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MONOTONE ITERATIVE METHOD FOR FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this article, by using the lower and upper solution method, we prove the existence of iterative solutions for a class of fractional initial value problem with non-monotone term

$$D_{0+}^{\alpha}u(t) = f(t,u(t)), \quad t \in (0,h),$$
$$t^{1-\alpha}u(t)\big|_{t=0} = u_0 \neq 0,$$

where $0 < h < +\infty$, $f \in C([0,h] \times \mathbb{R}, \mathbb{R})$, $D_{0+}^{\alpha}u(t)$ is the standard Riemann-Liouville fractional derivative, $0 < \alpha < 1$. A new condition on the nonlinear term is given to guarantee the equivalence between the solution of the IVP and the fixed-point of the corresponding operator. Moreover, Moreover, we show the existence of maximal and minimal solutions.

1. Introduction

Fractional differential equations have recently proved to be useful tools in the modeling of many physical phenomena. It draws a great application in nonlinear oscillations of earthquakes, many physical phenomena such as seepage flow in porous media and in fluid dynamic traffic model. For more details on fractional calculus theory, one can see [1-6, 8-18]. Some recent contributions to the theory of fractional differential equations initial value problems can be seen in [1].

In [15], the lower and upper solution method was used to study the IVP

$$D_{0+}^{\alpha}u(t) = f(t,u(t)), \quad t \in (0,1),\ (0 < \alpha < 1),$$
$$u(0) = 0,$$

where $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and $f(t,\cdot)$ is nondecreasing for each $t \in [0,1]$.

In [9, 13], the existence and uniqueness of solution of the initial value problem

$$D_{0+}^{\alpha}u(t) = f(t,u(t)),\ (0 < \alpha < 1;\ t > 0), \tag{1.1}$$
$$D_{0+}^{\alpha-1}u(0+) = u_0. \tag{1.2}$$

was obtained under the assumption that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is Lipchitz continuous, by using the Banach concentration mapping principle.

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In [17], the existence and uniqueness of solution of the initial value problem

$$D_{0+}^{\alpha}u(t) = f(t, u(t)), \quad t \in (0, T],$$
$$t^{1-\alpha}u(t)\big|_{t=0} = u_0$$

was discussed by using the method of lower and upper solutions and its associated monotone iterative method. In [12], a new proof of the maximum principle was given by using the completely monotonicity of the Mittag-Leffler type function. We refer the readers to [10] for other applications of monotone method to various fractional differential equations.

In the previous works, the nonlinear term has to satisfy the monotone or other control conditions. In fact, the nonlinear fractional differential equation with non-monotone term can respond better to impersonal law, so it is very important to weaken control conditions of the nonlinear term.

Motivated by the above references, we focus our attention on the problem

$$D_{0+}^{\alpha}u(t) = f(t, u(t)), \quad t \in (0, h), \tag{1.3}$$
$$t^{1-\alpha}u(t)\big|_{t=0} = u_0, \tag{1.4}$$

where $f \in C([0, h] \times \mathbb{R}, \mathbb{R})$, $D_{0+}^{\alpha}u(t)$ is the standard Riemann-Liouville fractional derivative, $0 < \alpha < 1$. The existence of the blow-up solution, that is to say $u \in C(0, h]$ and $\lim_{t\to 0+} u(t) = \infty$, is obtained by the use of the lower and upper solution method.

This paper is organized as follows. In section 2, we recall briefly some notion of fractional calculus and theory of the operators for integration and differentiation of fractional order. Section 3 is devoted to the study of the existence of solution for utilizing the method of upper and lower solutions. The existence of maximal and minimal solutions is also given.

2. Preliminaries

Given $0 \le a < b < +\infty$ and $r > 0$, define a set

$$C_r[a, b] = \{u : u \in C(a, b], (t - a)^r u(t) \in C[a, b]\}.$$

Clearly, $C_r[a, b]$ is a linear space with the normal multiplication and addition. Given $u \in C_r[a, b]$, define

$$\|u\| = \max_{t\in[a,b]} (t - a)^r |u(t)|,$$

then $(C_r[a, b], \|\cdot\|)$ is a normed space. Moreover, if $\{u_n\} \subset C_r[a, b]$ and $\|u_n - u\| \to 0$, then one has $u \in C_r[a, b]$. In fact, setting $v_n(t) = (t - a)^r u_n(t)$, $v(t) = (t - a)^r u(t)$, then $v_n \in C[a, b]$ and

$$\|u_n - u\| \to 0 \Leftrightarrow \|v_n - v\|_\infty \to 0.$$

By the completeness of the space $C[a, b]$, one has $v \in C[a, b]$, so $u(t) = (t - a)^{-r}v(t) \in C_r[a, b]$. Thus, $(C_r[a, b], \|\cdot\|)$ is a Banach space.

Lemma 2.1 ([9]). *The linear initial value problem*

$$D_{0+}^{\alpha}u(t) + \lambda u(t) = q(t),$$
$$t^{1-\alpha}u(t)\big|_{t=0} = u_0,$$

where $\lambda \geq 0$ is a constant and $q \in L(0,h)$, has the following integral representation for a solution

$$u(t) = \Gamma(\alpha) u_0 t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^\alpha) + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t-s)^\alpha) q(s) ds.$$

Here, $E_{\alpha,\alpha}(t)$ is a Mittag-Leffler function.

Lemma 2.2. *For $0 < \alpha \leq 1$, the Mittag-Leffler type function $E_{\alpha,\alpha}(-\lambda t^\alpha)$ satisfies*

$$0 \leq E_{\alpha,\alpha}(-\lambda t^\alpha) \leq \frac{1}{\Gamma(\alpha)}, \quad t \in [0,\infty),\ \lambda \geq 0.$$

Proof. According to [12, 14], the function $g(t) := E_{\alpha,\alpha}(-\lambda t^\alpha)$, $t \in (0,+\infty)$ is completely monotonic, that is to say that $g(t)$ possesses of derivatives $g^{(n)}(t)$ for all $n = 0, 1, 2, \dots$, and $(-1)^n f^{(n)}(t) \geq 0$ for all $t \in (0,\infty)$. This combined with the fact that $E_{\alpha,\alpha}(-\lambda t^\alpha)$ is continuous on $\mathbb{R}$ and $E_{\alpha,\alpha}(0) = 1/\Gamma(\alpha)$ yields the conclusion. □

Lemma 2.3 ([7]). *Suppose that E is an ordered Banach space, $x_0, y_0 \in E$, $x_0 \leq y_0$, $D = [x_0, y_0]$, $T : D \to E$ is an increasing completely continuous operator and $x_0 \leq Tx_0$, $y_0 \geq Ty_0$. Then the operator T has a minimal fixed point x^* and a maximal fixed point y^*. If we let*

$$x_n = Tx_{n-1}, \quad y_n = Ty_{n-1}, \quad n = 1, 2, 3, \dots,$$

then

$$x_0 \leq x_1 \leq x_2 \leq \dots \leq x_n \leq \dots \leq y_n \leq \dots \leq y_2 \leq y_1 \leq y_0,$$
$$x_n \to x^*, \quad y_n \to y^*.$$

Definition 2.4. A function $v(t) \in C_{1-\alpha}[0,h]$ is called as a lower solution of (1.3), (1.4), if it satisfies

$$D^\alpha_{0+} v(t) \leq f(t, v(t)), \quad t \in (0,h), \tag{2.1}$$
$$t^{1-\alpha} v(t)\big|_{t=0} \leq u_0. \tag{2.2}$$

Definition 2.5. A function $w(t) \in C_{1-\alpha}[0,h]$ is called as an upper solution of (1.3), (1.4), if it satisfies

$$D^\alpha_{0+} w(t) \geq f(t, w(t)), \quad t \in (0,h), \tag{2.3}$$
$$t^{1-\alpha} w(t)\big|_{t=0} \geq u_0. \tag{2.4}$$

3. Existence of solutions

The following assumptions will be used in our main results:

(A1) $f : [0,h] \times \mathbb{R} \to \mathbb{R}$ and there exist constants $A, B \geq 0$ and $0 < r_1 \leq 1 < r_2 < 1/(1-\alpha)$ such that for $t \in [0,h]$

$$|f(t,u) - f(t,v)| \leq A|u-v|^{r_1} + B|u-v|^{r_2}, \quad u, v \in \mathbb{R}. \tag{3.1}$$

(A2) Assume that $f : [0,h] \times \mathbb{R} \to \mathbb{R}$ satisfies

$$f(t,u) - f(t,v) + \lambda(u-v) \geq 0, \quad \text{for } \hat{u} \leq v \leq u \leq \tilde{u}, \tag{3.2}$$

where $\lambda \geq 0$ is a constant and $\hat{u}, \tilde{u}$ are lower and upper solutions of Problem (1.3), (1.4) respectively.

Remark 3.1. Assume that $f(t,u)=a(t)g(u)$ and g is a Hölder continuous function, $a(t)$ is bounded, then (3.1) holds.

Theorem 3.2. *Suppose* (A1) *holds. The function u solves problem* (1.3), (1.4) *if and only if it is a fixed-point of the operator $T: C_{1-\alpha}[0,h] \to C_{1-\alpha}[0,h]$ defined by*

$$\begin{aligned}(Tu)(t) &= \Gamma(\alpha)u_0 t^{\alpha-1}E_{\alpha,\alpha}(-\lambda t^\alpha)\\ &\quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)[f(s,u(s))+\lambda u(s)]ds.\end{aligned}$$

Proof. Firstly, we need to show that the operator T is well defined, i.e., for every $u \in C_{1-\alpha}[0,h]$ and $t>0$, the integral

$$\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)[f(s,u(s))+\lambda u(s)]ds$$

belongs to $C_{1-\alpha}[0,h]$.

Under condition (3.1),

$$|f(t,u)| \le A|u|^{r_1} + B|u|^{r_2} + C,$$

where $C = \max_{t\in[0,h]} f(t,0)$.

By Lemma 2.2, for $u(t) \in C_{1-\alpha}[0,h]$, we have

$$\begin{aligned}&\Big|t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)[f(s,u(s))+\lambda u(s)]ds\Big|\\
&\le t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)|f(s,u(s))+\lambda u(s)|ds\\
&\le t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)\big(A|u|^{r_1}+\lambda|u|+B|u|^{r_2}+C\big)ds\\
&\le t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)\Big\{As^{(\alpha-1)r_1}[s^{1-\alpha}|u(s)|]^{r_1}\\
&\quad + \lambda s^{\alpha-1}s^{1-\alpha}|u(s)| + Bs^{(\alpha-1)r_2}[s^{1-\alpha}|u(s)|]^{r_2}+C\}ds\\
&\le \frac{A\|u\|^{r_1}t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{(\alpha-1)r_1}ds + \frac{\lambda\|u\|t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{\alpha-1}ds\\
&\quad + \frac{B\|u\|^{r_2}t^{1-\alpha}}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}s^{(\alpha-1)r_2}ds + \frac{Ct}{\Gamma(\alpha+1)}\\
&\le A\|u\|^{r_1}\frac{\Gamma((\alpha-1)r_1+1)}{\Gamma((\alpha-1)r_1+\alpha+1)}t^{(\alpha-1)r_1+\alpha+1-\alpha} + \lambda\|u\|\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}t^\alpha\\
&\quad + B\|u\|^{r_2}\frac{\Gamma((\alpha-1)r_2+1)}{\Gamma((\alpha-1)r_2+\alpha+1)}t^{(\alpha-1)r_2+\alpha+1-\alpha} + \frac{Ct}{\Gamma(\alpha+1)}\\
&\le \frac{\Gamma[(\alpha-1)r_1+1]Ah^{(\alpha-1)r_1+1}}{\Gamma[(\alpha-1)r_1+\alpha+1]}\|u\|^{r_1} + \lambda\|u\|\frac{\Gamma(\alpha)}{\Gamma(2\alpha)}h^\alpha\\
&\quad + \frac{\Gamma[(\alpha-1)r_2+1]Bh^{(\alpha-1)r_2+1}}{\Gamma[(\alpha-1)r_2+\alpha+1]}\|u\|^{r_2} + \frac{Ch}{\Gamma(\alpha+1)}.\end{aligned}$$

That is to say that the integral exists and belongs to $C_{1-\alpha}[0,h]$.

The the above inequality and the assumption $0 < r_1 \le 1 < r_2 < 1/(1-\alpha)$ imply that

$$\lim_{t\to 0+} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)[f(s,u(s))+\lambda u(s)]ds = 0.$$

Combining with the fact that $\lim_{t\to 0+} E_{\alpha,\alpha}(-\lambda t^\alpha) = E_{\alpha,\alpha}(0) = 1/\Gamma(\alpha)$ yields that $\lim_{t\to 0+} t^{1-\alpha}(Tu)(t) = u_0$.

The above arguments combined with Lemma 2.1 implies that the fixed-point of the operator T solves (1.3), (1.4). And the vice versa. The proof is complete. $\square$

In the following, we consider the compactness of a set of the space $C_r[0,h]$. Let $F \subset C_r[0,h]$ and $E = \{g(t) = t^r h(t) \mid h(t) \in F\}$, then $E \subset C[0,h]$. It is clear that F is a bounded set of $C_r[0,h]$ if and only if E is a bounded set of $C[0,h]$.

Therefore, to proof that $F \subset C_r[0,h]$ is a compact set, it is sufficient to prove that $E \subset C[0,h]$ is a bounded and equicontinuous set.

Theorem 3.3. *Suppose* (A1) *holds. Then T is a completely continuous operator.*

Proof. Given $u_n \to u \in C_{1-\alpha}[0,h]$, with the definition of T and condition (A1), one has

$$\begin{aligned}
&\|Tu_n - Tu\| \\
&= \|t^{1-\alpha}(Tu_n - Tu)\|_\infty \\
&= \max_{0\le t\le h}\Big|t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)[f(s,u_n)-f(s,u)+\lambda(u_n-u)]ds\Big| \\
&\le \frac{1}{\Gamma(\alpha)}\max_{0\le t\le h} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}[A|u_n-u|^{r_1} + B|u_n-u|^{r_2} + \lambda|u_n-u|]ds \\
&\le \frac{1}{\Gamma(\alpha)}\Big[A\max_{0\le t\le h} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{-r_1(1-\alpha)}s^{r_1(1-\alpha)}|u_n-u|^{r_1}ds \\
&\quad + \lambda\max_{0\le t\le h} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{-(1-\alpha)}s^{(1-\alpha)}|u_n-u|ds \\
&\quad + B\max_{0\le t\le h} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{-r_2(1-\alpha)}s^{r_2(1-\alpha)}|u_n-u|^{r_2}ds\Big] \\
&\le \frac{1}{\Gamma(\alpha)}\Big[A\|u_n-u\|^{r_1}\max_{0\le t\le h} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{-r_1(1-\alpha)}ds \\
&\quad + \lambda\|u_n-u\|\max_{0\le t\le h} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{-(1-\alpha)}ds \\
&\quad + B\|u_n-u\|^{r_2}\max_{0\le t\le h} t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}s^{-r_2(1-\alpha)}ds\Big] \\
&\le \frac{A\|u_n-u\|^{r_1}\Gamma[1-r_1(1-\alpha)]}{\Gamma[1-r_1(1-\alpha)+\alpha]}h^{1-r_1(1-\alpha)} + \frac{\lambda\|u_n-u\|\Gamma[\alpha]}{\Gamma[2\alpha]}h^\alpha \\
&\quad + \frac{B\|u_n-u\|^{r_2}\Gamma[1-r_2(1-\alpha)]}{\Gamma[1-r_2(1-\alpha)+\alpha]}h^{1-r_2(1-\alpha)} \\
&\to 0, \quad (n\to\infty).
\end{aligned}$$

That is to say that T is continuous.

Suppose that $F \subset C_{1-\alpha}[0,h]$ is a bounded set. The argument as in the proof of Theorem 3.2 shows that $T(F) \subset C_{1-\alpha}[0,h]$ is bounded.

At last, we prove the equicontinuity of $T(F)$. Let $f_1(t,u) = f(t,u) + \lambda u$. Given $\epsilon > 0$, for every $u \in F$ and $t_1, t_2 \in [0,h], t_1 \le t_2$,

$$\begin{aligned}
&\big|[t^{1-\alpha}(Tu)(t)]_{t=t_2} - [t^{1-\alpha}(Tu)(t)]_{t=t_1}\big| \\
&\le \big[\Gamma(\alpha)u_0 E_{\alpha,\alpha}(-\lambda t^\alpha)\big]_{t_1}^{t_2} + \Big[t^{1-\alpha}\int_0^t (t-s)^{\alpha-1}E_{\alpha,\alpha}(-\lambda(t-s)^\alpha)f_1(s,u(s))ds\Big]_{t_1}^{t_2} \\
&\le \big[\Gamma(\alpha)u_0 E_{\alpha,\alpha}(-\lambda t^\alpha)\big]_{t_1}^{t_2} + \frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} t_2^{1-\alpha}(t_2-s)^{\alpha-1}|f_1(s,u(s))|ds \\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^{t_1}\big[t_2^{1-\alpha}(t_2-s)^{\alpha-1} - t_1^{1-\alpha}(t_1-s)^{\alpha-1}\big]s^{\alpha-1}\big|s^{1-\alpha}f_1(s,u(s))\big|ds.
\end{aligned}$$

For the first term of the above formula, by the function $E_{\alpha,\alpha}(-\lambda t^\alpha)$ is continuous and therefore uniformly continuous on $[0,h]$, there exists $\delta_1 > 0$ such that when $|t_2 - t_1| < \delta_1$, there is

$$\big[\Gamma(\alpha)u_0 E_{\alpha,\alpha}(-\lambda t^\alpha)\big]_{t_1}^{t_2} < \frac{\epsilon}{3};$$

For the second term, by the continuity of $t_2^{1-\alpha}(t_2-s)^{\alpha-1}|f_1(s,u(s))|$, there is a positive M_1 such that

$$\frac{1}{\Gamma(\alpha)}|t_2^{1-\alpha}(t_2-s)^{\alpha-1}f_1(s,u(s))| < M_1,$$

Thus, letting $\delta_2 = \epsilon/(3M_1)$, when $|t_2 - t_1| < \delta_2$, we have

$$\frac{1}{\Gamma(\alpha)}\int_{t_1}^{t_2} t_2^{1-\alpha}(t_2-s)^{\alpha-1}|f_1(s,u(s))|ds < \frac{\epsilon}{3};$$

For the third term, $\int_0^{t_1} s^{\alpha-1}ds = (t_1)^{\alpha-1}/(\alpha-1)$. By the continuity of the function $s^{1-\alpha}f_1(s,u(s))$, there is a positive constant M_2 such that

$$|s^{1-\alpha}f_1(s,u(s))| \le M_2;$$

The function $t_2^{1-\alpha}(t_2-s)^{\alpha-1} - t_1^{1-\alpha}(t_1-s)^{\alpha-1}$ is continuity and therefore uniform continuity on $[0,h]^3$, so there exists $\delta_3 > 0$ such that when $|t_2 - t_1| < \delta_3$, there is

$$\frac{1}{\Gamma(\alpha)}|t_2^{1-\alpha}(t_2-s)^{\alpha-1} - t_1^{1-\alpha}(t_1-s)^{\alpha-1}| < \frac{\epsilon(\alpha-1)}{3M_2h^{\alpha-1}},$$

thus

$$\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\big[t_2^{1-\alpha}(t_2-s)^{\alpha-1} - t_1^{1-\alpha}(t_1-s)^{\alpha-1}\big]s^{\alpha-1}\big|s^{1-\alpha}f(s,u(s))\big|ds < \frac{\epsilon}{3}.$$

To sum up, Given $\epsilon > 0$, for every $u \in F$ and $t_1, t_2 \in [0,h]$, let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, when $|t_2 - t_1| < \delta$, there holds

$$\big|[t^{1-\alpha}(Tu)(t)]_{t=t_2} - [t^{1-\alpha}(Tu)(t)]_{t=t_1}\big| < \epsilon.$$

That is to say that $T(F)$ is equicontinuous. The proof is complete. □

Theorem 3.4. *Assume* (A1), (A2) *hold, and $v, w \in C_{1-\alpha}[0,h]$ are lower and upper solutions of* (1.3) *and* (1.4) *respectively, such that*

$$v(t) \le w(t), \quad 0 \le t \le h. \tag{3.3}$$

Then, the fractional IVP (1.3), (1.4) *has a minimal solution* x^* *and a maximal solution* y^* *such that*

$$x^* = \lim_{n\to\infty} T^n v, \quad y^* = \lim_{n\to\infty} T^n w.$$

Proof. Clearly, if functions v, w are lower and upper solutions of IVP (1.3), (1.4), then there are $v \le Tv, w \ge Tw$. In fact, by the definition of the lower solution, there exist $q(t) \ge 0$ and $\epsilon \ge 0$ such that

$$\begin{aligned} D_{0+}^{\alpha} v(t) = f(t, v(t)) - q(t), \quad t \in (0, h), \\ t^{1-\alpha} v(t) = u_0 - \epsilon. \end{aligned}$$

Using Theorem 3.2 and Lemma 2.2, one has

$$\begin{aligned} v(t) &= \Gamma(\alpha)(u_0 - \epsilon)t^{\alpha-1} E_{\alpha,\alpha}(-\lambda t^{\alpha}) \\ &\quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha}(-\lambda (t-s)^{\alpha})[f(s, v(s)) + \lambda v(s) - q(s)]ds \\ &\le (Tv)(t). \end{aligned}$$

Similarly, there is $w \ge Tw$.

By Theorem 3.3 the operator $T : C_{1-\alpha}[0, h] \to C_{1-\alpha}[0, h]$ is increasing and completely continuous. Setting $D := [v, w]$, by the use of Lemma 2.3, the existence of x^*, y^* is obtained. The proof is complete. □

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