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# EXISTENCE OF SOLUTIONS TO QUASILINEAR SCHRÖDINGER EQUATIONS WITH INDEFINITE POTENTIAL

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Abstract. In this article, we study the existence and multiplicity of solutions of the quasilinear Schrödinger equation

$$-u'' + V(x)u - (|u|^2)''u = f(u)$$

on $\mathbb{R}$, where the potential $V$ allows sign changing and the nonlinearity satisfies conditions weaker than the classical Ambrosetti-Rabinowitz condition. By a local linking theorem and the fountain theorem, we obtain the existence and multiplicity of solutions for the equation.

## 1. Introduction

We study the existence and multiplicity of solutions for the quasilinear elliptic equation

$$- u'' + V(x)u - (|u|^2)''u = f(u), \quad x \in \mathbb{R}. \tag{1.1}$$

Solutions of the equation are related to standing wave solutions for quasilinear Schrödinger equation of the form

$$i\partial_t z = -z'' + \tilde{V}(x)z - (|z|^2)''z - \tilde{f}(|z|^2)z \tag{1.2}$$

which arises in various fields of physics, like the theory of superfluids or in dissipative quantum mechanics, plasma physics, fluid mechanics and in the theory of Heisenberg ferromagnets, etc. For further physical motivations and a more complete list of references, we refer to [6, 9, 11] and the references therein.

As far as we know, the first existence result for equation (1.1) by variational methods is due to [11], where by a constrained minimization argument the authors proved the existence of a positive ground state solution with an unknown Lagrange multiplier $\lambda$ in front of the nonlinear term. Ambrosetti and Wang [12] considered the existence of positive solutions of perturbation to the equation with a particular nonlinearity $g(u) = u^p$. Alves et al [1], considered the existence and concentration of positive solutions as $\epsilon \to 0$ for a related equation with $\epsilon^2$. Some related problems on $\mathbb{R}$ are also considered in [2].

There is also much work devoting to the corresponding high dimensional equation; e.g. see [6, 10, 8, 9]. The solutions of equation (1.1) correspond to the critical

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points of the functional on $H^1(\mathbb{R})$:

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}} u'^2 + V(x)u^2 dx + \int_{\mathbb{R}} u'^2 u^2 dx - \int_{\mathbb{R}} F(u)dx, \tag{1.3}$$

where $F(t) = \int_0^t f(s)ds$. All the above mentioned papers require that the potential $V(x)$ is positive. So that the energy functional $\Phi$ possesses the mountain pass geometry. Therefore, the mountain pass lemma can be applied. In [13], the authors consider the case which the potential $V(x)$ allows sign-change. However, in order to satisfy the conditions of mountain pass theory, they need additional conditions on nonlinearity. The similar assumptions are added in [14] (See Remark ). The aim of the paper is to investigate equation (1.1) where the potential $V(x)$ can be sign changing and the nonlinearity does not need to satisfy Ambrosetti-Rabinowitz condition. The term $\int_{\mathbb{R}} u'^2 u^2 dx$ in (1.3) is homogeneous of order 4 and non-convex, it prevents the linking geometric structure of the energy functional under our assumption. Inspired by the recent work of Chen and Liu [5] we make use of the local linking theory to overcome this difficulty. To state our main results, we list the assumptions on $f$ and $V$ as follows.

- (V1) The potential $V(x) \in C(\mathbb{R})$ is bounded from below and $\mu(V^{-1}(-\infty, M)) < \infty$ for every $M > 0$.
- (F1) There exists $C > 0$ and $p > 2$ such that
$$|f(t)| \le C|t|^{p-1} \quad \text{for all } t \in \mathbb{R}.$$
- (F2) $4F(t) \le f(t)t$ and
$$\lim_{t\to\infty} \frac{F(t)}{t^4} = +\infty \quad \text{for all } t \in \mathbb{R}.$$

We are now ready to state our results.

**Theorem 1.1.** *Suppose that* (V1), (F1)–(F2) *are satisfied and $f$ is odd. Then equation* (1.1) *has a sequence of solutions such that $\Phi(u_k) \to +\infty$.*

**Theorem 1.2.** *Suppose that* (V1), (F1)–(F2) *are satisfied. Then equation* (1.1) *has at least one nontrivial solution.*

**Remark 1.3.** In [13, 14], the authors need assumptions (F2) and

- (G2) $\widetilde{F}(x,u) := \frac{1}{4}f(x,u)u - F(x,u) \ge 0$, and there exist $c_0 > 0$ and $\sigma > max\{1, \frac{2N}{N+2}\}$ such that
$$|F(x,u)|^\sigma \le c_0|u|^{2\sigma}\widetilde{F}(x,u)$$
for all $(x,u) \in \mathbb{R}^N$.

In this paper, $(G_2)$ is not needed.

Throughout this paper, the letters $C$ and $C_i$ denote positive constants, which may be different from place to place. The usual norm in $L^p(\mathbb{R})$ with $1 \le p \le +\infty$ is denoted by $|\cdot|_p$

## 2. Preliminaries

To overcome the non-compactness of the embedding $H^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$, we consider a linear subspace $X$ of $H^1(\mathbb{R})$:

$$X := \{u \in H^1(\mathbb{R}) : \int_{\mathbb{R}} \bar{V}(x)u^2 dx < \infty\}$$

equipped with the inner product

$$\langle u, v\rangle = \int_{\mathbb{R}} u'v' + \bar{V}uvdx$$

and the corresponding norm $\|u\| = \langle u, v\rangle^{1/2}$, where $\bar{V} = V(x) + m > 1$ for a fixed positive number $m$. It is well known that the imbedding $X \hookrightarrow L^2(\mathbb{R})$ is compact under the condition (V1). See [4]. Therefore, the eigenvalues of the operator

$$S := -\Delta + V$$

can be numbered as

$$-\infty < \lambda_1 \le \lambda_2 \le \lambda_3 \dots, \lambda_l \to \infty$$

and the corresponding eigenfunctions are denoted by $\phi_1,\ \phi_2 \dots$. We assume that $0 \in (\lambda_l, \lambda_{l+1})$ for some $l > 1$. Let

$$X^- = \operatorname{span}\{\phi_1, \dots, \phi_l\}, \quad X^+ = (X^-)^\perp.$$

Then $X^-$ and $X^+$ are the negative and positive spaces of the quadratic form

$$Q(u) = \frac{1}{2}\int_{\mathbb{R}} u'(x)^2 + V(x)u^2(x)dx.$$

. It is well known that there is a positive constant $\alpha > 0$ such that

$$\pm Q(u) \ge \alpha\|u\|^2, \quad u \in X^\pm. \tag{2.1}$$

Let

$$J(u) = \frac{1}{2}\int_{\mathbb{R}} u'^2 + V(x)u^2dx - \int_{\mathbb{R}} F(u)dx, \quad I(u) = \int_{\mathbb{R}} u'^2u^2dx.$$

Then

$$\Phi(u) = J(u) + I(u).$$

By the continuous imbedding $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$ and $X \hookrightarrow H^1(\mathbb{R})$, $I$ is well defined on $X$ and

$$|I(u)| \le |u|_\infty^2\|u\|_{H^1}^2 \le C\|u\|^4 \quad \text{for all } u \in X. \tag{2.2}$$

To verify that the functional $\Phi$ is $C^1$, it is sufficient to prove this for $I(u)$.

**Lemma 2.1.** *$I(u)$ belongs to $C^1$ in $X$.*

The proof of the above lemma is similar to that of [11, Lemma 1]. We omit it here. From the above discussions, the functional $\Phi(u)$ is a $C^1$ functional with derivative given by

$$\begin{aligned}(\Phi'(u), v) &= \int_{\mathbb{R}} u'v' + \bar{V}uvdx + \int_{\mathbb{R}} 2u'^2uv + 2u^2u'v'dx - \int_{\mathbb{R}} g(u)vdx \\ &= \langle u, v\rangle + \int_{\mathbb{R}} 2u'^2uv + 2u^2u'v'dx - \int_{\mathbb{R}} g(u)vdx,\end{aligned} \tag{2.3}$$

where $g(t) = f(t) + mt$. To prove our main results, we need to introduce some definitions and theorems.

**Definition.** We say that $\Phi \in C^1(X)$ satisfies condition (PS) if any sequence $(u_n) \subset X$ such that

$$\Phi(u_n) \to c, \quad \Phi'(u_n) \to 0$$

has a convergent subsequence.

For the proof of Theorem 1.1, we use the following fountain theorem by Bartsch [3]. For $k = 1, 2 \dots$, let

$$Y_k = \operatorname{span}\{\phi_1, \dots \phi_k\}, \quad Z_k = \overline{\operatorname{span}\{\phi_k, \phi_{1+k} \dots\}}. \tag{2.4}$$

**Theorem 2.2.** *Assume that the even functional $\Phi \in C^1(X)$ satisfies the (PS) condition, if there exists $k_0 > 0$ such that for $k \geq k_0$ there exists $\rho_k > r_k > 0$ such that*

(i) $b_k = \inf_{u \in Z_k, \|u\| = r_k} \Phi(u) \to +\infty$ *as $k \to \infty$,*
(ii) $\alpha_k = \max_{u \in Y_k, \|u\| = \rho_k} \Phi(u) \leq 0$.

*Then $\Phi$ has a sequence of critical points $\{u_k\}$ such that $\Phi(u_k) \to +\infty$.*

For the proof of Theorem 1.2, we will use the local linking theorem. Recall that the definition of local linking at 0 with respect to the direct sum decomposition $X = X^+ \oplus X^-$, if there is $\rho > 0$ such that for $u \in X^-$

$$\begin{aligned} \Phi(u) \leq 0, &\quad \text{for } u \in X^-,\ \|u\| \leq \rho \\ \Phi(u) \geq 0, &\quad \text{for } u \in X^+,\ \|u\| \leq \rho. \end{aligned} \tag{2.5}$$

Next, we consider two sequences of finite dimensional subspaces

$$X_0^{\pm} \subset X_1^{\pm} \subset \dots \subset X^{\pm}$$

such that

$$X^{\pm} = \overline{\cup_{n \in \mathbb{N}} X_n^{\pm}}.$$

For multi-index $\alpha = (\alpha^-, \alpha^+) \in \mathbb{N}^2$, we set $X_\alpha = X_\alpha^- \oplus X_\alpha^+$ and denote by $\Phi_\alpha$ the restriction of $\Phi$ on $\mathrm{X}_\alpha$. A sequence $\{\alpha_n\} \subset \mathbb{N}^2$ is admissible if, for any $\alpha \in \mathbb{N}^2$, there is $m \in \mathbb{N}$ such that $\alpha \leq \alpha_n$ for $n \geq m$, where for $\alpha, \beta \in \mathbb{N}^2$, $\alpha \leq \beta$ means $\alpha^{\pm} \leq \beta^{\pm}$. Obviously, if $\{\alpha_n\}$ is admissible, then any subsequence of $\{\alpha_n\}$ is also admissible.

**Definition.** We say that $\Phi \in C^1(X)$ satisfies condition $(\mathrm{C})^*$ if, whenever $\{\alpha_n\} \subset \mathbb{N}^2$ admissible, any sequence $\{u_n\} \subset X$ such that

$$u_n \in X_{\alpha_n}, \quad \sup_n \Phi(u_n) < \infty, \quad (1 + \|u_n\|)\|\Phi'_{\alpha_n}(u_n)\| \to 0 \tag{2.6}$$

contains a subsequence which converges to a critical point of $\Phi$.

**Theorem 2.3** (Local linking theorem [7])**.** *Suppose that $\Phi \in C^1(X)$ has a local linking at 0, $\Phi$ satisfies (C)$^*$, $\Phi$ maps bounded sets into bounded sets and for every $m \in \mathbb{N}$,*

$$\Phi(u) \to -\infty \quad \textit{as } \|u\| \to \infty,\ u \in X^- \oplus X_m^+. \tag{2.7}$$

*Then $\Phi$ has a nontrivial critical point.*

## 3. Proofs of Theorems 1.1 and 1.2

It is reasonable to write the functional $\Phi$ in a form in which the quadratic part is $\|u\|^2$. Let $g(t) = f(t) + mt$. By (F1)–(F2), it is known that $G(t)$ satisfies the following properties

$$G(t) \leq \frac{t}{4} g(t) + \frac{m}{4} t^2, \tag{3.1}$$

$$\lim_{|t| \to \infty} \frac{G(t)}{t^4} = +\infty, \tag{3.2}$$

and hence there is a $\Lambda > 0$ such that

$$G(t) \geq -\Lambda t^4 \quad \text{for all } t \in \mathbb{R}. \tag{3.3}$$

**Lemma 3.1.** *Suppose that* (V1), (F1)–(F2) *are satisfied, then* $\Phi$ *satisfies the (PS) condition.*

*Proof.* Suppose that $\{u_n\}$ is a (PS) sequence. We claim that $\{u_n\}$ is bounded. By contradiction, we may assume that $\|u_n\| \to \infty$. Using (2.3) and (3.1),we have

$$\begin{aligned} 4\sup_n \Phi(u_n) + \|u_n\| &\geq 4\Phi(u_n) - (\Phi'(u_n), u_n) \\ &= \|u_n\|^2 + \int_{\mathbb{R}} g(u_n)u_n - 4G(u_n)dx \\ &\geq \|u_n\|^2 - m\int_{\mathbb{R}} u_n^2 dx. \end{aligned} \tag{3.4}$$

By (3.4), we obtain

$$\|u_n\| = O(|u_n|_2). \tag{3.5}$$

Let $v_n = \|u_n\|^{-1}u_n$. Up to a subsequence, by the compact embedding $X \hookrightarrow L^2(\mathbb{R})$ we can assume that

$$v_n \rightharpoonup v \text{ in } X, \quad v_n \to v \text{ in } L^2(\mathbb{R}), \quad v_n(x) \to v(x) \text{ a.e in } \mathbb{R}.$$

By (3.5), we have

$$|v_n|_2 \geq \frac{|u_n|_2}{c|u_n|_2} = \frac{1}{c} > 0$$

for some positive constant $c > 0$. Therefor $v \neq 0$. Using (3.3), we obtain

$$\begin{aligned} \int_{\mathbb{R}} \frac{G(u_n)}{\|u_n\|^4}dx &= \int_{v=0} \frac{G(u_n)}{\|u_n\|^4}dx + \int_{v\neq 0} \frac{G(u_n)}{\|u_n\|^4}dx \\ &= \int_{v=0} \frac{G(u_n)}{u_n^4}v_n^4 dx + \int_{v\neq 0} \frac{G(u_n)}{u_n^4}v_n^4 dx \\ &\geq -\Lambda \int_{v=0} v_n^4 \, dx + \int_{v\neq 0} \frac{G(u_n)}{u_n^4}v_n^4 dx \\ &= I_1 + I_2. \end{aligned}$$

Obviously, $I_1 \geq -c > -\infty$. For $x \in \{x \in \mathbb{R} | v \neq 0\}$, we have $|u_n| \to \infty$. By (3.2), we obtain

$$\frac{G(u_n)}{\|u_n\|^4} = \frac{G(u_n)}{u_n^4}v_n^4 \to +\infty.$$

By Fatou's lemma, $I_2 \to +\infty$. Then we have

$$\int_{\mathbb{R}} \frac{G(u_n)}{\|u_n\|^4}dx \to +\infty. \tag{3.6}$$

On the other hand

$$\begin{aligned} \int_{\mathbb{R}} G(u_n)dx &= \frac{1}{2}\|u_n\|^2 + \int_{\mathbb{R}} {u_n'}^2 u_n^2 dx - \Phi(u_n) \\ &\leq \frac{1}{2}\|u_n\|^2 + c\|u_n\|^4 + C. \end{aligned}$$

Then we have

$$\int_{\mathbb{R}} G(u_n)dx = O(\|u_n\|^4), \tag{3.7}$$

a contradiction to (3.6). So $\{u_n\}$ is bounded.

Next, we show that such sequence $\{u_n\}$ has a subsequence converging to a critical point of $\Phi$. Because $\{u_n\}$ is bounded in $X$, we may assume $u_n \rightharpoonup u$ in $X$. Since the imbedding $X \hookrightarrow L^p(\mathbb{R})$ is compact, we have $u_n \to u$ in $L^p(\mathbb{R})$. By a simple computation, we have

$$\Phi'(u_n)(u_n-u) = \|u_n-u\|^2+2\int_{\mathbb{R}} {u'}^2(u_n-u)^2+2\int_{\mathbb{R}} u^2({u'}_n-u')^2-\int_{\mathbb{R}} g(u_n)(u_n-u).$$

By condition (F1) and Holder's inequality, we have

$$\int_{\mathbb{R}} f(u_n)(u_n-u) \le C|u|_p^{p-1}|u_n-u|_p.$$

Since $u_n \to u$ in $L^p(\mathbb{R})$ and $p \ge 1$, we have

$$\int_{\mathbb{R}} f(u_n)(u_n-u)dx \to 0 \quad \text{as } n \to \infty. \tag{3.8}$$

Similarly,

$$\int_{\mathbb{R}} mu_n(u_n-u)dx \to 0 \quad \text{as } n \to \infty. \tag{3.9}$$

By (3.8) and (3.9), we obtain

$$\int_{\mathbb{R}} g(u_n)(u_n-u) \to 0 \quad \text{as } n \to \infty. \tag{3.10}$$

By the assumptions we have

$$\Phi'(u_n)(u_n-u) = o(\|u_n-u\|). \tag{3.11}$$

From (3.10) we obtain

$$\Phi'(u_n)(u_n-u) = \|u_n-u\|^2+2\int_{\mathbb{R}} {u'}^2(u_n-u)^2+2\int_{\mathbb{R}} u^2({u'}_n-u')^2+o(1).$$

Hence we obtain $\|u_n-u\| \to 0$ as $n \to \infty$. This completes the proof. $\square$

**Lemma 3.2.** *Under assumptions* (V1), (F1)–(F2), *the functional* $\Phi$ *has a local linking at 0 with respect to the decomposition* $X = X^+ \oplus X^-$.

*Proof.* By (F1), there exists a $C > 0$ such that

$$|F(u)| \le C|u|^p. \tag{3.12}$$

Using (2.1), (2.2), for $u \in X^-$, there exists a $\delta > 0$. Then we have

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}} {u'}^2+V(x)u^2dx+\int_{\mathbb{R}} {u'}^2u^2dx-\int_{\mathbb{R}} F(u)dx \\
&\le \frac{1}{2}\int_{\mathbb{R}} {u'}^2+V(x)u^2dx+C\|u\|^4+C|u|_p^p \\
&\le -\alpha\|u\|^2+C\|u\|^4+C_1|u|_p^p \\
&\le -\delta\|u\|^2+C\|u\|^4+C_1\|u\|^p.
\end{aligned} \tag{3.13}$$

If $u \in X^+$, then there exists $\xi > 0$ such that

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}} {u'}^2+V(x)u^2dx+\int_{\mathbb{R}} {u'}^2u^2dx-\int_{\mathbb{R}} F(u)dx \\
&\ge \xi\|u\|^2-C_1\|u\|^p.
\end{aligned} \tag{3.14}$$

By (3.13) and (3.14), there exists $0 < \rho < 1$, such that

$$\Phi(u) \le 0 \quad \text{for } u \in X^-, \|u\| \le \rho,$$
$$\Phi(u) \ge 0 \quad \text{for } u \in X^+, \|u\| \le \rho.$$

This completes the proof. □

**Lemma 3.3.** *Let $Y$ be a finite dimensional subspace of $X$. Then $\Phi$ is anti-coercive on $Y$; that is,*

$$\Phi(u) \to -\infty \quad \text{as } \|u\| \to \infty \ u \in Y.$$

*Proof.* A similar lemma was proved in [5], we sketch the proof here for the reader's convenience. If the conclusion were not true, we can choose $\{u_n\} \subset Y$ and $\varsigma \in \mathbb{R}$ such that

$$\|u_n\| \to \infty, \quad \Phi(u_n) \ge \varsigma. \tag{3.15}$$

Let $v_n = \|u_n\|^{-1}u_n$. Since $\dim Y < \infty$, up to a subsequence, we have

$$\|v_n - v\| \to 0, \quad v_n(x) \to v(x) \quad \text{a.e. } \mathbb{R}$$

for some $v \in Y$ with $\|v\| = 1$. If $v(x) \ne 0$, we have $|u_n(x)| \to \infty$. Using (2.2) and (3.6), we deduce

$$\Phi(u_n) \le \|u_n\|^4\Big(\frac{1}{2\|u_n\|^2} + C - \int_{\mathbb{R}} \frac{G(u_n)}{\|u_n\|^4}dx\Big) \to -\infty,$$

a contradiction with (3.15). □

**Lemma 3.4.** *Suppose* , (V1), (F1)-(F2) *are satisfied. Then $\Phi$ satisfies condition (C)$^*$.*

This proof is similar to the Lemma 3.1 and is omitted here. See also [5].

*Proof of Theorem 1.1.* It suffices to verify that

(i) $b_k = \inf_{u\in Z_k, \|u\|=r_k} \Phi(u) \to +\infty$ as $k \to \infty$,
(ii) $\alpha_k = \max_{u\in Y_k, \|u\|=\rho_k} \Phi(u) \le 0$.

(i) We claim that for any $2 \le p$, we have

$$\beta_k := \sup_{u\in Z_k, \|u\|=1} \|u\|_{L^p} \to 0, \quad \text{as } k \to \infty. \tag{3.16}$$

If the conclusion were not true, we may assume that $\beta_k \to \beta > 0$ as $k \to \infty$. Then there exists a $u_k \in Z_k$ with $\|u_k\| = 1$ and $\|u_k\|_p \ge \frac{\beta}{2}$ for large $k$. By the Parseval equality we have

$$\begin{aligned}
\langle u, u_k\rangle &= \big|\langle \sum_{j=k}^{\infty} \alpha_j\phi_j, u_k\rangle\big| \\
&\le \|\sum_{j=k}^{\infty} \alpha_j\phi_j\| \, \|u_k\| \\
&= \Big(\sum_{j=k}^{\infty} \alpha_j^2\Big)^{1/2} \to 0
\end{aligned}$$

where $\langle , \rangle$ denotes the inner product in $X$. Using the Riesz-Frechet representation theorem, we obtain that $u_k \rightharpoonup 0$ and thus $u_k \to 0$ in $L^p$. This is a contradiction. For $u \in Z_k$ with $\|u\| = r_k$, for enough small $\epsilon$,

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\int_{\mathbb{R}} u'^2 + V(x)u^2 dx + \int_{\mathbb{R}} u'^2 u^2 dx - \int_{\mathbb{R}} F(u)dx \\
&\geq k\|u\|^2 - \int_{\mathbb{R}} F(u)dx \\
&\geq k\|u\|^2 - C|u|_p^p \\
&\geq k\|u\|^2 - C\beta_k^p \|u\|^p.
\end{aligned}$$

Choosing $r_k = \beta_k^{-1}$, we have

$$\Phi(u) \geq k\beta_k^{-2} - C \to +\infty.$$

This proves (i).

(ii) Since $\dim Y_k < \infty$, using Lemma 3.3, we have

$$\Phi(u) \to -\infty \quad \text{for } u \in Y_k \text{ and } \rho_k \to \infty.$$

Then we obtain

$$\alpha_k = \max_{u \in Y_K, \|u\| = \rho_k} \Phi(u) \leq 0.$$

This completes the proof. □

*Proof of Theorem 1.2.* In Lemmas 3.2 and 3.4, we see that $\Phi$ satisfies condition $(C)^*$, and has a local linking at 0. Since $\dim(X^- \oplus X_m^+) < \infty$, By Lemma 3.3, we have $\Phi(u) \to -\infty$, as $\|u\| \to \infty$, $u \in X^- \oplus X_m^+$. By Theorem 2.3, equation (1.1) has at least one nontrivial solution. □

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