GLOBAL STABILITY OF A VACCINATION MODEL WITH IMMIGRATION

SARAH HENSHAW, C. CONNELL MCCLUSKEY

ABSTRACT. We study an $SVIR$ model of disease transmission with immigration into all four classes. Vaccinated individuals may only receive partial immunity to the disease, giving a leaky vaccine. The incidence function permits a nonlinear response to the number of infectives, so that mass action and saturating incidence are included as special cases. Because of the immigration of infected individuals, there is no disease-free equilibrium and hence no basic reproduction number. We use the Brouwer Fixed Point Theorem to show that an endemic equilibrium exists and the Poincaré-Hopf Theorem to show that it is unique. We show the equilibrium is globally asymptotically stable by using a Lyapunov function.

1. INTRODUCTION

With modern levels of travel and migration between various parts of the world, it is inevitable that disease will be carried across international borders. Fighting infectious disease has become a global issue.

Vaccination against measles has been available for over 50 years. In Canada, the number of measles cases has fallen from about 350,000 per year in 1963 to less than 2000 per year in 1995 [2]. For various reasons, a portion of the population, including children, remains unvaccinated. A consequence of this is that there have been several measles outbreaks in Canada in recent years (2007-2014). These outbreaks are believed to be initiated by unvaccinated individuals travelling abroad and bringing the infection home with them [10].

In this article, we consider an infectious disease for which there is a vaccine that may not be fully effective. The population is divided into susceptible, vaccinated, infectious and recovered classes, giving an SVIR model. A key feature of the model is that there is immigration into each class, accounting for infected individuals entering the study population. Additionally, the incidence rate is permitted to have a non-linear dependence on the number of infectives, including mass action and saturating incidence as special cases.

Ordinary differential equation compartmental models for infectious disease with immigration, but without vaccination are studied in [1, 3, 4, 7, 12, 13]. In each of these models, the disease is not explicitly modelled outside the main population

being studied. Instead, contact with external regions arises as a constant influx into various classes of the model. This can be interpreted as the disease being at a constant endemic level in some other region, so that a constant immigration influx with no screening (or imperfect screening) leads to a steady flow of infectious (and other) individuals into the study population.

A delay differential equation model with migration was introduced in [15], and studied further in [11].

This paper is laid out as follows. In Section 2, we present our model. In Section 3, we show that there exists a unique equilibrium, which corresponds to the disease remaining endemic, and we show that the equilibrium is locally asymptotically stable. The global stability of the equilibrium is shown in Section 4. In Section 5, we show that increased vaccination always leads to a lower prevalence of the disease (at equilibrium). The results are discussed in Section 6.

2. The Model

The population is divided into four classes, susceptible, vaccinated, infectious and recovered individuals, with the sizes of the groups given by the variables S, V, I and R, respectively.

We assume nonlinear incidence $Sf(I)$ for the susceptible population. Furthermore, we allow that the vaccine is imperfect so that infection of vaccinated individuals is possible, however, we assume that the vaccine is not deleterious. Thus, the incidence for the vaccinated population is $Vg(I)$, with $g(I) \leq f(I)$, as stated below in (H2). Additionally, we assume that the incidence functions f and g satisfy certain reasonable restrictions, stated below in (H1).

We assume that there is constant recruitment into each of the four classes at rates Ω_S, Ω_V, Ω_I and Ω_R. These rates represent the total of births and immigration into the population. For example, in the absence of vertical transmission, Ω_S and Ω_V would include all births, with some vaccinated and some unvaccinated. Additionally, immigration into all four groups would also be included in these parameters.

The per capita death rate for non-disease related reasons is μ, and for disease related reasons is γ. Individuals who recover from the disease do so after an average duration $\frac{1}{\delta}$, and develop permanent immunity.

Susceptibles are vaccinated at a per capita rate α. We allow that vaccinated individuals may develop full immunity to the disease after an average duration $\frac{1}{\gamma_1}$, so that the per capita rate at which individuals move from V to R is γ_1. The case that full immunity is never achieved is included in the model by having $\gamma_1 = 0$.
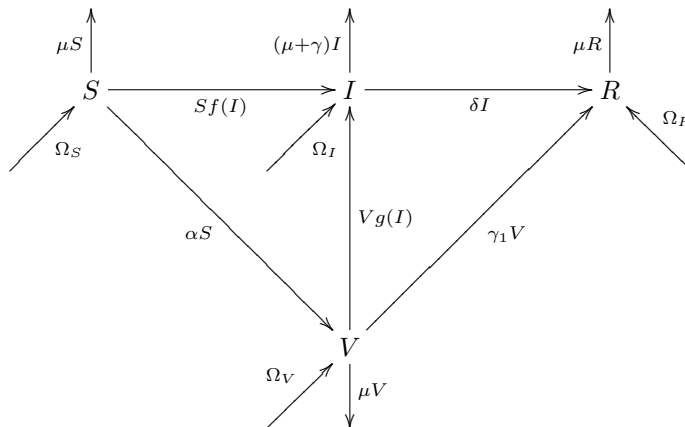
An incidence function h is assumed to be continuous for $I \geq 0$, and to satisfy the following criteria:

$$\begin{aligned} h(0) = 0, \quad h(I) \geq 0, \\ h'(I) \geq 0, \quad h''(I) \leq 0, \end{aligned} \tag{2.1}$$

for $I > 0$.

(H1) The functions f and g satisfy the criteria given in (2.1).
(H2) $g(I) \leq f(I)$ for all $I \geq 0$.
(H3) $\Omega_S, \Omega_V, \Omega_I, \Omega_R, \mu > 0$ and $\alpha, \delta, \gamma, \gamma_1 \geq 0$.

The transfer diagram is:

The corresponding differential equations are:

$$\frac{dS}{dt} = \Omega_S - Sf(I) - (\mu + \alpha)S$$
$$\frac{dV}{dt} = \Omega_V + \alpha S - Vg(I) - (\mu + \gamma_1)V$$
$$\frac{dI}{dt} = \Omega_I + Sf(I) + Vg(I) - (\mu + \gamma + \delta)I \qquad (2.2)$$
$$\frac{dR}{dt} = \Omega_R + \gamma_1 V + \delta I - \mu R.$$

Because R does not appear in the equations for the other variables, it can be omitted; we work with S, V, and I. The initial conditions are $(S(0), V(0), I(0)) \in \mathbb{R}^3_{\geq 0}$. Let $\Omega = \Omega_S + \Omega_V + \Omega_I$. For any $\epsilon \geq 0$, let

$$D_\epsilon = \big\{(S, V, I) : S, V, I \geq \epsilon \text{ and } S + V + I \leq \frac{\Omega}{\mu}\big\}.$$

Proposition 2.1. *The non-negative octant $\mathbb{R}^3_{\geq 0}$ is positively invariant. There exists $\bar{\epsilon} > 0$ such that $D_{\bar{\epsilon}}$ is nonempty, attracting, and positively invariant.*

Proof. Clearly the vector field defined by equations (2.2) is smooth and therefore solution is unique. Since $\frac{dS}{dt}\big|_{S=0} = \Omega_S > 0$, $\frac{dI}{dt}\big|_{I=0} = \Omega_I > 0$ and $\frac{dV}{dt}\big|_{V=0} = \Omega_V + \alpha S > 0$ within $\mathbb{R}^3_{\geq 0}$, [5, Proposition 2.1] implies that $\mathbb{R}^3_{\geq 0}$ is positively invariant.

Let $N = S + V + I$. Then

$$\frac{dN}{dt} = \Omega_S + \Omega_V + \Omega_I - \mu(S + V + I) - \gamma_1 V - (\gamma + \delta)I \leq \Omega - \mu N.$$

Thus, $\limsup_{t \to \infty} N(t) \leq \frac{\Omega}{\mu}$, and so D_0 is attracting. If $N(t_0) \leq \frac{\Omega}{\mu}$ for some $t_0 \in \mathbb{R}$, then $N(t) \leq \frac{\Omega}{\mu}$ for all $t \geq t_0$, and so D_0 is positively invariant.

In D_0, $\frac{dS}{dt}\big|_{S=0} = \Omega_S > 0$. Since the vector field is continuous and D_0 is compact, there exists $\epsilon_S > 0$ such that $\frac{dS}{dt} > 0$ for $S \leq \epsilon_S$. Therefore, S will increase to be larger than ϵ_S, and will remain above that level for all future time.

A similar argument holds for V and I, allowing us to define ϵ_V and ϵ_I. Let $\bar{\epsilon} = \min\{\epsilon_S, \epsilon_V, \epsilon_I\}$. Then $D_{\bar{\epsilon}}$ is positively invariant and attracting. $\quad\square$

3. Existence, local stability and uniqueness of an equilibrium point

Note that $\frac{dI}{dt}$ is strictly positive when I is zero. Thus, there is no disease-free equilibrium and therefore there is no basic reproduction number $\mathcal{R}_0$.

Solving for equilibria directly gives the existence of an endemic equilibrium, but showing the uniqueness of this solution is problematic. Here, we take a different approach. First we show the existence of at least one equilibrium. Next we show that each equilibrium is locally asymptotically stable. Then we demonstrate the uniqueness of the equilibrium.

Proposition 3.1. *There exists at least one equilibrium. Furthermore, all equilibria lie in the set $D_{\bar{\epsilon}}$.*

Proof. By Proposition 2.1, the set $D_{\bar{\epsilon}}$ is nonempty and positively invariant. Since $D_{\bar{\epsilon}}$ is also compact and convex, it follows from the Brouwer Fixed Point Theorem (see [14, Theorem 4.8], for example) that $D_{\bar{\epsilon}}$ contains at least one equilibrium.

By Proposition 2.1, $D_{\bar{\epsilon}}$ is attracting, and so all equilibria are elements of $D_{\bar{\epsilon}}$. $\quad\square$

In the proof of Proposition 3.4, the following result will be applied to f and g. It appears in [12, Proposition 4.1].

Proposition 3.2. *Suppose h satisfies the criteria in (2.1). Then $h'(I) \leq \frac{h(I)}{I}$ for all $I > 0$.*

Proof. Let $I > 0$. By the Mean Value Theorem, there exists $c \in (0, I)$ with $h'(c) = \frac{h(I)-h(0)}{I-0}$. The criteria in (2.1) imply $h(0) = 0$, and so $h'(c) = \frac{h(I)}{I}$. Because $h''(I) \leq 0$, it follows that $h'(I)$ is decreasing and so $h'(I) \leq h'(c) = \frac{h(I)}{I}$. $\quad\square$

The following lemma, [7, Lemma 3], will also be used in the proof of Proposition 3.4. For 3×3 real matrices, it is an alternative to the Routh-Hurwitz criteria. To state the lemma, it is necessary to define the second compound $M^{[2]}$ of a square matrix M, which we do here only for the case of 3×3 matrices. If

$$M = \begin{bmatrix} A & a & b \\ c & B & d \\ e & f & C \end{bmatrix}, \quad \text{then } M^{[2]} = \begin{bmatrix} A+B & d & -b \\ f & A+C & a \\ -e & c & B+C \end{bmatrix}.$$

The eigenvalues of $M^{[2]}$ are sums of pairs of eigenvalues of M. Further reading on compound matrices can be found in [9].

Lemma 3.3. *Let M be a 3×3 real matrix. If $\operatorname{trace}(M)$, $\det(M)$ and $\det(M^{[2]})$ are all negative, then all of the eigenvalues of M have negative real part.*

Proposition 3.4. *Each equilibrium is locally asymptotically stable.*

Proof. Let $X^* = (S^*, V^*, I^*) \in D_{\bar{\epsilon}}$ denote an equilibrium and let $\tau = (\mu + \gamma + \delta) - S^* f'(I^*) - V^* g'(I^*)$. The Jacobian at X^* is

$$J = \begin{bmatrix} -J_1 & 0 & -S^* f'(I^*) \\ \alpha & -J_2 & -V^* g'(I^*) \\ f(I^*) & g(I^*) & -J_3 \end{bmatrix},$$

where

$$\begin{bmatrix} J_1 \\ J_2 \\ J_3 \end{bmatrix} = \begin{bmatrix} f(I^*) + \mu + \alpha \\ g(I^*) + \mu + \gamma_1 \\ (\mu + \gamma + \delta) - S^* f'(I^*) - V^* g'(I^*) \end{bmatrix}.$$

We begin by showing that J_3 is positive. Applying Proposition 3.2 to f and to g, we have

$$\begin{aligned} J_3 &= (\mu + \gamma + \delta) - S^* f'(I^*) - V^* g'(I^*) \\ &\geq (\mu + \gamma + \delta) - S^* \frac{f(I^*)}{I^*} - V^* \frac{g(I^*)}{I^*} \\ &= \frac{(\mu + \gamma + \delta)I^* - S^* f(I^*) - V^* g(I^*)}{I^*} \\ &= \frac{\Omega_I}{I^*} > 0. \end{aligned}$$

Thus, each of J_1, J_2 and J_3 is positive. Thus, we can now see that $\text{trace}(J) = -(J_1 + J_2 + J_3) < 0$. The determinant of J satisfies

$$\det(J) = -J_1 J_2 J_3 - S^* f'(I^*) g(I^*) \alpha - J_1 g(I^*) V^* g'(I^*) - S^* f'(I^*) f(I^*) J_2 < 0.$$

The second compound of J is

$$J^{[2]} = \begin{bmatrix} -(J_1 + J_2) & -V^* g'(I^*) & S^* f'(I^*) \\ g(I^*) & -(J_1 + J_3) & 0 \\ -f(I^*) & \alpha & -(J_2 + J_3) \end{bmatrix},$$

with

$$\begin{aligned} \det(J^{[2]}) &= -(J_1 + J_2)(J_1 + J_2)(J_2 + J_3) + S^* f'(I^*) g(I^*) \alpha \\ &\quad - V^* g'(I^*) g(I^*)(J_2 + J_3) - S^* f'(I^*) f(I^*)(J_1 + J_3) \\ &< S^* f'(I^*) g(I^*) \alpha - S^* f'(I^*) f(I^*)(J_1 + J_3). \end{aligned}$$

Using (H2), $g(I) \leq f(I)$ and therefore

$$\det(J^{[2]}) < S^* f'(I^*) f(I^*)[\alpha - (J_1 + J_3)] < 0.$$

Thus, Lemma 3.3 implies that each eigenvalue of J has negative real part. Hence, the equilibrium X^* is locally asymptotically stable. $\square$

Proposition 3.5. *The equilibrium is unique.*

Proof. Let $\mathcal{G}(x)$ denote the vector field described by the differential equation given in (2.2). Let $\mathcal{F} = -\mathcal{G}$. Then the vector field $\mathcal{F}$ is outward pointing on the boundary of $D_{\bar{\epsilon}}$. Note that $D_{\bar{\epsilon}}$ is homeomorphic to the ball in $\mathbb{R}^3$, and therefore has Euler characteristic $+1$. The zeroes of $\mathcal{G}$ (given by equilibria) are also zeroes of $\mathcal{F}$. Since the equilibria are locally asymptotically stable for $\mathcal{G}$, they are all isolated and each has an index of $+1$ for $\mathcal{F}$. Thus, the sum of the indices of the equilibria is equal to the number of equilibria.

However, by the Poincaré-Hopf Theorem [8, Chapter 6], the sum of the indices of the equilibria is equal to the Euler characteristic of $D_{\bar{\epsilon}}$. Thus, there is only one equilibrium. $\square$

4. Global stability

In this section, a Lyapunov function will be used to show the global stability of the unique equilibrium. Let

$$\omega(x) = x - 1 - \ln x$$

The following result appears in [6, Lemma 2.3].

Proposition 4.1. *If $a, b, c, d > 0$, then $(a-b)(c-d) = \omega(ac) - \omega(ad) - \omega(bc) + \omega(bd)$.*

Proof. Note that

$$
\begin{aligned}
&\omega(ac) - \omega(ad) - \omega(bc) + \omega(bd) \\
&= ac - 1 - \ln(ac) - ad + 1 + \ln(ad) - bc + 1 + \ln(bc) + bd - 1 - \ln(bd) \\
&= ac - ad - bc + bd - \ln(ac) + \ln(ad) + \ln(bc) - \ln(bd) \\
&= ac - ad - bc + bd \\
&= (a - b)(c - d).
\end{aligned}
$$

$\square$

The following result appears in [12, Proposition A.1].

Proposition 4.2. *Suppose h satisfies the criteria in (2.1). If $I > 0$, then*

$$\omega\left(\frac{h(I)}{h(I^*)}\right) \le \omega\left(\frac{I}{I^*}\right).$$

Proof. Suppose $I \ge I^*$. Let $m(I) = \frac{h(I)}{I}$. Then

$$m'(I) = \frac{h'(I)I - h(I)}{I^2} \le \frac{h(I) - h(I)}{I^2} = 0.$$

Thus m is decreasing. Thus $m(I) \le m(I^*)$. Thus, $\frac{h(I)}{I} \le \frac{h(I^*)}{I^*}$, and so $\frac{h(I)}{h(I^*)} \le \frac{I}{I^*}$. Since h is increasing, we have $1 \le \frac{h(I)}{h(I^*)} \le \frac{I}{I^*}$. Note that

$$\omega'(x) = 1 - \frac{1}{x}.$$

Thus, ω is increasing for $x > 1$. Hence, $\omega\left(\frac{h(I)}{h(I^*)}\right) \le \omega\left(\frac{I}{I^*}\right)$.

A similar argument works for $I \in (0, I^*)$. $\square$

In the following proof, the equilibrium equations

$$
\begin{aligned}
\Omega_S &= S^* f(I^*) + (\mu + \alpha)S^*, \\
\Omega_V &= V^* g(I^*) + (\mu + \gamma_1)V^* - \alpha S^*, \\
\mu + \gamma + \delta &= \frac{\Omega_I + S^* f(I^*) + V^* g(I^*)}{I^*}.
\end{aligned}
\tag{4.1}
$$

will be used.

Theorem 4.3. *The equilibrium is globally asymptotically stable.*

Proof. Let

$$U = S^*\omega\left(\frac{S}{S^*}\right) + V^*\omega\left(\frac{V}{V^*}\right) + I^*\omega\left(\frac{I}{I^*}\right).$$

Note that

$$\frac{dU}{dt} = \left(1 - \frac{S^*}{S}\right)\frac{dS}{dt} + \left(1 - \frac{V^*}{V}\right)\frac{dV}{dt} + \left(1 - \frac{I^*}{I}\right)\frac{dI}{dt}$$

$$= (1 - \frac{S^*}{S})[\Omega_S - Sf(I) - (\mu + \alpha)S]$$
$$+ (1 - \frac{V^*}{V})[\Omega_V + \alpha S - Vg(I) - (\mu + \gamma_1)V]$$
$$+ (1 - \frac{I^*}{I})[\Omega_I + Sf(I) + Vg(I) - (\mu + \gamma + \delta)I].$$

Using (4.1) to replace Ω_S, Ω_V and $\mu + \gamma + \delta$ gives

$$\frac{dU}{dt} = (1 - \frac{S^*}{S})[S^* f(I^*) + (\mu + \alpha)S^* - Sf(I) - (\mu + \alpha)S]$$
$$+ (1 - \frac{V^*}{V})[V^* g(I^*) + (\mu + \gamma_1)V^* - \alpha S^* + \alpha S - Vg(I) - (\mu + \gamma_1)V]$$
$$+ (1 - \frac{I^*}{I})\Big[\Omega_I + Sf(I) + Vg(I) - \frac{\Omega_I + S^* f(I^*) + V^* g(I^*)}{I^*}I\Big]$$
$$= (1 - \frac{S^*}{S})\Big[S^* f(I^*)(1 - \frac{Sf(I)}{S^* f(I^*)}) + (\mu + \alpha)S^*(1 - \frac{S}{S^*})\Big]$$
$$+ (1 - \frac{V^*}{V})\Big[V^* g(I^*)(1 - \frac{Vg(I)}{V^* g(I^*)}) + (\mu + \gamma_1)V^*(1 - \frac{V}{V^*}) + \alpha S^*(\frac{S}{S^*} - 1)\Big]$$
$$+ (1 - \frac{I^*}{I})\Big[\Omega_I(1 - \frac{I}{I^*}) + S^* f(I^*)(\frac{Sf(I)}{S^* f(I^*)} - \frac{I}{I^*})$$
$$+ V^* g(I^*)(\frac{Vg(I)}{V^* g(I^*)} - \frac{I}{I^*})\Big].$$

Next, we apply Proposition (4.1) (noting that $\omega(1) = 0$), and group and cancel terms to obtain

$$\frac{dU}{dt} = (\mu + \alpha)S^*\Big[-\omega(\frac{S^*}{S}) - \omega(\frac{S}{S^*})\Big] - \Omega_I \frac{(I - I^*)^2}{II^*}$$
$$+ S^* f(I^*)\Big[-\omega(\frac{S^*}{S}) + \omega(\frac{f(I)}{f(I^*)}) - \omega(\frac{I}{I^*}) - \omega(\frac{Sf(I)I^*}{S^* f(I^*)I})\Big]$$
$$+ V^* g(I^*)\Big[-\omega(\frac{I}{I^*}) - \omega(\frac{Vg(I)I^*}{V^* g(I^*)I}) - \omega(\frac{V^*}{V}) + \omega(\frac{g(I)}{g(I^*)})\Big]$$
$$+ (\mu + \gamma_1)V^*\Big[-\omega(\frac{V^*}{V}) - \omega(\frac{V}{V^*})\Big] + \alpha S^*\Big[\omega(\frac{S}{S^*}) - \omega(\frac{SV^*}{S^*V}) + \omega(\frac{V^*}{V})\Big].$$
$$(4.2)$$

It follows from the second line of (4.1), that $\alpha S^* < V^* g(I^*) + (\mu + \gamma_1)V^*$. Thus, the three terms in (4.2) involving $\omega(\frac{V^*}{V})$ combine to give a quantity that is less than or equal to zero. Also, the term $\alpha S^* \omega(\frac{S}{S^*})$ in the final line of (4.2) cancels with part of the term $(\mu + \alpha)S^* \omega(\frac{S}{S^*})$ in the first line. Thus,

$$\frac{dU}{dt} \le - (\mu + \alpha)S^* \omega(\frac{S^*}{S}) - \mu S^* \omega(\frac{S}{S^*}) - \Omega_I \frac{(I - I^*)^2}{II^*}$$
$$+ S^* f(I^*)\Big[-\omega(\frac{S^*}{S}) + \omega(\frac{f(I)}{f(I^*)}) - \omega(\frac{I}{I^*}) - \omega(\frac{Sf(I)I^*}{S^* f(I^*)I})\Big]$$
$$+ V^* g(I^*)\Big[-\omega(\frac{I}{I^*}) - \omega(\frac{Vg(I)I^*}{V^* g(I^*)I}) + \omega(\frac{g(I)}{g(I^*)})\Big]$$
$$- (\mu + \gamma_1)V^* \omega(\frac{V}{V^*}) - \alpha S^* \omega(\frac{SV^*}{S^*V}).$$

The functions f and g each satisfy the criteria in (2.1). Thus, Proposition 4.2 implies $\omega\big(\frac{f(I)}{f(I^*)}\big)$, $\omega\big(\frac{g(I)}{g(I^*)}\big) \leq \omega(\frac{I}{I^*})$. Therefore,

$$
\begin{aligned}
\frac{dU}{dt} \leq &-(\mu+\alpha)S^*\omega(\frac{S^*}{S}) - \mu S^*\omega(\frac{S}{S^*}) - \Omega_I\frac{(I-I^*)^2}{II^*} \\
&- S^*f(I^*)\big[\omega(\frac{S^*}{S}) + \omega\big(\frac{Sf(I)I^*}{S^*f(I^*)I}\big)\big] - V^*g(I^*)\omega\big(\frac{Vg(I)I^*}{V^*g(I^*)I}\big) \\
&- (\mu+\gamma_1)V^*\omega(\frac{V}{V^*}) - \alpha S^*\omega\big(\frac{SV^*}{S^*V}\big) \leq 0,
\end{aligned}
$$

since ω is non-negative. Furthermore, we only obtain $\frac{dU}{dt} = 0$ at the equilibrium. Thus, by Lyapunov's Direct Method, the equilibrium (S^*, V^*, I^*) is globally asymptotically stable. □

5. Effect of vaccination on disease prevalence

We now explore the connection between higher vaccination levels (i.e. increasing α) and disease prevalence at equilibrium. In doing so, we consider S^* and V^* to be functions of I^* and α, where I^* is in turn considered to be a function of α. Throughout this section, the variables S, V and I are only considered at the equilibrium, and so we omit the superscripts * in order to present a tidier calculation. For similar reasons, we also define

$$F = f(I) + \mu + \alpha, \quad G = g(I) + \mu + \gamma_1. \tag{5.1}$$

From $\frac{dS}{dt} = 0$ and $\frac{dV}{dt} = 0$, at equilibrium we have

$$S = \frac{\Omega_S}{f(I)+\mu+\alpha} = \frac{\Omega_S}{F}, \quad V = \frac{\Omega_V + \alpha S}{g(I)+\mu+\gamma_1} = \frac{\Omega_V + \alpha\frac{\Omega_S}{F}}{G}. \tag{5.2}$$

Adding $\frac{dS}{dt}$, $\frac{dV}{dt}$ and $\frac{dI}{dt}$ at equilibrium gives

$$0 = \Omega_S + \Omega_V + \Omega_I - \mu S - (\mu+\gamma_1)V - (\mu+\gamma+\delta)I.$$

Differentiating with respect to α and using (5.2) gives

$$
\begin{aligned}
0 = &-\mu\frac{dS}{d\alpha} - (\mu+\gamma_1)\frac{dV}{d\alpha} - (\mu+\gamma+\delta)\frac{dI}{d\alpha} \\
= &\;\mu\frac{\Omega_S}{F^2}\big(f'(I)\frac{dI}{d\alpha}+1\big) - (\mu+\gamma_1)\frac{(S+\alpha\frac{dS}{d\alpha})G - (\Omega_V+\alpha S)g'(I)\frac{dI}{d\alpha}}{G^2} \\
&- (\mu+\gamma+\delta)\frac{dI}{d\alpha}.
\end{aligned}
$$

Using (5.2) to replace the last instance of $\frac{dS}{d\alpha}$, and also to replace terms involving Ω_S and Ω_V, we obtain

$$0 = \frac{\mu S(f'(I)\frac{dI}{d\alpha}+1)}{F} - (\mu+\gamma_1)\frac{S - \frac{\alpha S}{F}(f'(I)\frac{dI}{d\alpha}+1) - Vg'(I)\frac{dI}{d\alpha}}{G} - (\mu+\gamma+\delta)\frac{dI}{d\alpha}.$$

Solving for $\frac{dI}{d\alpha}$, using (5.1) to replace F and G as needed, and then using the third line of (4.1), gives

$$
\begin{aligned}
\frac{dI}{d\alpha} &= S\frac{\gamma_1 f(I) + \mu(f(I) - g(I))}{(\mu + \gamma_1)FVg'(I) + (\mu + \gamma_1)\alpha Sf'(I) + \mu GSf'(I) - (\mu + \gamma + \delta)FG} \\
&= S\frac{\gamma_1 f(I) + \mu(f(I) - g(I))}{(\mu + \gamma_1)FVg'(I) + [(\mu + \gamma_1)\alpha + \mu G]Sf'(I) - (\frac{\Omega_I}{I} + \frac{Sf(I)}{I} + \frac{Vg(I)}{I})FG}.
\end{aligned}
\tag{5.3}
$$

By (H1), we can apply Proposition 3.2 to f and g, allowing us to conclude that the denominator in (5.3) is negative. The numerator in (5.3) is non-negative by (H2). Thus, it follows that

$$
\frac{dI}{d\alpha} \leq 0.
\tag{5.4}
$$

In fact, equality is only obtained in (5.4) in degenerate cases: either f and g are identically zero, in which case there is no disease transmission, or $\gamma_1 = 0$ and $f(I^*) = g(I^*)$, in which case there is no difference between susceptible and vaccinated individuals (at equilibrium). Outside of these cases, we have $\frac{dI}{d\alpha} < 0$, and so vaccination has an impact on decreasing disease prevalence.

6. Discussion

We have studied an SVIR infectious disease model, including vaccination of susceptibles and immigration into each group. The vaccine is allowed to be imperfect, but not deleterious, so that the force of infection for vaccinated individuals is less than or equal to the force of infection for susceptibles. We allow that the incidence may have a nonlinear dependence on the size of the infectious population.

For all parameter values, there is no disease-free equilibrium and there is a unique endemic equilibrium, which is globally asymptotically stable. Existence and uniqueness are shown using the Brouwer Fixed Point Theorem and the Poincaré-Hopf Theorem. The global stability is shown using Lyapunov's Direct Method.

The global stability means that the disease cannot be fully eradicated from the population when there is an influx of infected individuals from a region where the disease is endemic. Given the high level of interconnectedness between all regions of the world, it is inevitable that disease will travel between countries. Control of the disease in any one region requires treating the disease as a global problem, and controlling it in all regions.

References

[1] F. Brauer, P. van den Driessche; Models for transmission of disease with immigration of infectives. *Math. Biosci.*, 171:143–154, 2001.

[2] Immunize Canada; Measles, 2014. www.immunize.ca/en/diseases-vaccines/measles.aspx.

[3] H. Guo, M. Y. Li; Impacts of migration and immigration on disease transmission dynamics in heterogeneous populations. *Discrete Contin. Dyn. Syst. Ser. B*, 17(7):2413–2430, 2012.

[4] H. Guo, J. Wu; Persistent high incidence of tuberculosis among immigrants in a low-incidence country: impact of immigrants with early or late latency. *Math. Biosci. and Eng.*, 8(3):695–709, 2011.

[5] W. M. Haddad, V. Chellaboina, Q. Hui; *Nonnegative and Compartmental Dynamical Systems*. Princeton University Press, Princeton, New Jersey, 2010.

[6] C. C. McCluskey; Using Lyapunov functions to construct Lyapunov functionals for delay differential equations. *SIAM J. App. Dyn. Sys.*, 14(1):1–24, 2015.

[7] C. C. McCluskey, P. van den Driessche; Global analysis of two tuberculosis models. *J. Dynam. Differential Equations*, 16(1):139–166, 2004.

[8] J. W. Milnor; *Topology from the Differentiable Viewpoint*. University Press of Virginia, Charlottesville, 1965.

[9] J. S. Muldowney; Compound matrices and ordinary differential equations. *Rocky Mountain J. Math.*, 20:857–872, 1990.

[10] Public Health Agency of Canada; Public health notice: Measles, 2014. www.phac-aspc.gc.ca/phn-asp/2013/measles-0717-eng.php.

[11] R. P. Sigdel, C. C. McCluskey; Disease dynamics for the hometown of migrant workers. *Math. Biosci. and Eng.*, 11(5):1175–1180, 2014.

[12] R. P. Sigdel, C. C. McCluskey; Global stability for an SEI model of infectious disease with immigration. *Appl. Math. Comput.*, 243:684–689, 2014.

[13] J. Tumwiine, J. Y. T. Mugisha, L. S. Luboobi; A host-vector model for malaria with infective immigrants. *J. Math. Anal. Appl.*, 361:139–149, 2010.

[14] F. Verhulst; *Nonlinear Differential Equations and Dynamical Systems*. Springer-Verlag, New York, 1990.

[15] Lin Wang, Xiao Wang; Influence of temporary migration on the transmission of infectious diseases in a migrants' home village. *J. Theoret. Biol.*, 300:100–109, 2012.

Sarah Henshaw

Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, Canada
    E-mail address: hens3420@mylaurier.ca

C. Connell McCluskey

Department of Mathematics, Wilfrid Laurier University, Waterloo, Ontario, Canada
    E-mail address: ccmcc8@gmail.com