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EXACT CONTROLLABILITY FOR A STRING EQUATION IN DOMAINS WITH MOVING BOUNDARY IN ONE DIMENSION

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Abstract. We consider a string equation in a domain with moving boundary. By using the multiplier method in non-cylindrical domains, we establish the exact boundary controllability in domains with moving boundary, and obtain a weaker condition on the time controllability.

1. Introduction and main results

Let $T > 0$. For any given $k \in (0,1)$, set $\alpha_k(t) = 1 + kt$ for $t \in [0,T]$. Denote by $\hat{Q}_T^k$ the non-cylindrical domain in R^2,

$$\hat{Q}_T^k = \{(x,t) \in R^2; 0 < x < \alpha_k(t), t \in (0,T)\}.$$

Consider the controlled string equation in $\hat{Q}_T^k$:

$$\begin{gathered} u_{tt} - u_{xx} = 0 \quad \text{in } \hat{Q}_T^k, \\ u(0,t) = 0, \quad u(\alpha_k(t),t) = v \quad \text{on } (0,T), \\ u(0) = u^0, \quad u_t(0) = u^1 \quad \text{in } (0,1), \end{gathered} \tag{1.1}$$

where u is the state variable, v is the control variable and $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ is any given initial data. (1.1) may describe the motion of a string with a fixed endpoint and a moving one. By Milla Miranda [11], for $0 < k < 1$, any $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ and $v \in L^2(0,T)$, (1.1) admits a unique solution in the sense of transposition.

The goal of this article is to study the exact controllability of (1.1) in the following sense.

Definition 1.1. Problem (1.1) is said to be exactly controllable at time T, if for any initial state $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$ and any preassigned state $(u_T^0, u_T^1) \in L^2(0, \alpha_k(T)) \times H^{-1}(0, \alpha_k(T))$, there is a control $v \in L^2(0,T)$ such that the corresponding solution of (1.1) in the sense of transposition satisfies

$$u(T) = u_T^0, \quad u_t(T) = u_T^1.$$

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For $k \in (0,1)$, set

$$\bar{T}_k = \frac{2}{1-k}\,. \tag{1.2}$$

Our main result is stated as follows.

Theorem 1.2. *Suppose that $k \in (0,1)$. Then for any given $T > \bar{T}_k$, Problem* (1.1) *is exactly controllable at time T.*

Remark 1.3. It is easy to check that

$$\bar{T} \triangleq \lim_{k\to 0} \bar{T}_k = \lim_{k\to 0} \frac{2}{1-k} = 2.$$

And we can easily find that $\bar{T}_k < T_k^*$, where $T_k^* = \frac{e^{\frac{2k(1+k)}{1-k}}-1}{k}$ is the controllability time in [6]. Indeed, we have

$$T_k^* = \frac{e^{\frac{2k(1+k)}{1-k}}-1}{k} > \frac{2(1+k)}{1-k} > \bar{T}_k\,,$$

for $0 < k < 1$.

Similar to [8, Theorem 4.1], we use the Hilbert Uniqueness Method (HUM) to seek a control v in the special form $v(t) = w_x(\alpha_k(t), t)$, where w is the solution of the homogeneous problem

$$\begin{gathered} w_{tt} - w_{xx} = 0 \quad \text{in } \hat{Q}_T^k, \\ w(0,t) = 0, \quad w(\alpha_k(t), t) = 0 \quad \text{on } (0,T), \\ w(0) = w^0, \quad w_t(0) = w^1 \quad \text{in } (0,1), \end{gathered} \tag{1.3}$$

where $k \in (0,1), (w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$ is any given initial value, $\alpha_k(t) = 1 + kt$ is the function given in (1.1). According to the existence theorem in [4], Problem (1.3) has a unique solution

$$w \in C([0,T]; H_0^1(0,\alpha_k(t))) \cap C^1([0,T]; L^2(0,\alpha_k(t))).$$

We define the energy of the above system as

$$E(t) = \frac{1}{2}\int_0^{\alpha_k(t)} (w_x^2 + w_t^2)dx \quad \text{for } t \geq 0\,. \tag{1.4}$$

Next, we deduce a growth estimate of the energy of problem (1.3).

Theorem 1.4. *Let $w(x,t)$ be a solution of the problem* (1.3) *in $\hat{Q}_T^k$. Then $E(t)$ is nonincreasing for $t \geq 0$ and*

$$\frac{1-k}{(1+k)(1+kt)}E(0) \leq E(t) \leq \frac{1+k}{(1-k)(1+kt)}E(0) \quad \text{for } t \geq 0. \tag{1.5}$$

Remark 1.5. In the case of $k = 1$, some results have been obtained in [6]. However, we do not extend the approach developed in this paper to the case $k = 1$.

Remark 1.6. We can obtain the same results as in this article for a more general function $\alpha_k(t)$, as long as it meets the condition $0 < \alpha_k'(t) < 1$.

Remark 1.7. It would be quite interesting to the study Dirichlet control for multi-dimensional wave equations in non-cylindrical domains by the same approach as this paper. We shall consider Neumann control for wave equations in domains with moving boundary in the forthcoming papers.

Control and stabilization for wave equations in domains with moving boundary has been widely studied and many results are given, see [3, 4, 5, 9, 11, 12] and the references therein. Under a restrictive assumption on the domain with moving boundary, Bardos et al [1] considered the exact controllability and stabilization of (1.1) for multi-dimensional wave equations in non-cylindrical domains. Using a suitable change of variables, Cavalcanti et al [2] studied the existence and asymptotic behavior of global regular solutions of the mixed problem for the Kirchhoff nonlinear model. By using the same transformation, Cui et al [6] considered the exact boundary controllability for a one-dimensional wave equation and obtained the controllability time. A damped Klein-Gordon equation in a non-cylindrical domain was studied in [7], there the authors obtained the existence of global solutions and the exponential decay of the energy. The stabilization and controllability for the wave equation with variable coefficients in domains with moving boundary was investigated in [10]. Under some appropriate geometric conditions, the energy decay estimates was established and the exact controllability was also obtained by the Rimannian geometry method. It is well known that the Riemannian geometry method was first introduced in [13] for the controllability of the wave equation with variable coefficients.

Motivated by [1, 6, 10], we study the exact boundary controllability of (1.1). Instead of transforming the problem from a non-cylindrical domain into a cylindrical domain, we study the problem directly in non-cylindrical domains. By using a modified multiplier method, we obtain a controllability time which is smaller than that in [6].

The rest of this article is organized as follows. In section 2, we prove three lemmas which will be needed in the sequel. Section 3 we prove our main results.

2. Preliminaries

In this section, we establish three key lemmas which are needed in proving our main results. The first lemma gives a equality on the energy of the solution to problem (1.3).

Lemma 2.1. *Let $w(x,t)$ be a solution of (1.3) in $\hat{Q}_T^k$. Then*

$$E(T) - E(0) = \frac{k(k^2-1)}{2\sqrt{1+k^2}} \int_0^T w_x^2(\alpha_k(t),t)dt. \tag{2.1}$$

Proof. Multiply(1.3) by w_t and integrate on $\hat{Q}_T^k$, we obtain

$$\begin{aligned} 0 &= \int_0^T \int_0^{\alpha_k(t)} (w_{tt} - w_{xx})w_t \,dx\,dt \\ &= \int_0^T \int_0^{\alpha_k(t)} [(\frac{1}{2}w_t^2)_t + (\frac{1}{2}w_x^2)_t - (w_x w_t)_x] \,dx\,dt \end{aligned}$$

Let us denote by Σ the boundary of $\hat{Q}_T^k$ and by $\nu = (\nu_x, \nu_t)$ the unit outward normal at (x,t) on Σ. First, Gauss-Green formula implies

$$0 = \int_\Sigma [\frac{1}{2}(w_t^2 + w_x^2)\nu_t - (w_x w_t)\nu_x] d\Sigma. \tag{2.2}$$

Since w is a solution to (1.3), $w(0,t) = 0$ and $w(\alpha_k(t),t) = 0$, then we have $kw_x(\alpha_k(t),t) + w_t(\alpha_k(t),t) = 0$ and $w_t(0,t) = 0$. Substituting these equalities in (2.2), (2.1) is proved. $\square$

Remark 2.2. Without loss of generality, let $0 < t_1 < t_2 < T$, replacing T by t_2 and 0 by t_1, we have

$$E(t_2) - E(t_1) = \frac{k(k^2-1)}{2\sqrt{1+k^2}} \int_{t_1}^{t_2} w_x^2(\alpha_k(t), t)dt.$$

Moreover it is easy to check that the energy of (1.3) is decreasing for $k \in (0,1)$ and is conserved for $k = 1$.

Lemma 2.3. *Suppose that w is the solution of* (1.3). *Let $T > 0$ be given, we have the equality*

$$\begin{aligned} &\frac{(1-k^2)}{\sqrt{1+k^2}} \int_0^T w_x^2(\alpha_k(t), t)dt \\ &= 2TE(T) - 2\int_0^1 xw_t(0)w_x(0)dx + 2\int_0^{1+kT} xw_t(T)w_x(T)dx. \end{aligned} \tag{2.3}$$

Proof. (i) Multiplying (1.3) by $2xw_x$ and integrating by parts on $\hat{Q}_T^k$, we have

$$\begin{aligned} 0 &= \int_0^T \int_0^{\alpha_k(t)} (w_{tt} - w_{xx})2xw_x \, dx \, dt \\ &= \int_0^T \int_0^{\alpha_k(t)} [(2xw_xw_t)_t - (xw_t^2 + xw_x^2)_x + w_t^2 + w_x^2] \, dx \, dt \\ &= \int_\Sigma [(2xw_tw_x)\nu_t - (xw_t^2 + xw_x^2)\nu_x] d\Sigma + 2\int_0^T E(t)dt \\ &= \int_0^{1+kT} 2xw_t(T)w_x(T)dx - \int_0^1 2xw^1 w_x(x,0) \\ &\quad + \frac{k^2-1}{\sqrt{1+k^2}} \int_0^T (1+kt)w_x^2(\alpha_k(t), t)dt + 2\int_0^T E(t)dt. \end{aligned}$$

This implies

$$\begin{aligned} &\frac{1-k^2}{\sqrt{1+k^2}} \int_0^T (1+kt)w_x^2(\alpha_k(t), t)dt \\ &= 2\int_0^T E(t)dt - \int_0^1 2xw_t(0)w_x(0,x)dx + \int_0^{1+kT} 2xw_t(T)w_x(T)dx. \end{aligned} \tag{2.4}$$

(ii) Multiplying (1.3) by $2tw_t$ and integrating by parts on $\hat{Q}_T^k$ yields

$$\frac{1-k^2}{\sqrt{1+k^2}} \int_0^T ktw_x^2(\alpha_k(t), t)dt = 2\int_0^T E(t)dt - 2TE(T). \tag{2.5}$$

Equality (2.3) follows easily from (2.4) and (2.5). □

Using Cauchy's inequality, we can obtain easily the following result.

Lemma 2.4. *Denote by w the solution of* (1.3). *For $t \in (0,T)$ and $k \in (0,1)$, we have the estimate*

$$\Big| \int_0^{1+kt} 2xw_tw_x dx \Big| \le 2(1+kt)E(t). \tag{2.6}$$

3. Proof of main results

In this section, we prove the exact controllability for the string equation (1.1) in the non-cylindrical domain $\hat{Q}_T^k$ (Theorem 1.2) for $0 < k < 1$ by the Hilbert Uniqueness Method (HUM). We first need to prove Theorem 1.4.

Proof of Theorem 1.4. From (2.1) and (2.3), it is easy to check that

$$\begin{aligned} &\frac{2}{k}(E(0) - E(T)) \\ &= 2TE(T) - 2\int_0^1 xw_t(0)w_x(0)dx + \int_0^{1+kT} 2xw_t(T)w_x(T)dx. \end{aligned} \tag{3.1}$$

Rearranging this equality, we have

$$E(0) + k\int_0^1 xw^1 w_x(0,x)dx = (1+kT)E(T) + k\int_0^{1+kT} xw_t w_x\,dx.$$

Using (2.6), one can easily obtain

$$\begin{gathered} (1-k)E(0) \le (1+k)(1+kT)E(T), \\ (1+k)E(0) \ge (1-k)(1+kT)E(T), \end{gathered}$$

for $T > 0$. Then we have the estimates:

$$\frac{1-k}{(1+k)(1+kT)}E(0) \le E(T) \le \frac{1+k}{(1-k)(1+kT)}E(0).$$

□

Proof of Theorem 1.2. We apply HUM as in [8, chapter 4]. For any $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$, let w be the solution of (1.3). Consider the problem

$$\begin{gathered} u_{tt} - u_{xx} = 0 \quad \text{in } \hat{Q}_T^k, \\ u(0,t) = 0, \quad u(\alpha_k(t),t) = w_x(\alpha_k(t),t) \quad \text{on } (0,T), \\ u(T) = 0, \quad u_t(T) = 0 \quad \text{in } (0,1). \end{gathered} \tag{3.2}$$

It is well known that (3.2) admits a unique solution such that

$$(u(0), u_t(0)) \in L^2(0,1) \times H^{-1}(0,1).$$

Then, we introduce a map $\Lambda : H_0^1(0,1) \times L^2(0,1) \to H^{-1}(0,1) \times L^2(0,1)$ defined by

$$\Lambda(w^0, w^1) = (u^1, -u^0),$$

where $u^0 = u(0),\ u^1 = u_t(0)$.

Then the map Λ is an isomorphism of $H_0^1(0,1)\times L^2(0,1)$ onto $H^{-1}(0,1)\times L^2(0,1)$. To simplify our analysis, we introduce the following notation:

$$F := H_0^1(0,1) \times L^2(0,1) \quad F' := H^{-1}(0,1) \times L^2(0,1).$$

In fact, multiplying equation (3.2) by w and integrating on Q_T^k, we obtain that

$$\int_0^1 (w^0u^1 - w^1u^0)dx = \frac{1-k^2}{\sqrt{1+k^2}}\int_0^T w_x^2(\alpha_k(t),t)dt. \tag{3.3}$$

Hence we have

$$\langle \Lambda(w^0, w^1), (w^0, w^1)\rangle_{F',F} = \frac{1-k^2}{\sqrt{1+k^2}}\int_0^T w_x^2(\alpha_k(t),t)dt, \tag{3.4}$$

for every $(w^0, w^1) \in F$.

Recalling estimate (1.5) and equality (2.1) , we have

$$\frac{(1-k)T-2}{(1-k)(1+kT)}E(0) \leq \langle \Lambda(w^0, w^1), (w^0, w^1) \rangle_{F',F} \leq \frac{(1+k)T+2}{(1+k)(1+kT)}E(0). \quad (3.5)$$

From these inequalities, we conclude that Λ is a coercive linear map for $T > \bar{T}_k$ and is bounded. Therefore, Λ is a surjection by Lax-Milgram Theorem. It follows that Λ is an isomorphism.

Since Λ is an isomorphism, for any initial value $(u^0, u^1) \in L^2(0,1) \times H^{-1}(0,1)$, there exists $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$ such that

$$\Lambda(w^0, w^1) = (u^1, -u^0).$$

Then u is the solution of (1.1) with $v = w_x(\alpha_k(t), t)$. Furthermore, $(u(0), u_t(0)) = (u^0, u^1)$ and $(u(T), u_t(T)) = (0, 0)$. This completes the proof. $\square$

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