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BOUNDEDNESS AND LARGE-TIME BEHAVIOR OF SOLUTIONS FOR A GIERER-MEINHARDT SYSTEM OF THREE EQUATIONS

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Abstract. The aim of this work is to prove the uniform boundedness and the existence of global solutions for Gierer-Meinhardt model of three substance described by reaction-diffusion equations with Neumann boundary conditions. Based on a Lyapunov functional we establish the asymptotic behaviour of the solutions.

1. Introduction

In this article, we consider the Gierer-Meinhardt type system of three equations

$$\begin{aligned} \frac{\partial u}{\partial t} - a_1 \Delta u &= -b_1 u + f(u,v,w), \quad \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial v}{\partial t} - a_2 \Delta v &= -b_2 v + g(u,v,w), \quad \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial w}{\partial t} - a_3 \Delta w &= -b_3 w + h(u,v,w), \quad \text{in } \mathbb{R}^+ \times \Omega, \end{aligned} \tag{1.1}$$

where

$$\begin{aligned} f(u,v,w) &= \rho_1(x,u,v,w) \frac{u^{p_1}}{v^{q_1}(w^{r_1}+c)} + \sigma_1(x), \\ g(u,v,w) &= \rho_2(x,u,v,w) \frac{u^{p_2}}{v^{q_2} w^{r_2}} + \sigma_2(x), \\ h(u,v,w) &= \rho_3(x,u,v,w) \frac{u^{p_3}}{v^{q_3} w^{r_3}} + \sigma_3(x), \end{aligned}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \eta} = \frac{\partial v}{\partial \eta} = \frac{\partial w}{\partial \eta} = 0 \quad \text{on } \mathbb{R}^+ \times \partial\Omega, \tag{1.2}$$

and initial data

$$u(0,x) = \varphi_1(x), \quad v(0,x) = \varphi_2(x), \quad w(0,x) = \varphi_3(x), \quad \text{in } \Omega. \tag{1.3}$$

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Here Ω is an open bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and outer normal $\eta(x)$. The constants c, p_i, q_i, r_i, a_i and b_i, $i = 1, 2, 3$ are real numbers such that

$$c, p_i, q_i, r_i \geq 0, \quad \text{and} \quad a_i, b_i > 0,$$

and

$$0 < p_1 - 1 < \max\big\{p_2 \min\big(\frac{q_1}{q_2+1}, \frac{r_1}{r_2}, 1\big), p_3 \min\big(\frac{r_1}{r_3+1}, \frac{q_1}{q_3}, 1\big)\big\}. \tag{1.4}$$

The initial data are assumed to be positive and continuous functions on $\bar{\Omega}$. For $i = 1, 2, 3$, we assume that σ_i are positive functions in $C(\bar{\Omega})$, and ρ_i are positive bounded functions in $C^1(\bar{\Omega} \times \mathbb{R}^3_+)$.

In 1972, following the ingenious idea of Turing [15], Gierer and Meinhardt [2] proposed a mathematical model for pattern formations of spatial tissue structure of hydra in morphogenesis, a biological phenomenon discovered by Trembley in 1744 [14]. It can be expressed in the following system

$$\begin{aligned} \frac{\partial u}{\partial t} &= a_1 \Delta u - \mu_1 u + \frac{u^p}{v^q} + \sigma, \quad \text{in } \mathbb{R}^+ \times \Omega, \\ \frac{\partial v}{\partial t} &= a_2 \Delta v - \mu_2 v + \frac{u^r}{v^s}, \quad \text{in } \mathbb{R}^+ \times \Omega, \end{aligned} \tag{1.5}$$

on a bounded $\Omega \subset \mathbb{R}^N$, with the homogeneous Neumann boundary conditions and positive initial data: a_1, a_2, μ_1, μ_2 and σ are positive constants, and p, q, r, s are non negative constants satisfying the relation

$$\frac{p-1}{r} < \frac{q}{s+1}.$$

The existence of global solutions to the system (1.5) is proved by Rothe [11] with special cases $N = 3, p = 2, q = 1, r = 2$ and $s = 0$. The Rothe's method can not be applied (at least directly) to general p, q, r, s. Wu and Li [16] obtained the same results for the problem (1.5) so long as u, v^{-1} and σ are suitably small. Li, Chen and Qin [7] showed that the solutions of this problem are bounded all the time for each pair of initial values in $L^\infty(\Omega)$ if

$$\frac{p-1}{r} < \min\big\{1, \frac{q}{s+1}\big\}. \tag{1.6}$$

Masuda and Takahashi [8] considered the generalized Gierer-Meinhardt system

$$\frac{\partial u_i}{\partial t} = a_i \Delta u_i - \mu_i u_i + g_i(x, u_1, u_2), \quad \text{in } \mathbb{R}^+ \times \Omega \ (i = 1, 2), \tag{1.7}$$

where a_i, μ_i, $i = 1, 2$ are positive constants, and

$$\begin{aligned} g_1(x, u_1, u_2) &= \rho_1(x, u_1, u_2) \frac{u_1^p}{u_2^q} + \sigma_1(x), \\ g_2(x, u_1, u_2) &= \rho_2(x, u_1, u_2) \frac{u_1^r}{u_2^s} + \sigma_2(x), \end{aligned}$$

with $\sigma_1(\cdot)$ (resp. $\sigma_2(\cdot)$) is a positive (resp. non-negative) C^1 function on $\bar{\Omega}$, and ρ_1 (resp. ρ_2) is a non negative (resp. positive) bounded and C^1 function on $\bar{\Omega} \times \mathbb{R}^2_+$. They extended the result of global existence of solutions for (1.7) of Li, Chen and Qin [7] to

$$\frac{p-1}{r} < \frac{2}{N+2}, \tag{1.8}$$

and

$$\begin{gathered}\varphi_1,\varphi_2\in W^{2,l}(\Omega),\quad l>\max\{N,2\},\\ \frac{\partial\varphi_1}{\partial\eta}=\frac{\partial\varphi_2}{\partial\eta}=0\quad\text{on }\partial\Omega\quad\text{and}\quad\varphi_1\ge 0,\varphi_2>0\quad\text{in }\bar\Omega.\end{gathered}\tag{1.9}$$

Jiang [6] obtained the same results as Masuda and Takahashi [8] by another method such that (1.6) and (1.9) are satisfied.

Abdelmalek, et al [1] considered the following Gierer-Meinhardt system of three equations

$$\begin{gathered}\frac{\partial u}{\partial t}-a_1\Delta u=-b_1u+\frac{u^{p_1}}{v^{q_1}(w^{r_1}+c)}+\sigma,\quad\text{in }\mathbb{R}^+\times\Omega,\\ \frac{\partial v}{\partial t}-a_2\Delta v=-b_2v+\frac{u^{p_2}}{v^{q_2}w^{r_2}},\quad\text{in }\mathbb{R}^+\times\Omega,\\ \frac{\partial w}{\partial t}-a_3\Delta w=-b_3w+\frac{u^{p_3}}{v^{q_3}w^{r_3}},\quad\text{in }\mathbb{R}^+\times\Omega,\end{gathered}\tag{1.10}$$

with homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial\eta}=\frac{\partial v}{\partial\eta}=\frac{\partial w}{\partial\eta}=0\quad\text{on }\mathbb{R}^+\times\partial\Omega,\tag{1.11}$$

and the initial data

$$\begin{gathered}u(0,x)=\varphi_1(x)>0,\\ v(0,x)=\varphi_2(x)>0,\\ w(0,x)=\varphi_3(x)>0\end{gathered}\tag{1.12}$$

in Ω, and $\varphi_i\in C(\bar\Omega)$ for all $i=1,2,3$. Under the condition (1.4) and by using a suitable Lyapunov functional, they studied the global existence of solutions for the system (1.10)–(1.12). Their method gave only the existence of global solutions, and they did not obtain results about the uniform boundedness of solutions on $(0,+\infty)$.

For the asymptotic behavior of the solutions, Wu and Li [16] considered the system

$$\begin{gathered}\frac{\partial u_1}{\partial t}=a_1\Delta u_1-u_1+\frac{u_1^p}{u_2^q}+\sigma_1(x),\quad\text{in }\mathbb{R}^+\times\Omega,\\ \tau\frac{\partial u_2}{\partial t}=a_2\Delta u_2-u_2+\frac{u_1^r}{u_2^s}+\sigma_2(x),\quad\text{in }\mathbb{R}^+\times\Omega,\end{gathered}\tag{1.13}$$

with the constant of relaxation time $\tau>0$, and they proved that if $\sigma_1\equiv\sigma_2\equiv 0$ and $\tau>\frac{q}{p-1}$, then $(u(t,x),v(t,x))\to(0,0)$ uniformly on $\bar\Omega$ as $t\to+\infty$.

Under suitable conditions on τ and on the initial data, Suzuki and Takagi [12,13] also studied the behavior of the solutions for (1.13) with the constant of relaxation time τ.

We first treat the uniform boundedness of the solutions for Gierer-Meinhardt system of three equations by proving that the Lyapunov function argument proposed in [1] can be adapted to our situation. Interestingly, we show that the same Lyapunov function satisfies a differential inequality from which the uniform boundedness of the solutions is deduced for any positive time. Then under reasonable conditions on the coefficients b_1,b_2 and b_3, and by using the uniform boundedness of the solutions and the Lyapunov function which is non-increasing function, we deal with the long-time behavior of solutions as the time goes to $+\infty$. In particular we are concerned with $\sigma_1\equiv 0$, σ_2 and σ_3 are non-negative constants to assure that

$$\lim_{t\to+\infty}\|u(t,.)\|_\infty=\lim_{t\to+\infty}\|v(t,.)-\frac{\sigma_2}{b_2}\|_\infty=\lim_{t\to+\infty}\|w(t,.)-\frac{\sigma_3}{b_3}\|_\infty=0\,.$$

2. Notation and preliminary results

2.1. **Existence of local solutions.** For $i = 1, 2, 3$ we set

$$\begin{gathered}
\underline{\varphi_i} = \min_{x\in\bar{\Omega}} \varphi_i(x), \quad \bar{\varphi}_i = \max_{x\in\bar{\Omega}} \varphi_i(x), \\
\underline{\rho_i} = \min_{x\in\bar{\Omega},\xi\in\mathbb{R}^3_+} \rho_i(x,\xi), \quad \bar{\rho}_i = \max_{x\in\bar{\Omega},\xi\in\mathbb{R}^3_+} \rho_i(x,\xi), \\
\underline{\sigma_i} = \min_{x\in\bar{\Omega}} \sigma_i(x), \quad \bar{\sigma}_i = \max_{x\in\bar{\Omega}} \sigma_i(x).
\end{gathered}$$

The basic existence theory for abstract semi linear differential equations directly leads to a local existence result to system (1.1)–(1.3) (see, Henry [5]). All solutions are classical on $(0, T) \times \Omega$, $T < T_{\max}$, where $T_{\max}(\|u_0\|_\infty, \|w_0\|_\infty)$ denotes the eventual blowing-up time in $L^\infty(\Omega)$.

2.2. **Positivity of solutions.**

Lemma 2.1. *If (u, v, w) is a solution of the problem (1.1)–(1.3), then for all $(t, x) \in (0, T_{mmax}) \times \Omega$, we have*

(1)

$$\begin{aligned}
u(t,x) &\geq e^{-b_1 t}\underline{\varphi_1} > 0, \\
v(t,x) &\geq e^{-b_2 t}\underline{\varphi_2} > 0, \\
w(t,x) &\geq e^{-b_3 t}\underline{\varphi_3} > 0.
\end{aligned}$$

(2)

$$\begin{aligned}
u(t,x) &\geq \min\left(\underline{\sigma_1}/b_1, \underline{\varphi_1}\right) = m_1, \\
v(t,x) &\geq \min\left(\underline{\sigma_2}/b_2, \underline{\varphi_2}\right) = m_2, \\
w(t,x) &\geq \min\left(\underline{\sigma_3}/b_3, \underline{\varphi_3}\right) = m_3.
\end{aligned}$$

The proof of the above lemma follows immediate from the maximum principle.

3. Boundedness of solutions

For proving the existence of global solutions for (1.1)–(1.3), it suffices to prove that the solutions remains bounded in $(0, T) \times \bar{\Omega}$. One of the main results of this paper reads as follows.

Theorem 3.1. *Assume that (1.4) holds. Let (u, v, w) be a solution to (1.1)–(1.3), and let*

$$L(t) = \int_\Omega \frac{u^\alpha(t,x)}{v^\beta(t,x)w^\gamma(t,x)}dx, \quad \textit{for all } t \in (0,T), \tag{3.1}$$

where α, β and γ are positive constants satisfying the following conditions:

$$\alpha > 2\max\Big(1, \frac{3b_2 + b_3}{b_1}\Big), \quad \frac{1}{\beta} > \frac{(a_1 + a_2)^2}{2a_1 a_2}, \tag{3.2}$$

and

$$\Big(\frac{1}{2\beta} - \frac{(a_1+a_2)^2}{4a_1a_2}\Big)\Big(\frac{1}{2\gamma} - \frac{(a_1+a_3)^2}{4a_1a_3}\Big) > \Big(\frac{(\alpha-1)(a_2+a_3)}{2\alpha\sqrt{a_2a_3}} - \frac{(a_1+a_2)(a_1+a_3)}{4\sqrt{a_1^2a_2a_3}}\Big)^2. \tag{3.3}$$

Then there exists a positive constant C such that for all $t \in (0,T)$,

$$\frac{d}{dt}L(t) \le -(\alpha b_1 - 3b_2\beta - \gamma b_3)L(t) + C. \tag{3.4}$$

Corollary 3.2. *Under the assumptions of Theorem 3.1, all solutions of* (1.1)–(1.3) *with positive initial data in $C(\bar{\Omega})$ are global and uniformly bounded on $(0,+\infty)\times\bar{\Omega}$.*

Before proving the above theorem we first need the following technical lemma.

Lemma 3.3. *Suppose that $x>0$, $y>0$ and $z>0$, then for each group of indices $r,p,q,\delta,\theta,\lambda$ and ξ satisfies $\lambda<p<\delta$ (not necessarily positive), and any constant $\Lambda>0$, we have*

$$\frac{x^p}{y^q z^r} \le \Lambda \frac{x^\delta}{y^\theta z^\xi} + \Lambda^{-\frac{p-\lambda}{\delta-p}} \frac{x^\lambda}{y^{\eta_1} z^{\eta_2}},$$

where

$$\begin{aligned}
\eta_1 &= [q(\delta-\lambda) - \theta(p-\lambda)](\delta-p)^{-1},\\
\eta_2 &= [r(\delta-\lambda) - \xi(p-\lambda)](\delta-p)^{-1}.
\end{aligned}$$

Proof. We can write

$$\frac{x^p}{y^q z^r} = \Big(x^{\frac{\delta(p-\lambda)}{\delta-\lambda}} y^{-\frac{\theta(p-\lambda)}{\delta-\lambda}} z^{-\frac{\xi(p-\lambda)}{\delta-\lambda}}\Big)\Big(x^{\frac{\lambda(\delta-p)}{\delta-\lambda}} y^{\frac{\theta(p-\lambda)}{\delta-\lambda}-q} z^{\frac{\xi(p-\lambda)}{\delta-\lambda}-r}\Big).$$

By using Young's inequality we obtain

$$\frac{x^p}{y^q z^r} \le \varepsilon \frac{x^\delta}{y^\theta z^\xi} + \varepsilon^{-\frac{p-\lambda}{\delta-p}} \frac{x^\lambda}{y^{\eta_1} z^{\eta_2}},$$

where

$$\begin{aligned}
\eta_1 &= [q(\delta-\lambda) - \theta(p-\lambda)](\delta-p)^{-1},\\
\eta_2 &= [r(\delta-\lambda) - \xi(p-\lambda)](\delta-p)^{-1}.
\end{aligned}$$

Then Lemma 3.3 is proved. □

Proof of Theorem 3.1. Let (u,v,w) be the solution of system (1.1)–(1.3) in $(0,T)$. Differentiating $L(t)$ respect to t, we obtain $L'(t) = I + J$, where

$$\begin{aligned}
I &= a_1\alpha \int_\Omega \frac{u^{\alpha-1}}{v^\beta w^\gamma}\Delta u\,dx - a_2\beta\int_\Omega \frac{u^\alpha}{v^{\beta+1}w^\gamma}\Delta v\,dx - a_3\gamma\int_\Omega \frac{u^\alpha}{v^\beta w^{\gamma+1}}\Delta w\,dx,\\
J &= (-\alpha b_1 + \beta b_2 + \gamma b_3)L(t) + \alpha\int_\Omega \rho_1(x,u,v,w)\frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}w^{\gamma+r_1}}dx\\
&\quad - \beta\int_\Omega \rho_2(x,u,v,w)\frac{u^{\alpha+p_2}}{v^{\beta+1+q_2}w^{\gamma+r_2}}dx - \gamma\int_\Omega \rho_3(x,u,v,w)\frac{u^{\alpha+p_3}}{v^{\beta+q_3}w^{\gamma+1+r_3}}dx\\
&\quad + \alpha\int_\Omega \sigma_1(x)\frac{u^{\alpha-1}}{v^\beta w^\gamma}dx - \beta\int_\Omega \sigma_2(x)\frac{u^\alpha}{v^{\beta+1}w^\gamma}dx - \gamma\int_\Omega \sigma_3(x)\frac{u^\alpha}{v^\beta w^{\gamma+1}}dx.
\end{aligned}$$

Using Green's formula, for all $t\in(0,T)$, we obtain (see [1])

$$I \le 0. \tag{3.5}$$

Now let us estimate the term J. For all $t \in (0,T)$ we have

$$\begin{aligned} J \le \big(-\alpha b_1 + \beta b_2 + \gamma b_3\big)L(t) + \alpha\bar{\rho}_1 \int_\Omega \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}w^{\gamma+r_1}}dx - \beta\underline{\rho_2}\int_\Omega \frac{u^{\alpha+p_2}}{v^{\beta+1+q_2}w^{\gamma+r_2}}dx \\ - \underline{\rho_3}\gamma\int_\Omega \frac{u^{\alpha+p_3}}{v^{\beta+q_3}w^{\gamma+1+r_3}}dx + \alpha\bar{\sigma}_1\int_\Omega \frac{u^{\alpha-1}}{v^\beta w^\gamma}dx. \end{aligned} \tag{3.6}$$

Applying Lemma 3.3 with $p=\alpha-1$, $q=\theta=\beta$, $r=\gamma$, $\delta=\alpha$, $\xi=\gamma$ and $\lambda=0$, one gets

$$\alpha\bar{\sigma}_1\int_\Omega \frac{u^{\alpha-1}}{v^\beta w^\gamma}dx \le \beta b_2\int_\Omega \frac{u^\alpha}{v^\beta w^\gamma}dx + C_1\int_\Omega \frac{1}{v^\beta w^\gamma}dx, \tag{3.7}$$

where $C_1 = \alpha\bar{\sigma}_1(\frac{\beta b_2}{\alpha\bar{\sigma}_1})^{1-\alpha}$.
Now, we choose $\epsilon_1 \in (0,\alpha)$ such that

$$\begin{gathered} \beta + \alpha\frac{q_1p_2-(p_1-1)(1+q_2)}{\epsilon_1(p_2+1-p_1)} + \alpha\frac{q_1-1-q_2}{p_2+1-p_1} \ge 0, \\ \gamma + \alpha\frac{r_1p_2-r_2(p_1-1)}{\epsilon_1(p_2-p_1+1)} + \alpha\frac{r_1-r_2}{p_2-p_1+1} \ge 0. \end{gathered}$$

Again, applying Lemma 3.3 for $p=\alpha-1+p_1$, $q=\beta+q_1$, $r=\gamma+r_1$, $\delta=\alpha+p_2$, $\theta=\beta+1+q_2$, $\xi=\gamma+r_2$ and $\lambda=\alpha-\epsilon_1$, we obtain

$$\alpha\bar{\rho}_1\int_\Omega \frac{u^{\alpha-1+p_1}}{v^{q_1+\beta}w^{r_1+\gamma}}dx \le \beta\underline{\rho_2}\int_\Omega \frac{u^{p_2+\alpha}}{v^{q_2+\beta+1}w^{r_2+\gamma}}dx + C_2\int_\Omega \frac{u^{\alpha-\epsilon_1}}{v^{\eta_1}w^{\eta_2}}dx, \tag{3.8}$$

where

$$\begin{gathered} \eta_1 = \beta + [q_1p_2 - (q_2+1)(p_1-1) + \epsilon_1(q_1-q_2-1)](p_2-p_1+1)^{-1}, \\ \eta_2 = \gamma + [r_1p_2 - r_2(p_1-1) + \epsilon_1(r_1-r_2)](p_2-p_1+1)^{-1}, \end{gathered}$$

and $C_2 = \alpha\bar{\rho}_1(\frac{\beta\underline{\rho_2}}{\alpha\bar{\rho}_1})^{-\frac{p_1-1+\epsilon_1}{p_2-p_1+1}}$.

In an analoguous way, we have

$$C_2\int_\Omega \frac{u^{\alpha-\epsilon_1}}{v^{\eta_1\eta_2}}dx \le b_2\beta\int_\Omega \frac{u^\alpha}{v^\beta w^\gamma}dx + C_3\int_\Omega \frac{1}{v^{\eta_3\eta_4}}dx, \tag{3.9}$$

where

$$\begin{gathered} \eta_3 = \beta + \alpha[\epsilon_1^{-1}(q_1p_2-(q_2+1)(p_1-1)) + q_1 - q_2 - 1](p_2-p_1+1)^{-1} \ge 0, \\ \eta_4 = \gamma + \alpha[\epsilon_1^{-1}(r_1p_2 - r_2(p_1-1)) + r_1 - r_2](p_2-p_1+1)^{-1} \ge 0, \end{gathered}$$

and $C_3 = C_2(\frac{b_2\beta}{C_2})^{-\frac{\alpha-\epsilon_1}{\epsilon_1}}$.

Or, we choose $\epsilon_2 \in (0,\alpha)$ such that

$$\begin{gathered} \beta + \alpha\frac{q_1p_3 - q_3(p_1-1)}{\epsilon_2(p_3-p_1+1)} + \alpha\frac{q_1-q_3}{p_3-p_1+1} \ge 0, \\ \gamma + \alpha\frac{r_1p_3-(r_3+1)(p_1-1)}{\epsilon_2(p_3-p_1+1)} + \alpha\frac{r_1-r_2-1}{p_3-p_1+1} \ge 0. \end{gathered}$$

Now, applying Lemma 3.3 with $p=p_1+\alpha-1$, $q=q_1+\beta$, $r=r_1+\gamma$, $\delta=p_3+\alpha$, $\theta=q_3+\beta$, $\xi=r_3+\gamma+1$ and $\lambda=\alpha-\epsilon_2$, we find that

$$\alpha\bar{\rho}_1\int_\Omega \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}w^{\gamma+r_1}}dx \le \gamma\underline{\rho_3}\int_\Omega \frac{u^{\alpha+p_3}}{v^{\beta+q_3}w^{\gamma+1+r_3}}dx + C_4\int_\Omega \frac{u^{\alpha-\epsilon_2}}{v^{\eta_5}w^{\eta_6}}dx, \tag{3.10}$$

where

$$\eta_5 = \beta + [q_1p_3 - q_3(p_1-1) + \epsilon_2(q_1-q_3)](p_3-p_1+1)^{-1},$$
$$\eta_6 = \gamma + [r_1p_3 - (r_3+1)(p_1-1) + \epsilon_2(r_1-r_3-1)](p_3-p_1+1)^{-1},$$

and $C_4 = \alpha\bar{\rho}_1(\frac{\gamma\rho_3}{\alpha\bar{\rho}_1})^{-\frac{p_1-1+\epsilon_2}{p_3-p_1+1}}$. In the same way, we obtain

$$C_4\int_\Omega \frac{u^{\alpha-\epsilon_2}}{v^{\eta_5}w^{\eta_6}}dx \le b_2\beta\int_\Omega \frac{u^\alpha}{v^\beta w^\gamma}dx + C_5\int_\Omega \frac{1}{v^{\eta_7}w^{\eta_8}}dx, \tag{3.11}$$

where

$$\eta_7 = \beta + \alpha[\epsilon_2^{-1}(q_1p_3 - q_3(p_1-1)) + q_1 - q_3](p_3-p_1+1)^{-1} \ge 0,$$
$$\eta_8 = \gamma + \alpha[\epsilon_2^{-1}(r_1p_3 - (r_3+1)(p_1-1)) + r_1 - r_3 - 1](p_3-p_1+1)^{-1} \ge 0,$$

and $C_5 = C_4(\frac{b_2\beta}{C_4})^{-\frac{\alpha-\epsilon_2}{\epsilon_2}}$.

From (3.6)–(3.11) there exists a positive constant C such that

$$L'(t) \le -(b_1\alpha - 3\beta b_2 - \gamma b_3)L(t) + C, \quad \forall t\in(0,T).$$

Then the proof is complete. □

Proof of Corollary 3.2. Since

$$L(t) \le L(0) + \frac{C}{\alpha b_1 - 3b_2\beta - \gamma b_3} \quad \text{for all } t\in(0,T),$$

then there exist non-negative constants C_6, C_7 and C_8 independent of t such that

$$\|f(u,v,w) - b_1u\|_N \le C_6,$$
$$\|g(u,v;w) - b_2v\|_N \le C_7,$$
$$\|h(u,v,w) - b_3w\|_N \le C_8.$$

Since $(\varphi_1,\varphi_2,\varphi_3)\in(C(\bar{\Omega}))^3$, we conclude from the L^p-L^q-estimate (see Henry [5], Haraux and Kirane [4]) that

$$u\in L^\infty((0,T),L^\infty(\Omega)), \quad v\in L^\infty((0,T),L^\infty(\Omega)), \quad w\in L^\infty((0,T),L^\infty(\Omega)).$$

Finally, we deduce that the solutions of the system (1.1)–(1.3) are global and uniformly bounded on $(0,+\infty)\times\bar{\Omega}$. □

Remark 3.4. It is clear that the results of this section are valid when $\sigma_1\equiv\sigma_2\equiv\sigma_3\equiv 0$.

4. Asymptotic behavior of the solutions

In this section, we study the asymptotic behavior of the solutions for the system

$$\begin{gathered}\frac{\partial u}{\partial t} - a_1\Delta u = -b_1u + f(u,v,w), \quad \text{in } \mathbb{R}^+\times\Omega,\\ \frac{\partial v}{\partial t} - a_2\Delta v = -b_2v + g(u,v,w), \quad \text{in } \mathbb{R}^+\times\Omega,\\ \frac{\partial w}{\partial t} - a_3\Delta w = -b_3w + h(u,v,w), \quad \text{in } \mathbb{R}^+\times\Omega,\end{gathered} \tag{4.1}$$

where

$$f(u,v,w) = \rho_1(x,u,v,w)\frac{u^{p_1}}{v^{q_1}(w^{r_1}+c)} + \sigma_1,$$

$$g(u,v,w)=\rho_2(x,u,v,w)\frac{u^{p_2}}{v^{q_2}w^{r_2}}+\sigma_2,$$
$$h(u,v,w)=\rho_3(x,u,v,w)\frac{u^{p_3}}{v^{q_3}w^{r_3}}+\sigma_3,$$

with homogeneous Neumann boundary conditions

$$\frac{\partial u}{\partial \eta}=\frac{\partial v}{\partial \eta}=\frac{\partial w}{\partial \eta}=0 \quad \text{on } \mathbb{R}^+\times\partial\Omega, \tag{4.2}$$

and initial data

$$u(0,x)=\varphi_1(x),\quad v(0,x)=\varphi_2(x),\quad w(0,x)=\varphi_3(x)\quad \text{in } \Omega. \tag{4.3}$$

Here σ_1, σ_2 and σ_3 are non negative constants.

Before stating the results, let us expose some simple facts concluded from the result of the previous section. From Theorem 3.1, and by using classical method of a semi group and a power fractional (see [5]) we can find the positive constants M_1, M_2 and M_3 explicitly (see [9]) such that

$$\|u(t,.)\|_\infty M_1,\quad \|v(t,.)\|_\infty\le M_2,\quad \|w(t,.)\|_\infty\le M_3.$$

Let us consider the same function as in Theorem 3.1,

$$L(t)=\int_\Omega \frac{u^\alpha(t,x)}{v^\beta(t,x)w^\gamma(t,x)}dx,\quad \forall t\in(0,+\infty),$$

where α,β and γ are positive constants satisfying the following conditions

$$\alpha>2\max(1,\frac{3b_2+b_3}{b_1}),\quad \frac{1}{\beta}>\frac{(a_1+a_2)^2}{2a_1a_2},$$

and

$$\Big(\frac{1}{2\beta}-\frac{(a_1+a_2)^2}{4a_1a_2}\Big)\Big(\frac{1}{2\gamma}-\frac{(a_1+a_3)^2}{4a_1a_3}\Big)>\Big(\frac{(\alpha-1)(a_2+a_3)}{2\alpha\sqrt{a_2a_3}}-\frac{(a_1+a_2)(a_1+a_3)}{4\sqrt{a_1^2a_2a_3}}\Big)^2.$$

The main result in this section reads as follows.

Theorem 4.1. *Assume* (1.4) *holds. Let* (u,v,w) *be the solution of* (4.1)–(4.3) *in* $(0,+\infty)$. *Suppose that* $\sigma_1=0$, *and*

$$b_1>\frac{\beta b_2+\gamma b_3+K}{2}, \tag{4.4}$$

where

$$K=\frac{\alpha\bar\rho_1(\frac{\beta\rho_2}{\alpha\bar\rho_1})^{-\frac{p_1-1}{p_2-p_1+1}}}{m_2^{[q_1p_2-(q_2+1)(p_1-1)](p_2-p_1+1)^{-1}}m_3^{[r_1p_2-r_2(p_1-1)](p_2-p_1+1)^{-1}}},$$

or

$$K=\frac{\alpha\bar\rho_1(\frac{\gamma\rho_3}{\alpha\bar\rho_1})^{-\frac{p_1-1}{p_3-p_1+1}}}{m_2^{[q_1p_3-q_3(p_1-1)](p_3-p_1+1)^{-1}}m_3^{[r_1p_3-(r_3+1)(p_1-1)](p_3-p_1+1)^{-1}}}.$$

Then for all $t\in(0,+\infty)$ *we have*

$$L(t)\le\int_\Omega\frac{\varphi_1^\alpha(x)}{\varphi_2^\beta(x)\varphi_3^\gamma(x)}dx.$$

Corollary 4.2. *Under the assumptions of Theorem 4.1, for all positive initial data in $C(\bar{\Omega})$ we have*

$$\begin{gathered}
\|u(t,.)\|_\infty \to 0 \quad \text{as } t \to +\infty, \\
\|v(t,.) - \frac{\sigma_2}{b_2}\|_\infty \to 0 \quad \text{as } t \to +\infty, \\
\|w(t,.) - \frac{\sigma_3}{b_3}\|_\infty \to 0 \quad \text{as } t \to +\infty.
\end{gathered}$$

Proof of Theorem 4.1. From (3.5) and (3.6), we obtain for all $t \in (0,+\infty)$

$$\begin{aligned}
L'(t) \leq &-(\alpha b_1 - \beta b_2 - \gamma b_2)L(t) + \alpha\bar{\rho_1}\int_\Omega \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}w^{\gamma+r_1}}dx \\
&- \beta\underline{\rho_2}\int_\Omega \frac{u^{\alpha+p_2}}{v^{\beta+1+q_2}w^{\gamma+r_2}}dx - \gamma\underline{\rho_3}\int_\Omega \frac{u^{\alpha+p_3}}{v^{\beta+q_3}w^{\gamma+1+r_3}}dx.
\end{aligned} \tag{4.5}$$

Now, we apply Lemma 3.3 for $p = \alpha - 1 + p_1$, $q = \beta + q_1$, $r = \gamma + r_1$, $\delta = \alpha + p_2$, $\theta = \beta + 1 + q_2$, $\xi = \gamma + r_2$ and $\lambda = \alpha$ we obtain

$$\alpha\bar{\rho_1}\int_\Omega \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}w^{\gamma+r_1}}dx \leq \beta\underline{\rho_2}\int_\Omega \frac{u^{\alpha+p_2}}{v^{\beta+1+q_2}w^{\gamma+r_2}}dx + A_1\int_\Omega \frac{u^\alpha}{v^{\eta_9}w^{\eta_{10}}}dx, \tag{4.6}$$

where

$$\begin{gathered}
\eta_9 = \beta + [q_1p_2 - (q_2+1)(p_1-1)](p_2 - p_1 + 1)^{-1} > 0, \\
\eta_{10} = \gamma + [r_1p_2 - r_2(p_1-1)](p_2 - p_1 + 1)^{-1} > 0,
\end{gathered}$$

and $A_1 = \alpha\bar{\rho_1}(\frac{\beta\underline{\rho_2}}{\alpha\bar{\rho_1}})^{-\frac{p_1-1}{p_2-p_1+1}}$.

Or, applying Lemma 3.3 for $p = \alpha - 1 + p_1$, $q = \beta + q_1$, $r = \gamma + r_1$, $\delta = \alpha + p_3$, $\theta = \beta + q_3$, $\xi = \gamma + 1 + r_3$ and $\lambda = \alpha$, we obtain

$$\alpha\bar{\rho_1}\int_\Omega \frac{u^{\alpha-1+p_1}}{v^{\beta+q_1}w^{\gamma+r_1}}dx \leq \gamma\underline{\rho_3}\int_\Omega \frac{u^{\alpha+p_3}}{v^{\beta+q_3}w^{\gamma+1+r_3}}dx + A_2\int_\Omega \frac{u^\alpha}{v^{\eta_{11}}w^{\eta_{12}}}dx, \tag{4.7}$$

where

$$\begin{gathered}
\eta_{11} = \beta + [q_1p_3 - q_3(p_1-1)](p_3 - p_1 + 1)^{-1} > 0, \\
\eta_{12} = \gamma + [r_1p_3 - (r_3+1)(p_1-1)](p_3 - p_1 + 1)^{-1} > 0,
\end{gathered}$$

and $A_2 = \alpha\bar{\rho_1}(\frac{\gamma\underline{\rho_3}}{\alpha\bar{\rho_1}})^{-\frac{p_1-1}{p_3-p_1+1}}$. By combining (4.5) with (4.6) and (4.7) we obtain

$$L'(t) \leq -(\alpha b_1 - \beta b_2 - \gamma b_3 - K)L(t), \quad \forall t \in (0,+\infty), \tag{4.8}$$

where

$$K = \frac{\alpha\bar{\rho_1}(\frac{\beta\underline{\rho_2}}{\alpha\bar{\rho_1}})^{-\frac{p_1-1}{p_2-p_1+1}}}{m_2^{[q_1p_2-(q_2+1)(p_1-1)](p_2-p_1+1)^{-1}}m_3^{[r_1p_2-r_2(p_1-1)](p_2-p_1+1)^{-1}}},$$

or

$$K = \frac{\alpha\bar{\rho_1}(\frac{\gamma\underline{\rho_3}}{\alpha\bar{\rho_1}})^{-\frac{p_1-1}{p_3-p_1+1}}}{m_2^{[q_1p_3-q_3(p_1-1)](p_3-p_1+1)^{-1}}m_3^{[r_1p_3-(r_3+1)(p_1-1)](p_3-p_1+1)^{-1}}}.$$

Using (4.4) we deduce that the function $t \longmapsto L(t)$ is a non-increasing function. This completes the proof of Theorem 4.1. $\square$

Proof of Corollary 4.2. Setting for all $(t,x) \in (0,+\infty)\times\Omega$:

$$\begin{aligned} h_1(t,x) &= u(t,x),\\ h_2(t,x) &= v(t,x) - \frac{\sigma_2}{b_2},\\ h_3(t,x) &= w(t,x) - \frac{\sigma_3}{b_3}. \end{aligned}$$

For $i = 1,2,3$ we have

$$\frac{dh_i}{dt} - a_i\Delta h_i = -b_i h_i + \rho_i(x,u,v,w)\frac{u^{p_i}}{v^{q_i}w^{r_i}}. \tag{4.9}$$

Multiplying (4.9) by $h_i(t,x)$, $i = 1,2,3$ and integrating over $[0,t]\times\Omega$ we obtain

$$\begin{aligned} &\frac{1}{2}\int_\Omega h_i^2 dx + a_i\int_0^t\int_\Omega |\nabla h_i|^2\,dx\,ds + b_i\int_0^t\int_\Omega h_i^2\,dx\,ds\\ &= \frac{1}{2}\int_\Omega h_i^2(0)dx + \int_0^t\int_\Omega h_i\rho_i(x,u,v)\frac{u^{p_i}}{v^{q_i}w^{r_i}}\,dx\,ds. \end{aligned}$$

From (4.8), for all $t \in (0,+\infty)$, and for $i = 1,2,3$ we obtain

$$\int_0^t\int_\Omega h_i\rho_i(x,u,v)\frac{u^{p_i}}{v^{q_i}w^{r_i}}\,dx\,ds \le \bar\rho_i M_i\frac{M_1^{p_i}M_2^{\beta}M_3^{\gamma}}{m_2^{q_i}m_1^{\alpha}m_3^{r_i}}\int_0^t\int_\Omega \frac{u^\alpha}{v^\beta w^\gamma}\,dx\,ds < +\infty.$$

One obviously deduces that for $i = 1,2,3$,

$$\begin{gathered} h_i(t,.) \in L^2(\Omega), \quad \int_0^{+\infty}\int_\Omega |\nabla h_i|^2 dxds < +\infty,\\ \int_0^{+\infty}\int_\Omega h_i^2\,dx\,ds < +\infty, \end{gathered}$$

so that Barbalate's lemma [3, Lemma 1.2.2] permits to conclude that

$$\lim_{t\to+\infty}\|h_i(t,.)\|_2 = 0, \quad i = 1,2,3.$$

On the other hand, since the orbits $\{h_i(t,.)/t \ge 0, i = 1,2,3\}$ are relatively compact in $C(\bar\Omega)$ (see [4]), it follows readily that

$$\lim_{t\to+\infty}\|h_i(t,.)\|_\infty = 0, \quad i = 1,2,3.$$

Then proof of Corollary 4.2 is complete. □

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