that $w_1\in\mathcal{N}_\lambda^-$ and $J_\lambda(w_1)=c^-$.

Proof. Let $u\in H$ be such that $\|u\|=1$. Then

$$t^+(u)u\in\mathcal{N}_\lambda^- \quad\text{and}\quad J_\lambda(t^+(u)u)=\max_{t\geq t_m}J_\lambda(tu).$$

The uniqueness of $t^+(u)$ and its extremal property give that $u\mapsto t^+(u)$ is a continuous function. We put

$$U_1=\{u=0 \text{ or } u\in H\setminus\{0\}:\|u\|<t^+(\frac{u}{\|u\|})\},$$
$$U_2=\{u\in H\setminus\{0\}:\|u\|>t^+(\frac{u}{\|u\|})\}.$$

Then $H\setminus\mathcal{N}_\lambda^-=U_1\cup U_2$ and $\mathcal{N}_\lambda^+\subset U_1$. In particular $w_0\in U_1$.

As in [8], there exists $R_0>0$ and $\varepsilon>0$ such that $w_0+R_0u_{\varepsilon,a}\in U_2$. We put

$$\mathcal{F}=\{h:[0,1]\to H \text{ continuous}, h(0)=w_0 \text{ and } h(1)=w_0+R_0u_{\varepsilon,a}\}.$$

It is clear that $h:[0,1]\to H$ with $h(t)=w_0+tR_0u_{\varepsilon,a}$ belongs to $\mathcal{F}$. Thus by Lemma 4.2, we conclude that

$$c_0=\inf_{h\in\mathcal{F}}\max_{t\in[0,1]}J_\lambda(h(t))<c+\frac{1}{N}(p_0S)^{N/2}. \tag{4.8}$$

Since $h(0) \in U_1$, $h(1) \in U_2$ and h is continuous, there exists $t_0 \in]0,1[$ such that $h(t_0) \in \mathcal{N}_\lambda^-$ Hence

$$c_0 \geq c^- = \inf_{u \in \mathcal{N}_\lambda^-} J_\lambda(u). \tag{4.9}$$

Applying again the Ekeland variational principle, we obtain a minimizing sequence $(u_n) \subset \mathcal{N}_\lambda^-$ such that (a) $J_\lambda(u_n) \to c^-$ and (b) $\|J'_\lambda(u_n)\|_{-1} \to 0$. Thus, we obtain a subsequence (u_n) such that

$$u_n \to w_1 \text{ strongly in } H.$$

This implies that w_1 is a critical point for J_λ, $w_1 \in \mathcal{N}_\lambda^-$ and $J_\lambda(w_1) = c^-$. □

Proof of Theorem 1.5. From the facts that $w_0 \in \mathcal{N}_\lambda^+$, $w_1 \in \mathcal{N}_\lambda^-$ and $\mathcal{N}_\lambda^+ \cap \mathcal{N}_\lambda^- = \emptyset$ for $\lambda \in (0, \frac{\Lambda_0}{2})$, we deduce that problem (1.1) admits at least two distinct solutions in H. □

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