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ROBIN BOUNDARY VALUE PROBLEMS FOR ELLIPTIC OPERATIONAL DIFFERENTIAL EQUATIONS WITH VARIABLE COEFFICIENTS

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Abstract. In this article we give some new results on abstract second-order differential equations of elliptic type with variable operator coefficients and general Robin boundary conditions, in the framework of Hölder spaces. We assume that the family of variable coefficients verify the well known Labbas-Terreni assumption used in the sum theory. We use Dunford calculus, interpolation spaces and the semigroup theory to obtain existence, uniqueness and maximal regularity results for the solution of the problem.

1. Introduction

In a complex Banach space E, we consider the second-order differential equation

$$u''(x) + A(x)u(x) - \omega u(x) = f(x), \quad x \in]0, 1[, \tag{1.1}$$

together with the general boundary conditions of Robin's type

$$u'(0) - Hu(0) = d_0, \quad u(1) = u_1. \tag{1.2}$$

Here ω is a positive real number, $f \in C^\theta([0, 1]; E)$, $0 < \theta < 1$, d_0 and u_1 are given elements in E, $(A(x))_{x\in[0,1]}$ is a family of closed linear operators whose domains $D(A(x))$ are not necessarily dense in E and H is a closed linear operator with $D(H) \subset E$. Set for $x \in [0, 1]$

$$A_\omega(x) = A(x) - \omega I, \quad \omega > 0.$$

For $f \in C^\theta([0, 1]; E)$, we search for a strict solution u of problem (1.1), (1.2); that is,

$$\begin{gathered} \forall x \in [0, 1] \quad u(x) \in D(A(x)), \quad u \in C^2([0, 1]; E) \\ x \mapsto A(x)u(x) \in C([0, 1]; E), \quad u(0) \in D(H). \end{gathered}$$

Our purpose is to establish existence, uniqueness and maximal regularity of the strict solution of this problem improving the results in [5] and completing the one in [7]. The method is essentially based on Dunford calculus, interpolation spaces, the semigroup theory and some techniques as in [7, 14].

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Throughout this work we suppose that the family $(A(x))_{x\in[0,1]}$ satisfies the following hypotheses:

(H1) There exist $\omega_0 > 0$ and $C > 0$ such that for all $x \in [0,1]$ and all $z \geq 0$, $(A_{\omega_0}(x) - zI)^{-1} \in L(E)$ and
$$\|(A_{\omega_0}(x) - zI)^{-1}\|_{L(E)} \leq \frac{C}{1+z}; \tag{1.3}$$
this estimate holds in some sector
$$\Pi_{\theta_0,r_0} = \{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| \leq \theta_0\} \cup \{z \in \mathbb{C} : |z| \leq r_0\},$$
where θ_0 and r_0 are small positive numbers.

Remark 1.1. It is well known that assumption (1.3) implies the same properties for $A_\omega(x)$, $\omega \geq \omega_0$.

On the other hand, it is well known that the square roots
$$Q_\omega(x) = -(-A_\omega(x))^{1/2}, \quad x \in [0,1], \ \ \omega \geq \omega_0,$$
are well defined and generate analytic semigroups not strongly continuous at zero, (see Balakrishnan [2] for dense domains and Martinez-Sanz [16] for non dense domains).

(H2) There exist C, α, $\mu > 0$ such that for all $x, \tau \in [0,1]$ and rall $\omega \geq \omega_0$:
$$\|Q_\omega(x)(Q_\omega(x) - zI)^{-1}(Q_\omega(x)^{-1} - Q_\omega(\tau)^{-1})\|_{L(E)} \leq \frac{C|x-\tau|^\alpha}{|z+\omega|^\mu} \tag{1.4}$$
with $\alpha + \mu - 2 > 0$. This hypothesis is known as the Labbas-Terreni assumption.

Remark 1.2. From (1.4) one can prove that there exist $C, \alpha, \mu > 0$ such that for all $x, \tau \in [0,1]$ and all $\omega \geq \omega_0$,
$$\|Q_\omega(x)(Q_\omega(x) - zI)^{-1}(Q_\omega(x)^{-2} - Q_\omega(\tau)^{-2})\|_{L(E)} \leq \frac{C|x-\tau|^\alpha}{|z+\omega|^\mu}$$
with $\alpha + \mu - 2 > 0$.

(H3) There exists $C > 0$ such that for all $x \in [0,1]$ and all $\omega \geq \omega_0$: $Q_\omega(x) - H$ is closable, $(\overline{Q_\omega(x) - H})$ is boundedly invertible and
$$\|(\overline{Q_\omega(x) - H})^{-1}\|_{L(E)} \leq C(1+\sqrt{\omega}) \tag{1.5}$$
see Cheggag et al [5] for the autonomous case.

(H4) For all $x \in [0,1]$ and all $\omega \geq \omega_0$,
$$\begin{gathered}(\overline{Q_\omega(x) - H})^{-1}((D(Q_\omega(x)), E)_{1-\theta,\infty}) \subset D(Q_\omega(x)) \cap D(H),\\ Q_\omega(x)(\overline{Q_\omega(x) - H})^{-1}((D(Q_\omega(x)), E)_{1-\theta,\infty}) \subset (D(Q_\omega(x)), E)_{1-\theta,\infty}.\end{gathered} \tag{1.6}$$

Recall that for all $x \in [0,1]$,
$$\begin{aligned} D_{Q_\omega(x)}(\theta, +\infty) &= \{\phi \in E : \sup_{r>0} \|r^\theta Q_\omega(x)(Q_\omega(x) - zI)^{-1}\phi\|_E < +\infty\} \\ &:= (D(Q_\omega(x)), E)_{1-\theta,\infty}. \end{aligned}$$
See Grisvard [9].

(H5) For all $x \in [0,1]$ and all $\omega \geq \omega_0$,
$$Q_\omega(x)^{-1}(\overline{Q_\omega(x) - H})^{-1} = (\overline{Q_\omega(x) - H})^{-1}Q_\omega(x)^{-1}. \tag{1.7}$$

(H6) There exists $C > 0$ such that for all $x, \tau \in [0, 1]$ and all $\omega \geq \omega_0$,
$$\|(\overline{Q_\omega(x) - H})^{-1} - (\overline{Q_\omega(\tau) - H})^{-1}\|_{L(E)} \leq C|x - \tau|^l, \tag{1.8}$$
with some $l \geq 2$.

Note that, from Remark 1.1 there exists a second sector
$$\Pi_{\theta_1+\frac{\pi}{2}, r_1} = \{z \in \mathbb{C}\backslash\{0\} : |\arg(z)| \leq \theta_1 + \frac{\pi}{2}\} \cup \{z \in \mathbb{C} : |z| \leq r_1\},$$
with small $\theta_1 > 0$ and $r_1 > 0$ such that the resolvent set of $-(-A_\omega(x))^{1/2}$ satisfies
$$\rho(-(-A_\omega(x))^{\frac{1}{2}}) \supset \Pi_{\theta_1+\frac{\pi}{2}, r_1}$$
for every $x \in [0, 1]$. Set
$$\Gamma = \{z = \rho e^{\pm i(\theta_1+\frac{\pi}{2})} : \rho \geq r_1\} \cup \{z \in \mathbb{C} : |z| = r_1,\ |\arg(z)| \geq \theta_1 + \frac{\pi}{2}\},$$
oriented from $\infty e^{-i\theta_1}$ to $\infty e^{i\theta_1}$.

Cheggag et al [5] studied the same problem with Robin condition for the autonomous case ($A(x) = A$) and used semigroups theory. Assumption (1.4) are used in many other situations. One can cite, for example, Bouziani et al [4].

There exists an other approach different from (1.4) which uses the differentiability of the resolvent operators, see Boutaous et al [3] for Dirichlet problems.

In this work, there are two difficulties: we do not assume the differentiability of the resolvent operators and one boundary condition contains an unbounded operator. Our essential result is summarized by Theorem 4.4.

This article is organized as follows. In Section 2, some preliminary technical results are proved. In Section 3, we obtain some new results verified by the solution u of (1.1) and (1.2) by using an heuristical reasoning. In Section 4, we give necessary and sufficient compatibility conditions in order to obtain the maximal regularity results of the solution. Section 5 is devoted to prove existence of the solution by the study of the associated approximating problem. Finally, we will present an example of a partial differential equation where our abstract results apply.

2. Technical lemmas

Lemma 2.1. *Assume* (1.3). *Then there exists a constant $C > 0$, such that for all $\omega \geq \omega_0$, $z \in \Gamma$ and $x \in [0, 1]$,*
$$\|(Q_\omega(x) - zI)^{-1}\|_{L(E)} \leq \frac{C}{\sqrt{\omega} + |z|}.$$

For a proof of the above lemma, see Haase [10, p. 57].

Lemma 2.2. *Assume* (1.3). *For any $\omega \geq \omega_0$ and $x \in [0, 1]$, the operator $I - e^{2Q_\omega(x)}$ has a bounded inverse and*
$$(I - e^{2Q_\omega(x)})^{-1} = \frac{1}{2\pi i}\int_\Gamma \frac{e^{2z}}{1 - e^{2z}}(zI - Q_\omega(x))^{-1}dz + I.$$

For a proof of the above lemma, see Lunardi [15, p. 60]. By using Lemma 2.1, there exists $C > 0$ such that for all $\omega \geq \omega_0$:
$$\|(I - e^{2Q_\omega(x)})^{-1}\|_{L(E)} \leq C.$$

Lemma 2.3. *There exists $C > 0$ such that*

(1) *for all $x > 0$ and $\varphi \in D(Q_\omega(0))$, $\|\varphi - e^{xQ_\omega(0)}\varphi\|_E \leq Cx\|Q_\omega(0)\varphi\|_E$*

(2) *for all* $x > 0$ *and* $\varphi \in D(Q_\omega(0)^2)$, $\|\varphi - e^{xQ_\omega(0)}\varphi\|_E \le Cx^2\|Q_\omega(0)^2\varphi\|_E$.

For a proof of the above lemma, see Boutaous [3, p. 8]. For $\omega \ge \omega_0$ and $x \in [0,1]$, we set

$$\Pi_\omega(x) = I + 2(I - e^{2Q_\omega(x)})^{-1}Q_\omega(x)e^{2Q_\omega(x)}(\overline{Q_\omega(x) - H})^{-1}.$$

Proposition 2.4. *Assume* (1.3)*,* (1.5) *and* (1.7)*. Then there exists* $\omega^* \ge \omega_0$ *such that for all* $\omega \ge \omega^*$ *and* $x \in [0,1]$*,* $\Pi_\omega(x)$ *is boundedly invertible.*

Proof. By Dore and Yakubov [6, p. 103], for $\alpha \in \mathbb{R}$ there exist constants C, $k > 0$ (which do not depend on ω) such that for any $y \ge 1$,

$$\|(-A(x) + \omega I)^\alpha e^{-y(-A(x)+\omega I)^{1/2}}\|_{L(E)} \le Ce^{-ky\sqrt{\omega}},$$

thus, for ω large enough,

$$\|Q_\omega(x)e^{2Q_\omega(x)}\|_{L(E)} \le Ce^{-2k\sqrt{\omega}},$$

from which we obtain

$$\|2(I - e^{2Q_\omega(x)})^{-1}Q_\omega(x)e^{2Q_\omega(x)}(\overline{Q_\omega(x) - H})^{-1}\|_{L(E)} \le C(1 + \sqrt{\omega})e^{-2k\sqrt{\omega}}.$$

Then there exists $\omega^* \ge \omega_0$ such that for any $\omega \ge \omega^*$,

$$\|2(I - e^{2Q_\omega(x)})^{-1}Q_\omega(x)e^{2Q_\omega(x)}(\overline{Q_\omega(x) - H})^{-1}\|_{L(E)} < 1.$$

Hence, for $\omega \ge \omega^*$, $\Pi_\omega(x)$ is boundedly invertible. □

Lemma 2.5. *Assume* (1.3) *and* (1.5)–(1.8) *and consider, for* $\omega \ge \omega_0$*, the linear operator*

$$\Lambda_\omega(x) = Q_\omega(x) - H + e^{2Q_\omega(x)}(Q_\omega(x) + H), \quad x \in [0,1],$$

of domain $D(\Lambda_\omega(x)) = D(Q_\omega(x)) \cap D(H)$*. Then there exists* $\omega^* \ge \omega_0$ *such that, for all* $\omega \ge \omega^*$ *and* $x \in [0,1]$*,* $\Lambda_\omega(x)$ *is closable, its closure is invertible with*

$$(\overline{\Lambda_\omega(x)})^{-1} = (\overline{Q_\omega(x) - H})^{-1}(\Pi_\omega(x))^{-1}(I - e^{2Q_\omega(x)})^{-1}, \tag{2.1}$$

$$(\overline{\Lambda_\omega(x)})^{-1} = (\overline{Q_\omega(x) - H})^{-1} + (\overline{Q_\omega(x) - H})^{-1}W(x), \tag{2.2}$$

where

$$W(x) \in L(E), \quad (\overline{Q_\omega(x) - H})^{-1}W(x) = W(x)(\overline{Q_\omega(x) - H})^{-1},$$
$$W(x)(E) \subset \cap_{k=1}^{\infty} D(Q_\omega(x)^k).$$

Proof. Here $\Lambda_\omega(x) = (I - e^{2Q_\omega(x)})(Q_\omega(x) - H) + 2Q_\omega(x)e^{2Q_\omega(x)}$; therefore $\Lambda_\omega(x)$ is closable (see Kato [12]) and

$$\begin{aligned}\overline{\Lambda_\omega(x)} &= (I - e^{2Q_\omega(x)})(\overline{Q_\omega(x) - H}) + 2Q_\omega(x)e^{2Q_\omega(x)}\\ &= (I - e^{2Q_\omega(x)})\Pi_\omega(x)(\overline{Q_\omega(x) - H}).\end{aligned}$$

The above equality together with (1.6) and Proposition 2.4 gives $0 \in \rho(\overline{\Lambda_\omega(x)})$ and (2.1) holds. By (2.1) we have

$$(\overline{\Lambda_\omega(x)})^{-1} = (\overline{Q_\omega(x) - H})^{-1}(I + M(x))^{-1}(I + N(x))^{-1},$$

with

$$M(x) = 2(I - e^{2Q_\omega(x)})^{-1}Q_\omega(x)e^{2Q_\omega(x)}(\overline{Q_\omega(x) - H})^{-1} \in L(E),$$
$$N(x) = -e^{2Q_\omega(x)} \in L(E), \quad [M(x)](E) \subset \cap_{k=1}^{\infty} D(Q_\omega(x)^k),$$
$$[N(x)](E) \subset \cap_{k=1}^{\infty} D(Q_\omega(x)^k).$$

Set $U(x) = -M(x)(I + M(x))^{-1} \in L(E)$ and $V(x) = -N(x)(I + N(x))^{-1} \in L(E)$. Then
$$(\overline{\Lambda_\omega(x)})^{-1} = (\overline{Q_\omega(x) - H})^{-1}(I + U(x))(I + V(x)),$$
with
$$\begin{gathered}
(\overline{Q_\omega(x) - H})^{-1}U(x) = U(x)(\overline{Q_\omega(x) - H})^{-1},\\
(\overline{Q_\omega(x) - H})^{-1}V(x) = V(x)(\overline{Q_\omega(x) - H})^{-1},\\
[U(x)](E), \quad [V(x)](E) \supset \cap_{k=1}^{\infty} D(Q_\omega(x)^k)\,.
\end{gathered}$$
Setting $W(x) = U(x) + V(x) + U(x)V(x)$ we deduce the result. $\square$

Lemma 2.6. *There exists $K > 0$ such that for each $z \in \Gamma$ and $r > 0$, we have*
$$\begin{gathered}
|z + r| \geq K|z|, \quad |z + r| \geq K|r|,\\
|z - r| \geq K|z|, \quad |z - r| \geq K|r|.
\end{gathered}$$
There exists $K > 0$ depending only on Γ such that for all $\lambda > 0$ and all $\nu \in [0, 1]$,
$$\int_\Gamma \frac{|dz|}{|z \pm \lambda||z|^\nu} \leq \frac{K}{\lambda^\nu}.$$

For a proof of the above lemma, see Labbas-Terreni [13, Lemmas 6.1 and 6.2].

3. Representation of the solution

3.1. **Heuristical reasoning.** Let us recall briefly the case when $A(x) \equiv A - \omega I \equiv A_\omega$ is a constant operator satisfying the natural ellipticity hypothesis mentioned above (we will take $Q_\omega = -(-A_\omega)^{1/2}$). In this case, the representation of the solution v of (1.1), (1.2) is given by the formula (see Cheggag [5]).
$$\begin{aligned}
v(x) &= e^{xQ_\omega}[(\overline{\Lambda_\omega})^{-1}d_0 + (Q_\omega + H)(\overline{\Lambda_\omega})^{-1}e^{Q_\omega}u_1]\\
&\quad + \frac{1}{2}e^{xQ_\omega}(Q_\omega + H)(\overline{\Lambda_\omega})^{-1}Q_\omega^{-1}\int_0^1 e^{sQ_\omega}f(s)ds\\
&\quad - \frac{1}{2}e^{xQ_\omega}(Q_\omega + H)(\overline{\Lambda_\omega})^{-1}e^{Q_\omega}Q_\omega^{-1}\int_0^1 e^{(1-s)Q_\omega}f(s)ds\\
&\quad + e^{(1-x)Q_\omega}[(I - (Q_\omega + H)(\overline{\Lambda_\omega})^{-1}e^{2Q_\omega})u_1 - (\overline{\Lambda_\omega})^{-1}e^{Q_\omega}d_0]\\
&\quad - \frac{1}{2}e^{(1-x)Q_\omega}(Q_\omega + H)(\overline{\Lambda_\omega})^{-1}e^{Q_\omega}Q_\omega^{-1}\int_0^1 e^{sQ_\omega}f(s)ds\\
&\quad - \frac{1}{2}e^{(1-x)Q_\omega}[I - (Q_\omega + H)(\overline{\Lambda_\omega})^{-1}e^{2Q_\omega}]Q_\omega^{-1}\int_0^1 e^{(1-s)Q_\omega}f(s)ds\\
&\quad + \frac{1}{2}Q_\omega^{-1}\int_0^x e^{(x-s)Q_\omega}f(s)ds + \frac{1}{2}Q_\omega^{-1}\int_x^1 e^{(s-x)Q_\omega}f(s)ds.
\end{aligned}$$
Set
$$\begin{aligned}
&L_{Q_\omega(x)}(x, f)\\
&= \frac{1}{2}e^{xQ_\omega(x)}(Q_\omega(x) + H)(\overline{\Lambda_\omega(x)})^{-1}Q_\omega(x)^{-1}\int_0^1 e^{sQ_\omega(x)}f(s)ds\\
&\quad - \frac{1}{2}e^{xQ_\omega(x)}(Q_\omega(x) + H)(\overline{\Lambda_\omega(x)})^{-1}e^{Q_\omega(x)}Q_\omega(x)^{-1}\int_0^1 e^{(1-s)Q_\omega(x)}f(s)ds
\end{aligned}$$

$$
\begin{aligned}
&-\frac{1}{2}e^{(1-x)Q_\omega(x)}(Q_\omega(x)+H)(\overline{\Lambda_\omega(x)})^{-1}e^{Q_\omega(x)}Q_\omega(x)^{-1}\int_0^1 e^{sQ_\omega(x)}f(s)ds\\
&-\frac{1}{2}e^{(1-x)Q_\omega(x)}[I-(Q_\omega(x)+H)(\overline{\Lambda_\omega(x)})^{-1}e^{2Q_\omega(x)}]Q_\omega(x)^{-1}\\
&\times\int_0^1 e^{(1-s)Q_\omega(x)}f(s)ds+\frac{1}{2}Q_\omega(x)^{-1}\int_0^x e^{(x-s)Q_\omega(x)}f(s)ds\\
&+\frac{1}{2}Q_\omega(x)^{-1}\int_x^1 e^{(s-x)Q_\omega(x)}f(s)ds.
\end{aligned}
$$

Our heuristical reasoning is the following. Assume that (1.1) and (1.2) has a strict solution u; that is,

$$
\begin{gathered}
\forall x\in[0,1]\ u(x)\in D(A_\omega(x)),\ u\in C^2([0,1];E),\\
x\mapsto A_\omega(x)u(x)\in C([0,1];E)\text{ and }u(0)\in D(H).
\end{gathered}
$$

Then we can write

$$
L_{Q_\omega(x)}(x,f)=L_{Q_\omega(x)}(x,u''(x)-Q_\omega(x)^2u(x)).
$$

After two integrations by parts and some formal calculus, one obtains

$$
\begin{aligned}
&u(x)+\frac{1}{2}e^{xQ_\omega(x)}T_\omega(x)\int_0^1 Q_\omega(x)e^{sQ_\omega(x)}(Q_\omega(s)^{-2}-Q_\omega(x)^{-2})Q_\omega(s)^2u(s)ds\\
&-\frac{1}{2}e^{xQ_\omega(x)}T_\omega(x)e^{Q_\omega(x)}\int_0^1 Q_\omega(x)e^{(1-s)Q_\omega(x)}(Q_\omega(s)^{-2}-Q_\omega(x)^{-2})Q_\omega(s)^2u(s)ds\\
&-\frac{1}{2}e^{(1-x)Q_\omega(x)}T_\omega(x)e^{Q_\omega(x)}\int_0^1 Q_\omega(x)e^{sQ_\omega(x)}(Q_\omega(s)^{-2}-Q_\omega(x)^{-2})Q_\omega(s)^2u(s)ds\\
&-\frac{1}{2}e^{(1-x)Q_\omega(x)}(I-T_\omega(x)e^{2Q_\omega(x)})\int_0^1 Q_\omega(x)e^{(1-s)Q_\omega(x)}(Q_\omega(s)^{-2}-Q_\omega(x)^{-2})\\
&\times Q_\omega(s)^2u(s)ds+\frac{1}{2}\int_0^x Q_\omega(x)e^{(x-s)Q_\omega(x)}(Q_\omega(s)^{-2}-Q_\omega(x)^{-2})Q_\omega(s)^2u(s)ds\\
&+\frac{1}{2}\int_x^1 Q_\omega(x)e^{(s-x)Q_\omega(x)}(Q_\omega(s)^{-2}-Q_\omega(x)^{-2})Q_\omega(s)^2u(s)ds\\
&=L_{Q_\omega(x)}(x,f)+e^{xQ_\omega(x)}[(\overline{\Lambda_\omega(x)})^{-1}d_0+T_\omega(x)e^{xQ_\omega(x)}u_1]\\
&\quad+e^{(1-x)Q_\omega(x)}[(I-T_\omega(x)e^{2Q_\omega(x)})u_1-(\overline{\Lambda_\omega(x)})^{-1}e^{Q_\omega(x)}d_0],
\end{aligned}
$$

where

$$
T_\omega(x)=(Q_\omega(x)+H)(\overline{\Lambda_\omega(x)})^{-1}.
$$

Applying $Q_\omega(x)^2$, we obtain

$$
\begin{aligned}
&Q_\omega(x)^2u(x)\\
&+\frac{1}{2}e^{xQ_\omega(x)}T_\omega(x)\int_0^1 Q_\omega(x)^3e^{sQ_\omega(x)}(Q_\omega(s)^{-2}-Q_\omega(x)^{-2})Q_\omega(s)^2u(s)ds\\
&-\frac{1}{2}e^{xQ_\omega(x)}T_\omega(x)e^{Q_\omega(x)}\int_0^1 Q_\omega(x)^3e^{(1-s)Q_\omega(x)}(Q_\omega(s)^{-2}-Q_\omega(x)^{-2})Q_\omega(s)^2u(s)ds\\
&-\frac{1}{2}e^{(1-x)Q_\omega(x)}T_\omega(x)e^{Q_\omega(x)}\int_0^1 Q_\omega(x)^3e^{sQ_\omega(x)}(Q_\omega(s)^{-2}-Q_\omega(x)^{-2})Q_\omega(s)^2u(s)ds
\end{aligned}
$$

$$
\begin{aligned}
&- \frac{1}{2} e^{(1-x)Q_\omega(x)}(I - T_\omega(x)e^{2Q_\omega(x)}) \int_0^1 Q_\omega(x)^3 e^{(1-s)Q_\omega(x)} \Big(Q_\omega(s)^{-2} \\
&\quad - Q_\omega(x)^{-2} \Big) Q_\omega(s)^2 u(s) ds \\
&+ \frac{1}{2} \int_0^x Q_\omega(x)^3 e^{(x-s)Q_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) Q_\omega(s)^2 u(s) ds \\
&+ \frac{1}{2} \int_x^1 Q_\omega(x)^3 e^{(s-x)Q_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) Q_\omega(s)^2 u(s) ds \\
&= Q_\omega(x)^2 L_{Q_\omega(x)}(x, f) + Q_\omega(x)^2 e^{xQ_\omega(x)} [(\overline{\Lambda_\omega(x)})^{-1} d_0 + T_\omega(x) e^{xQ_\omega(x)} u_1] \\
&\quad + Q_\omega(x)^2 e^{(1-x)Q_\omega(x)} [(I - T_\omega(x)e^{2Q_\omega(x)}) u_1 - (\overline{\Lambda_\omega(x)})^{-1} e^{Q_\omega(x)} d_0].
\end{aligned}
$$

By setting

$$
w(\cdot) = Q_\omega(\cdot)^2 u(\cdot),
$$

we obtain the new equation

$$
w + P_\omega w = G_{Q_\omega(x)}(d_0, u_1, f),
$$

where

$$
\begin{aligned}
&(P_\omega w)(x) \\
&= \frac{1}{2} T_\omega(x) e^{xQ_\omega(x)} \int_0^1 Q_\omega(x)^3 e^{sQ_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s) ds \\
&- \frac{1}{2} e^{xQ_\omega(x)} T_\omega(x) e^{Q_\omega(x)} \int_0^1 Q_\omega(x)^3 e^{(1-s)Q_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s) ds \\
&- \frac{1}{2} e^{(1-x)Q_\omega(x)} T_\omega(x) e^{Q_\omega(x)} \int_0^1 Q_\omega(x)^3 e^{sQ_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s) ds \\
&+ \frac{1}{2} e^{(1-x)Q_\omega(x)} T_\omega(x) e^{2Q_\omega(x)} \int_0^1 Q_\omega(x)^3 e^{(1-s)Q_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s) ds \\
&- \frac{1}{2} e^{(1-x)Q_\omega(x)} \int_0^1 Q_\omega(x)^3 e^{(1-s)Q_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s) ds \\
&+ \frac{1}{2} \int_0^x Q_\omega(x)^3 e^{(x-s)Q_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s) ds \\
&+ \frac{1}{2} \int_x^1 Q_\omega(x)^3 e^{(s-x)Q_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s) ds \\
&= \sum_{i=1}^{7} I_i,
\end{aligned}
$$

and

$$
\begin{aligned}
&G_{Q_\omega(x)}(d_0, u_1, f)(x) \\
&= Q_\omega(x)^2 L_{Q_\omega(x)}(x, f) + Q_\omega(x)^2 e^{xQ_\omega(x)} [(\overline{\Lambda_\omega(x)})^{-1} d_0 + T_\omega(x) e^{xQ_\omega(x)} u_1] \\
&\quad + Q_\omega(x)^2 e^{(1-x)Q_\omega(x)} [(I - T_\omega(x)e^{2Q_\omega(x)}) u_1 - (\overline{\Lambda_\omega(x)})^{-1} e^{Q_\omega(x)} d_0].
\end{aligned}
$$

Proposition 3.1. *Under Hypotheses* (1.3)–(1.8)*, there exists* $\omega^* > 0$*, such that for all* $\omega \geq \omega^*$*,*

$$
\|P_\omega\|_{L(C[0,1];E)} \leq \frac{1}{2}.
$$

Proof. We treat for example

$$\begin{aligned} I_1 &= \frac{1}{2}T_\omega(x)e^{xQ_\omega(x)}\int_0^x Q_\omega(x)^3 e^{sQ_\omega(x)}(Q_\omega(s)^{-2}-Q_\omega(x)^{-2})w(s)ds \\ &\quad + \frac{1}{2}T_\omega(x)e^{xQ_\omega(x)}\int_x^1 Q_\omega(x)^3 e^{sQ_\omega(x)}(Q_\omega(s)^{-2}-Q_\omega(x)^{-2})w(s)ds \\ &= a+b, \end{aligned}$$

and thus

$$\|a\|_E \le K\int_\Gamma |z|^2\Big(\sup_{x\in[0,1]}\int_0^x e^{-c(x+s)|z|}(x-s)^\alpha ds\Big)\frac{|dz|}{|z+\omega|^\mu}\|w\|_{C([0,1];E)}\,.$$

Using Hölder's inequality as in [4], [13], we prove that

$$\sup_{x\in[0,1]}\int_0^x e^{-c(x+s)|z|}(x-s)^\alpha ds \le \frac{K}{|z|^{1+\alpha}},$$

from which we deduce using Lemma 2.6, that

$$\begin{aligned} \|a\|_E &\le K\int_\Gamma \frac{|z|^2|dz|}{|z+\omega|^\mu|z|^{\alpha+1}}\|w\|_{C([0,1];E)} \\ &\le K\int_\Gamma \frac{|dz|}{|z+\omega|^\mu|z|^{\alpha-1}}\|w\|_{C([0,1];E)} \\ &\le \frac{K}{\omega^{\alpha+\mu-2}}\|w\|_{C([0,1];E)}, \end{aligned}$$

We obtain also a similar estimate for $\|b\|_E$. Therefore there exists $\omega^*>0$ such that for all $\omega\ge\omega^*$, $\|P_\omega\|_{L(C[0,1];E)}\le\frac{1}{2}$ which leads us to invert $I+P_\omega$ for $\omega\ge\omega^*$ in the space $C([0,1];E)$. ☐

3.2. **Regularity of the second member** $G_{Q_\omega(x)}(d_0,u_1,f)$**.** In this section we use the following lemmas (see [1, Lemma 1.8] and [8, Theorem 6 (4)]).

Lemma 3.2. *Fix $x\in[0,1]$ and $\beta\in]0,1[$. Then*

1. $s\to e^{sQ_\omega(x)}\varphi\in C([0,1];E)$ *if and only if* $\varphi\in\overline{D(Q_\omega(x))}$.
2. $s\to e^{sQ_\omega(x)}\varphi\in C^\beta([0,1];E)$ *if and only if* $\varphi\in D_{Q_\omega(x)}(\beta,+\infty)$.

Lemma 3.3. *Let $f\in C^\beta([0,1];E)$. Then*

$$Q_\omega(0)\int_0^1 e^{sQ_\omega(0)}(f(s)-f(0))ds\in(D(Q_\omega(0));E)_{1-\theta,\infty}.$$

The regularity results of $G_{Q_\omega(x)}(d_0,u_1,f)$ are as follows:

Proposition 3.4. *Assume (1.3)–(1.8) and let $f\in C^\theta([0,1];E)$ with $\theta\in[0,1]$. Then*

(1) $G_{Q_\omega(x)}(d_0,u_1,f)\in C([0,1];E)$ if and only if

$$\begin{gathered} u_1\in D(Q_\omega(1)^2),\quad (\overline{Q_\omega(0)-H})^{-1}d_0\in D(Q_\omega(0))\cap D(H), \\ Q_\omega(0)(\overline{Q_\omega(0)-H})^{-1}(d_0-Q_\omega(0)^{-1}f(0))\in D(Q_\omega(0)), \\ Q_\omega(0)^2(\overline{Q_\omega(0)-H})^{-1}(d_0-Q_\omega(0)^{-1}f(0))+f(0)\in\overline{D(Q_\omega(0))}, \\ Q_\omega(1)^2u_1+f(1)\in\overline{D(Q_\omega(1))}. \end{gathered} \tag{3.1}$$

(2) $G_{Q_\omega(x)}(d_0,u_1,f)\in C^\theta([0,1];E)$ *if and only if*

$$\begin{gathered}
u_1\in D(Q_\omega(1)^2),\quad (\overline{Q_\omega(0)-H})^{-1}d_0\in D(Q_\omega(0))\cap D(H),\\
Q_\omega(0)(\overline{Q_\omega(0)-H})^{-1}(d_0-Q_\omega(0)^{-1}f(0))\in D(Q_\omega(0)),\\
Q_\omega(0)^2(\overline{Q_\omega(0)-H})^{-1}(d_0-Q_\omega(0)^{-1}f(0))+f(0)\in (D(Q_\omega(0));E)_{1-\theta,\infty},\\
Q_\omega(1)^2u_1+f(1)\in (D(Q_\omega(1));E)_{1-\theta,\infty}.
\end{gathered}\tag{3.2}$$

Proof. Let $x\in[0,1]$. Then we have

$$\begin{aligned}
&G_{Q_\omega(x)}(d_0,u_1,f)(x)\\
&=Q_\omega(x)^2e^{xQ_\omega(x)}[(\overline{\Lambda_\omega(x)})^{-1}d_0+T_\omega(x)e^{Q_\omega(x)}u_1]\\
&\quad+Q_\omega(x)^2e^{(1-x)Q_\omega(x)}[(I-T_\omega(x)e^{2Q_\omega(x)})u_1-(\overline{\Lambda_\omega(x)})^{-1}e^{Q_\omega(x)}d_0]\\
&\quad+\frac12 e^{xQ_\omega(x)}T_\omega(x)\int_0^1 Q_\omega(x)e^{sQ_\omega(x)}f(s)ds\\
&\quad-\frac12 e^{xQ_\omega(x)}T_\omega(x)\int_0^1 Q_\omega(x)e^{(2-s)Q_\omega(x)}f(s)ds\\
&\quad-\frac12 e^{(1-x)Q_\omega(x)}T_\omega(x)\int_0^1 Q_\omega(x)e^{(1+s)Q_\omega(x)}f(s)ds\\
&\quad+\frac12 e^{(1-x)Q_\omega(x)}T_\omega(x)\int_0^1 Q_\omega(x)e^{(3-s)Q_\omega(x)}f(s)ds\\
&\quad-\frac12 e^{(1-x)Q_\omega(x)}\int_0^1 Q_\omega(x)e^{(1-s)Q_\omega(x)}f(s)ds\\
&\quad+\frac12\int_0^x Q_\omega(x)e^{(x-s)Q_\omega(x)}f(s)ds+\frac12\int_x^1 Q_\omega(x)e^{(s-x)Q_\omega(x)}f(s)ds\\
&=Q_\omega(x)^2e^{xQ_\omega(x)}[(\overline{\Lambda_\omega(x)})^{-1}d_0+T_\omega(x)e^{Q_\omega(x)}u_1]\\
&\quad+Q_\omega(x)^2e^{(1-x)Q_\omega(x)}[(I-T_\omega(x)e^{2Q_\omega(x)})u_1-(\overline{\Lambda_\omega(x)})^{-1}e^{Q_\omega(x)}d_0]\\
&\quad+\frac12 e^{xQ_\omega(x)}T_\omega(x)\int_0^1 Q_\omega(x)e^{sQ_\omega(x)}(f(s)-f(0))ds\\
&\quad-\frac12 e^{xQ_\omega(x)}T_\omega(x)\int_0^1 Q_\omega(x)e^{(2-s)Q_\omega(x)}f(s)ds\\
&\quad-\frac12 e^{(1-x)Q_\omega(x)}T_\omega(x)\int_0^1 Q_\omega(x)e^{(1+s)Q_\omega(x)}f(s)ds\\
&\quad+\frac12 e^{(1-x)Q_\omega(x)}T_\omega(x)\int_0^1 Q_\omega(x)e^{(3-s)Q_\omega(x)}f(s)ds\\
&\quad-\frac12 e^{(1-x)Q_\omega(x)}\int_0^1 Q_\omega(x)e^{(1-s)Q_\omega(x)}(f(s)-f(1))ds\\
&\quad+\frac12\int_0^x Q_\omega(x)e^{(x-s)Q_\omega(x)}(f(s)-f(0))ds\\
&\quad+\frac12\int_x^1 Q_\omega(x)e^{(s-x)Q_\omega(x)}(f(s)-f(1))ds\\
&\quad+\frac12 e^{xQ_\omega(x)}T_\omega(x)(e^{Q_\omega(x)}f(0)-f(0))+\frac12(e^{xQ_\omega(x)}f(0)-f(0))
\end{aligned}$$

$$+\frac{1}{2}e^{(1-x)Q_\omega(x)}(f(1)-e^{Q_\omega(x)}f(1))+\frac{1}{2}(e^{Q_\omega(x)}f(1)-f(1)).$$

Let us write

$$\begin{aligned}
f(0)-T_\omega(x)f(0)&=[I-T_\omega(x)]f(0)\\
&=[I-(Q_\omega(x)+H)(\overline{\Lambda_\omega(x)})^{-1}]f(0)\\
&=[(\overline{\Lambda_\omega(x)})-(Q_\omega(x)+H)](\overline{\Lambda_\omega(x)})^{-1}f(0)\\
&=-2H(\overline{\Lambda_\omega(x)})^{-1}f(0)+e^{2Q_\omega(x)}T_\omega(x)f(0).
\end{aligned}$$

Then we obtain

$$\begin{aligned}
&G_{Q_\omega(x)}(d_0,u_1,f)(x)\\
&=-e^{xQ_\omega(x)}H(\overline{\Lambda_\omega(x)})^{-1}f(0)+Q_\omega(x)^2e^{xQ_\omega(x)}(\overline{\Lambda_\omega(x)})^{-1}d_0\\
&\quad+e^{(1-x)Q_\omega(x)}f(1)+Q_\omega(x)^2e^{(1-x)Q_\omega(x)}u_1\\
&\quad+\frac{1}{2}e^{xQ_\omega(x)}T_\omega(x)\int_0^1 Q_\omega(x)e^{sQ_\omega(x)}(f(s)-f(0))ds\\
&\quad-\frac{1}{2}e^{(1-x)Q_\omega(x)}\int_0^1 Q_\omega(x)e^{(1-s)Q_\omega(x)}(f(s)-f(1))ds\\
&\quad+\frac{1}{2}\int_0^x Q_\omega(x)e^{(x-s)Q_\omega(x)}(f(s)-f(0))ds-\frac{1}{2}f(0)\\
&\quad+\frac{1}{2}\int_x^1 Q_\omega(x)e^{(s-x)Q_\omega(x)}(f(s)-f(1))ds-\frac{1}{2}f(1)\\
&\quad+Q_\omega(x)^2e^{xQ_\omega(x)}e^{Q_\omega(x)}T_\omega(x)\\
&\quad\times\Big[u_1-\frac{1}{2}Q_\omega(x)^{-1}\int_0^1 e^{(1-s)Q_\omega(x)}f(s)ds+\frac{1}{2}(I+e^{Q_\omega(x)})Q_\omega(x)^{-2}f(0)\Big]\\
&\quad-Q_\omega(x)^2e^{(1-x)Q_\omega(x)}e^{2Q_\omega(x)}T_\omega(x)[u_1-\frac{1}{2}Q_\omega(x)^{-1}\int_0^1 e^{(1-s)Q_\omega(x)}f(s)ds]\\
&\quad-Q_\omega(x)^2e^{(1-x)Q_\omega(x)}e^{Q_\omega(x)}\\
&\quad\times\Big[(\overline{\Lambda_\omega(x)})^{-1}d_0+\frac{1}{2}T_\omega(x)Q_\omega(x)^{-1}\int_0^1 e^{sQ_\omega(x)}f(s)ds+\frac{1}{2}Q_\omega(x)^{-2}f(1)\Big]\\
&=[-e^{xQ_\omega(x)}H(\overline{\Lambda_\omega(x)})^{-1}f(0)+Q_\omega(x)^2e^{xQ_\omega(x)}(\overline{\Lambda_\omega(x)})^{-1}d_0]\\
&\quad+[e^{(1-x)Q_\omega(x)}f(1)+Q_\omega(x)^2e^{(1-x)Q_\omega(x)}u_1]\\
&\quad+\frac{1}{2}e^{xQ_\omega(x)}T_\omega(x)\int_0^1 Q_\omega(x)e^{sQ_\omega(x)}(f(s)-f(0))ds\\
&\quad-\frac{1}{2}e^{(1-x)Q_\omega(x)}\int_0^1 Q_\omega(x)e^{(1-s)Q_\omega(x)}(f(s)-f(1))ds\\
&\quad+\frac{1}{2}\int_0^x Q_\omega(x)e^{(x-s)Q_\omega(x)}(f(s)-f(0))ds-\frac{1}{2}f(0)\\
&\quad+\frac{1}{2}\int_x^1 Q_\omega(x)e^{(s-x)Q_\omega(x)}(f(s)-f(1))ds-\frac{1}{2}f(1)\\
&\quad+A_\omega(x)R(x,f,d_0,u_1)
\end{aligned}$$

$$\begin{aligned}&= \sum_{k=1}^{4} I_k + [-e^{xQ_\omega(x)} H(\overline{\Lambda_\omega(x)})^{-1} f(0) + Q_\omega(x)^2 e^{xQ_\omega(x)} (\overline{\Lambda_\omega(x)})^{-1} d_0] \\ &\quad + [e^{(1-x)Q_\omega(x)} f(1) + Q_\omega(x)^2 e^{(1-x)Q_\omega(x)} u_1] + A_\omega(x) R(x, f, d_0, u_1),\end{aligned}$$

where

$$\begin{aligned}&R(x, f, d_0, u_1) \\ &= -e^{xQ_\omega(x)} e^{Q_\omega(x)} T_\omega(x) \\ &\quad \times \Big[u_1 - \frac{1}{2} Q_\omega(x)^{-1} \int_0^1 e^{(1-s)Q_\omega(x)} f(s) ds + \frac{1}{2}(I + e^{Q_\omega(x)}) Q_\omega(x)^{-2} f(0)\Big] \\ &+ e^{(1-x)Q_\omega(x)} e^{2Q_\omega(x)} T_\omega(x) [u_1 - \frac{1}{2} Q_\omega(x)^{-1} \int_0^1 e^{(1-s)Q_\omega(x)} f(s) ds] \\ &\quad + e^{(1-x)Q_\omega(x)} e^{Q_\omega(x)} \Big[(\overline{\Lambda_\omega(x)})^{-1} d_0 \\ &\quad + \frac{1}{2} T_\omega(x) Q_\omega(x)^{-1} \int_0^1 e^{sQ_\omega(x)} f(s) ds + \frac{1}{2} Q_\omega(x)^{-2} f(1)\Big].\end{aligned}$$

Each term in $R(\cdot, f, d_0, u_1)$ contains $e^{Q_\omega(x)}$ and $e^{Q_\omega(x)} \in L(E, D(Q_\omega(x)^m))$ for any $m \in \mathbb{N}$, so

$$\begin{gathered}R(\cdot, f, d_0, u_1) \in C^\infty([0,1]; E), \\ A_\omega(\cdot) R(\cdot, f, d_0, u_1) \in C^\infty([0,1]; E).\end{gathered}$$

For I_1, let us write

$$\begin{aligned}I_1 &= \frac{1}{2}[e^{Q_\omega(x)} T_\omega(x) Q_\omega(x) - e^{Q_\omega(0)} T_\omega(x) Q_\omega(0)] \int_0^1 e^{sQ_\omega(0)} (f(s) - f(0)) ds \\ &\quad + e^{Q_\omega(0)} T_\omega(x) Q_\omega(0) \int_0^1 e^{sQ_\omega(0)} (f(s) - f(0)) ds\,.\end{aligned}$$

From Lemma 3.3, we obtain

$$Q_\omega(0) \int_0^1 e^{sQ_\omega(0)} (f(s) - f(0)) ds \in (D(Q_\omega(0)), E)_{1-\theta,\infty},$$

from which we deduce that

$$e^{xQ_\omega(0)} Q_\omega(0) \int_0^1 e^{sQ_\omega(0)} (f(s) - f(0)) ds \in C^\theta([0,1]; E)\,.$$

For the other terms, we use the same technics as in [1].

Now, from Lemma 2.5, we have

$$\begin{aligned}&Q_\omega(x)^2 e^{xQ_\omega(x)} (\overline{\Lambda_\omega(x)})^{-1} d_0 - e^{xQ_\omega(x)} H(\overline{\Lambda_\omega(x)})^{-1} f(0) \\ &= Q_\omega(x)^2 e^{xQ_\omega(x)} (\overline{Q_\omega(x) - H})^{-1} d_0 - e^{xQ_\omega(x)} H(\overline{Q_\omega(x) - H})^{-1} f(0) \\ &\quad + Q_\omega(x)^2 e^{xQ_\omega(x)} (\overline{Q_\omega(x) - H})^{-1} W(x) d_0 - e^{xQ_\omega(x)} H(\overline{Q_\omega(x) - H})^{-1} W(x) f(0) \\ &= Q_\omega(x)^2 e^{xQ_\omega(x)} (\overline{Q_\omega(x) - H})^{-1} d_0 + e^{xQ_\omega(x)} (Q_\omega(x) - H)(\overline{Q_\omega(x) - H})^{-1} f(0) \\ &\quad - e^{xQ_\omega(x)} Q_\omega(x) (\overline{Q_\omega(x) - H})^{-1} f(0) \\ &\quad + Q_\omega(x)^2 e^{xQ_\omega(x)} (\overline{Q_\omega(x) - H})^{-1} W(x) d_0 - e^{xQ_\omega(x)} H(\overline{Q_\omega(x) - H})^{-1} W(x) f(0) \\ &= Q_\omega(x)^2 e^{xQ_\omega(x)} (\overline{Q_\omega(x) - H})^{-1} d_0 + e^{xQ_\omega(x)} f(0)\end{aligned}$$

$$
\begin{aligned}
&- e^{xQ_\omega(x)}Q_\omega(x)(\overline{Q_\omega(x)-H})^{-1}f(0)\\
&+ Q_\omega(x)^2e^{xQ_\omega(x)}(\overline{Q_\omega(x)-H})^{-1}W(x)d_0 - e^{xQ_\omega(x)}H(\overline{Q_\omega(x)-H})^{-1}W(x)f(0),
\end{aligned}
$$

where $W(x)\in L(E)$ and $R(W(x))\subset \cap_{k=1}^{\infty}D(Q_\omega(x)^k)$. So

$$
\begin{aligned}
&Q_\omega(x)^2e^{xQ_\omega(x)}(\overline{Q_\omega(x)-H})^{-1}d_0 + e^{xQ_\omega(x)}f(0) - e^{xQ_\omega(x)}Q_\omega(x)(\overline{Q_\omega(x)-H})^{-1}f(0)\\
&= [Q_\omega(x)^2e^{xQ_\omega(x)}(\overline{Q_\omega(0)-H})^{-1}d_0 - Q_\omega(0)^2e^{xQ_\omega(0)}(\overline{Q_\omega(0)-H})^{-1}d_0]\\
&\quad + [Q_\omega(x)^2e^{xQ_\omega(x)}(\overline{Q_\omega(x)-H})^{-1}d_0 - Q_\omega(x)^2e^{xQ_\omega(x)}(\overline{Q_\omega(0)-H})^{-1}d_0]\\
&\quad + [-e^{xQ_\omega(x)}Q_\omega(0)(\overline{Q_\omega(0)-H})^{-1}f(0) + e^{xQ_\omega(0)}Q_\omega(0)(\overline{Q_\omega(0)-H})^{-1}f(0)]\\
&\quad + [-e^{xQ_\omega(x)}Q_\omega(x)(\overline{Q_\omega(x)-H})^{-1}f(0) + e^{xQ_\omega(x)}Q_\omega(0)(\overline{Q_\omega(0)-H})^{-1}f(0)]\\
&\quad + [e^{xQ_\omega(x)}f(0) - e^{xQ_\omega(0)}f(0)]\\
&\quad + e^{xQ_\omega(0)}[Q_\omega(0)^2(\overline{Q_\omega(0)-H})^{-1}d_0 - Q_\omega(0)(\overline{Q_\omega(0)-H})^{-1}f(0) + f(0)].
\end{aligned}
$$

Using the algebraic identity

$$
\begin{aligned}
&Q_\omega(x)(Q_\omega(x)-zI)^{-1} - Q_\omega(0)(Q_\omega(0)-zI)^{-1}\\
&= zQ_\omega(x)(Q_\omega(x)-zI)^{-1}[Q_\omega(x)^{-1} - Q_\omega(0)^{-1}]Q_\omega(0)(Q_\omega(0)-zI)^{-1},
\end{aligned}
$$

I added the equal sign before zQ is it correct?

Then we obtain

$$
\begin{aligned}
&Q_\omega(x)^2e^{xQ_\omega(x)}(\overline{Q_\omega(x)-H})^{-1}d_0 + e^{xQ_\omega(x)}f(0)\\
&- e^{xQ_\omega(x)}Q_\omega(x)(\overline{Q_\omega(x)-H})^{-1}f(0)\\
&= -\frac{1}{2\pi i}\int_\Gamma z^2e^{xz}Q_\omega(x)(Q_\omega(x)-zI)^{-1}[Q_\omega(x)^{-1} - Q_\omega(0)^{-1}]\\
&\quad \times Q_\omega(0)(Q_\omega(0)-zI)^{-1}(\overline{Q_\omega(0)-H})^{-1}d_0\,dz\\
&\quad + \frac{1}{2\pi i}\int_\Gamma ze^{xz}Q_\omega(x)(Q_\omega(x)-zI)^{-1}[Q_\omega(x)^{-1} - Q_\omega(0)^{-1}]\\
&\quad \times Q_\omega(0)(Q_\omega(0)-zI)^{-1}(\overline{Q_\omega(0)-H})^{-1}f(0)dz\\
&\quad - \frac{1}{2\pi i}\int_\Gamma z^2e^{xz}Q_\omega(x)(Q_\omega(x)-zI)^{-1}[Q_\omega(x)^{-1} - Q_\omega(0)^{-1}]\\
&\quad \times Q_\omega(0)(Q_\omega(0)-zI)^{-1}[(\overline{Q_\omega(x)-H})^{-1} - (\overline{Q_\omega(0)-H})^{-1}]d_0dz\\
&\quad - \frac{1}{2\pi i}\int_\Gamma ze^{xz}Q_\omega(0)(Q_\omega(0)-zI)^{-1}[(\overline{Q_\omega(x)-H})^{-1} - (\overline{Q_\omega(0)-H})^{-1}]d_0dz\\
&\quad + \frac{1}{2\pi i}\int_\Gamma e^{xz}Q_\omega(x)(Q_\omega(x)-zI)^{-1}[(\overline{Q_\omega(x)-H})^{-1} - (\overline{Q_\omega(0)-H})^{-1}]f(0)dz\\
&\quad - \frac{1}{2\pi i}\int_\Gamma e^{xz}Q_\omega(x)(Q_\omega(x)-zI)^{-1}[Q_\omega(x)^{-1} - Q_\omega(0)^{-1}]\\
&\quad \times Q_\omega(0)(Q_\omega(0)-zI)^{-1}f(0)dz\\
&\quad + e^{xQ_\omega(0)}[Q_\omega(0)^2(\overline{Q_\omega(0)-H})^{-1}d_0 - Q_\omega(0)(\overline{Q_\omega(0)-H})^{-1}f(0) + f(0)]\\
&= \sum_{i=1}^{7} a_i\,.
\end{aligned}
$$

For a_1 we have

$$
\begin{aligned}
\|a_1\|_E &\leq K\int_\Gamma |z|e^{-cx|z|}\frac{x^\alpha}{|z|^\mu}|dz|\|Q_\omega(0)(\overline{Q_\omega(0)-H})^{-1}d_0\|_E\\
&\leq K\int_\Gamma e^{-\sigma}\frac{x^\alpha}{(\frac{\sigma}{x})^{\mu-1}}\frac{d\sigma}{x}\|Q_\omega(0)(\overline{Q_\omega(0)-H})^{-1}d_0\|_E\\
&\leq Kx^{\alpha+\mu-2}\|Q_\omega(0)(\overline{Q_\omega(0)-H})^{-1}d_0\|_E\\
&\leq K\|Q_\omega(0)(\overline{Q_\omega(0)-H})^{-1}d_0\|_E.
\end{aligned}
$$

The same technics are used for the other terms a_i, $i=1,2,\dots,6$. For the last term of G. One has

$$
\begin{aligned}
&[e^{(1-x)Q_\omega(x)}f(1)+Q_\omega(x)^2e^{(1-x)Q_\omega(x)}u_1]\\
&=[e^{(1-x)Q_\omega(x)}f(1)-e^{(1-x)Q_\omega(1)}f(1)]+[Q_\omega(x)^2e^{(1-x)Q_\omega(x)}u_1\\
&\quad -Q_\omega(1)^2e^{(1-x)Q_\omega(1)}u_1]+e^{(1-x)Q_\omega(1)}[f(1)+Q_\omega(1)^2u_1]\\
&=-\frac{1}{2\pi i}\int_\Gamma e^{(1-x)z}((Q_\omega(x)-zI)^{-1}-(Q_\omega(1)-zI)^{-1})f(1)dz\\
&\quad -\frac{1}{2\pi i}\int_\Gamma ze^{(1-x)z}(Q_\omega(x)(Q_\omega(x)-zI)^{-1}-Q_\omega(1)(Q_\omega(1)-zI)^{-1})u_1dz\\
&\quad +e^{(1-x)Q_\omega(1)}[f(1)+Q_\omega(1)^2u_1]\\
&=-\frac{1}{2\pi i}\int_\Gamma e^{(1-x)z}Q_\omega(x)(Q_\omega(x)-zI)^{-1}(Q_\omega(x)^{-1}-Q_\omega(1)^{-1})Q_\omega(1)\\
&\quad\times(Q_\omega(1)-zI)^{-1}f(1)dz-\frac{1}{2\pi i}\int_\Gamma z^2e^{(1-x)z}Q_\omega(x)(Q_\omega(x)-zI)^{-1}\\
&\quad\times(Q_\omega(x)^{-1}-Q_\omega(1)^{-1})Q_\omega(1)(Q_\omega(1)-zI)^{-1}u_1dz+e^{(1-x)Q_\omega(1)}\\
&\quad\times[f(1)+Q_\omega(1)^2u_1]\\
&=b_1+b_2+b_3.
\end{aligned}
$$

For the term b_1, we have

$$
\begin{aligned}
\|b_1\|_E &\leq K\int_\Gamma e^{-c(1-x)|z|}\frac{(1-x)^\alpha}{|z|^\mu}|dz|\|f(1)\|_E\\
&\leq K\int_\Gamma e^{-\sigma}\frac{(1-x)^\alpha}{(\frac{\sigma}{1-x})^\mu}\frac{(1-x)d\sigma}{(1-x)^2}\|f(1)\|_E\\
&\leq K(1-x)^{\alpha+\mu-2}\|f(1)\|_E\\
&\leq K\|f(1)\|_E.
\end{aligned}
$$

The same technic is used for the other terms. From a_7 and b_3 using Lemmas 3.2 and 3.3 we deduce (3.1) and (3.2). □

We can write for $\omega\geq\omega^*$ and $x\in[0,1]$,

$$
u(x)=Q_\omega(x)^{-2}(I+P_\omega)^{-1}G_{Q_\omega(x)}(d_0,u_1,f). \tag{3.3}
$$

4. Regularity of the solution

Throughout this section we assume that $\omega\geq\omega^*$.

4.1. **Regularity of P_ω.**

Proposition 4.1. *Under assumptions* (1.3)∼(1.8), *we have*

(1) $P_\omega \in L(C(E), C^{\alpha+\mu-2}(E))$
(2) $P_\omega \in L(C(E), C(E) \cap B(D_{A(\cdot)}(\frac{\beta}{2}, +\infty)))$, *where* $\beta \in]0, \alpha+\mu-2]$.

Proof. Let us prove the first statement. Let $0 \le \tau \le x \le 1$ and $w \in C([0,1]; E)$, we have

$$
\begin{aligned}
&(P_\omega w)(x) - (P_\omega w)(\tau)\\
&= \frac{1}{2} T_\omega(x) e^{xQ_\omega(x)} \int_0^1 Q_\omega(x)^3 e^{sQ_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s) ds\\
&\quad - \frac{1}{2} T_\omega(\tau) e^{\tau Q_\omega(\tau)} \int_0^1 Q_\omega(\tau)^3 e^{sQ_\omega(\tau)} (Q_\omega(s)^{-2} - Q_\omega(\tau)^{-2}) w(s) ds\\
&\quad - \frac{1}{2} e^{(1-x)Q_\omega(x)} \int_0^1 Q_\omega(x)^3 e^{(1-s)Q_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s) ds\\
&\quad + \frac{1}{2} e^{(1-\tau)Q_\omega(\tau)} \int_0^1 Q_\omega(\tau)^3 e^{(1-s)Q_\omega(\tau)} (Q_\omega(s)^{-2} - Q_\omega(\tau)^{-2}) w(s) ds\\
&\quad + \frac{1}{2} \int_0^x Q_\omega(x)^3 e^{(x-s)Q_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s) ds\\
&\quad - \frac{1}{2} \int_0^\tau Q_\omega(\tau)^3 e^{(\tau-s)Q_\omega(\tau)} (Q_\omega(s)^{-2} - Q_\omega(\tau)^{-2}) w(s) ds\\
&\quad + \frac{1}{2} \int_x^1 Q_\omega(x)^3 e^{(s-x)Q_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s) ds\\
&\quad - \frac{1}{2} \int_\tau^1 Q_\omega(\tau)^3 e^{(s-\tau)Q_\omega(\tau)} (Q_\omega(s)^{-2} - Q_\omega(\tau)^{-2}) w(s) ds\\
&\quad - \frac{1}{2} e^{xQ_\omega(x)} T_\omega(x) e^{Q_\omega(x)} \int_0^1 Q_\omega(x)^3 e^{(1-s)Q_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s) ds\\
&\quad + \frac{1}{2} e^{\tau Q_\omega(\tau)} T_\omega(\tau) e^{Q_\omega(\tau)} \int_0^1 Q_\omega(\tau)^3 e^{(1-s)Q_\omega(\tau)} (Q_\omega(s)^{-2} - Q_\omega(\tau)^{-2}) w(s) ds\\
&\quad - \frac{1}{2} e^{(1-x)Q_\omega(x)} T_\omega(x) e^{Q_\omega(x)} \int_0^1 Q_\omega(x)^3 e^{sQ_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s) ds\\
&\quad + \frac{1}{2} e^{(1-\tau)Q_\omega(\tau)} T_\omega(\tau) e^{Q_\omega(\tau)} \int_0^1 Q_\omega(\tau)^3 e^{sQ_\omega(\tau)} (Q_\omega(s)^{-2} - Q_\omega(\tau)^{-2}) w(s) ds\\
&+ \frac{1}{2} e^{(1-x)Q_\omega(x)} T_\omega(x) e^{2Q_\omega(x)} \int_0^1 Q_\omega(x)^3 e^{(1-s)Q_\omega(x)} (Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s) ds\\
&\quad - \frac{1}{2} e^{(1-\tau)Q_\omega(\tau)} T_\omega(\tau) e^{2Q_\omega(\tau)} \int_0^1 Q_\omega(\tau)^3 e^{(1-s)Q_\omega(\tau)} (Q_\omega(s)^{-2} - Q_\omega(\tau)^{-2}) w(s) ds\\
&= \sum_{i=1}^{14} I_i.
\end{aligned}
$$

One has $I_9 + I_{10}$, $I_{11} + I_{12}$ and $I_{13} + I_{14}$ are $o(|x-\tau|^\beta)$ where $0 < \beta \le \alpha + \mu - 2$.

For the other terms, we use the same technics, let us treat for example $I_5 + I_6$ we can write that

$$
\begin{aligned}
& I_5 + I_6 \\
& = -\frac{1}{4\pi i}\int_0^\tau \int_\Gamma (e^{(x-s)z} - e^{(\tau-s)z}) z^2 Q_\omega(\tau)(Q_\omega(\tau) - zI)^{-1} \\
& \quad \times (Q_\omega(s)^{-2} - Q_\omega(\tau)^{-2}) w(s)\, dz\, ds \\
& - \frac{1}{4\pi i}\int_0^\tau \int_\Gamma z^2 e^{(x-s)z}(Q_\omega(x)(Q_\omega(x) - zI)^{-1} - Q_\omega(\tau)(Q_\omega(\tau) - zI)^{-1}) \\
& \quad \times (Q_\omega(s)^{-2} - Q_\omega(\tau)^{-2}) w(s)\, dz\, ds \\
& - \frac{1}{4\pi i}\int_0^\tau \int_\Gamma z^2 e^{(x-s)z} Q_\omega(x)(Q_\omega(x) - zI)^{-1}(Q_\omega(\tau)^{-2} - Q_\omega(x)^{-2}) w(s)\, dz\, ds \\
& \quad - \frac{1}{4\pi i}\int_\tau^x \int_\Gamma z^2 e^{(x-s)z} Q_\omega(x)(Q_\omega(x) - zI)^{-1}(Q_\omega(s)^{-2} - Q_\omega(x)^{-2}) w(s)\, dz\, ds \\
& \sum_{i=1}^4 J_i,
\end{aligned}
$$

For J_1, we have

$$
\begin{aligned}
\|J_1\|_E &\le K \int_\Gamma \int_0^\tau \int_\tau^x e^{-c(\xi-s)|z|}|z|^3 \frac{(\tau-s)^\alpha}{|z|^\mu} |dz| d\xi ds \|w\|_{C([0,1];E)} \\
&\le K \int_\Gamma \int_0^\tau \int_\tau^x e^{-\sigma} \frac{(\tau-s)^\alpha}{(\frac{\sigma}{\xi-s})^{\mu-3}} \frac{d\sigma}{(\xi-s)} d\xi ds \|w\|_{C([0,1];E)} \\
&\le K \int_0^\tau \int_\tau^x (\tau-s)^\alpha (\xi-s)^{\mu-4} d\xi ds \|w\|_{C([0,1];E)} \\
&\le K \int_0^\tau (x-s)^{2\alpha+2\mu-4} ds \|w\|_{C([0,1];E)} \\
&\le K(x-\tau)^{\alpha+\mu-2} \|w\|_{C([0,1];E)},
\end{aligned}
$$

and

$$
\begin{aligned}
\|J_2\|_E &\le K \int_0^\tau \int_\Gamma e^{-c(x-s)|z|}|z|^3 \frac{(x-\tau)^\alpha}{|z|^\mu} \frac{(\tau-s)^\alpha}{|z|^\mu} |dz| ds \|w\|_{C([0,1];E)} \\
&\le K \int_0^\tau \int_\Gamma e^{-\sigma} \frac{(x-\tau)^\alpha (\tau-s)^\alpha}{(\frac{\sigma}{x-s})^{2\mu-3}} \frac{d\sigma}{(x-s)} ds \|w\|_{C([0,1];E)} \\
&\le K \int_0^\tau (x-\tau)^\alpha (\tau-s)^\alpha (x-s)^{2\mu-4} ds \|w\|_{C([0,1];E)} \\
&\le K(x-\tau)^{\alpha+\mu-2} \|w\|_{C([0,1];E)}.
\end{aligned}
$$

J_3 and J_4 are also treated as J_1. □

Corollary 4.2. *Under assumptions* (1.3)–(1.8) *and for all* $\omega \ge \omega^*$, *we have*

1. $(I + P_\omega)^{-1} \in L(C(E))$.
2. $(I + P_\omega)^{-1} \in L(C^\beta(E))$, *where* $\beta \in]0, \alpha + \mu - 2]$.
3. $(I + P_\omega)^{-1} \in L(C(E) \cap B(D_{A(\cdot)}(\frac{\beta}{2}, +\infty)))$, *where* $\beta \in]0, \alpha + \mu - 2]$.

4.2. **"mixed" regularity of $G_{Q_\omega(x)}(d_0,u_1,f)$.**

Proposition 4.3. *Assume* (1.3)–(1.8)*. Let* $\beta \in]0,\alpha+\mu-2]$,

$$(\overline{Q_\omega(0)-H})^{-1}d_0 \in D_{Q_\omega(0)} \cap D(H), \quad u_1 \in D_{A(1)}, \quad f \in C^\beta([0,1];E).$$

Then $G_{Q_\omega(x)}(d_0,u_1,f)(\cdot)+f(\cdot) \in B(D_{A(\cdot)}(\frac{\beta}{2},+\infty))$ *if and only if*

$$Q_\omega(0)(\overline{Q_\omega(0)-H})^{-1}(d_0-Q_\omega(0)^{-1}f(0)) \in D(Q_\omega(0)),$$
$$Q_\omega(0)^2(\overline{Q_\omega(0)-H})^{-1}(d_0-Q_\omega(0)^{-1}f(0))+f(0) \in D_{A(0)}(\frac{\beta}{2},+\infty),$$
$$A_\omega(1)u_1-f(1) \in D_{A(1)}(\frac{\beta}{2},+\infty).$$

Summarizing the above results we obtain the following theorem.

Theorem 4.4. *Assume* (1.3)–(1.8)*. Let* $\beta \in]0,\alpha+\mu-2]$,

$$(\overline{Q_\omega(0)-H})^{-1}d_0 \in D_{Q_\omega(0)} \cap D(H), \quad u_1 \in D_{A(1)}, \quad f \in C^\beta([0,1];E)$$

such that

$$Q_\omega(0)(\overline{Q_\omega(0)-H})^{-1}(d_0-Q_\omega(0)^{-1}f(0)) \in D(Q_\omega(0)),$$
$$Q_\omega(0)^2(\overline{Q_\omega(0)-H})^{-1}(d_0-Q_\omega(0)^{-1}f(0))+f(0) \in D_{A(0)}(\frac{\beta}{2},+\infty),$$
$$A_\omega(1)u_1-f(1) \in D_{A(1)}(\frac{\beta}{2},+\infty).$$

Then there exists $\omega^*>0$ *such that for all* $\omega \ge \omega^*$*, the equation* (3.3) *has a unique solution* $w(\cdot)=Q_\omega(\cdot)^2u(\cdot)$ *satisfies*

1. $Q_\omega(\cdot)^2u(\cdot) \in C([0,1];E)$.
2. $Q_\omega(\cdot)^2u(\cdot) \in C^\beta([0,1];E)$.
3. $u'' \in C^\beta([0,1];E)$.
4. $u'' \in C([0,1];E) \cap B(D_{A(\cdot)}(\frac{\beta}{2},+\infty))$.

Proof. We have

$$\begin{aligned} u''(\cdot) &= f(\cdot)+Q_\omega(\cdot)^2u(\cdot) \\ &= f(\cdot)+[G_{Q_\omega(x)}(d_0,u_1,f)(\cdot)-(P_\omega w)(\cdot)] \\ &= [f(\cdot)+G_{Q_\omega(x)}(d_0,u_1,f)(\cdot)]-(P_\omega w)(\cdot). \end{aligned}$$

□

5. Approximating problem

In our heuristical reasoning in section two, one proves that the solution of (1.1)–(1.2), when it exists, is given necessarily by (3.3).

To prove that the representation of u in (3.3) is the unique strict solution of (1.1)–(1.2), we consider the family of approximating problems

$$u_n''(x)+A_n(x)u_n(x)-\omega u_n(x)=f(x), \quad x \in]0,1[,$$
$$u_n'(0)-Hu_n(0)=d_0,$$
$$u_n(1)=u_1,$$

where $(A_n(x))_{x\in[0,1]}$ is the family of Yosida approximations of $(A(x))_{x\in[0,1]}$ defined by

$$A_n(x) = -nA(x)(A(x) - nI)^{-1},\ n \in \mathbb{N}^*.$$

Then we use the same arguments as in Labbas [14] or in Bouziani [4].

6. A concrete general example

Consider the complex Banach space $E = C([0,1])$ with its usual sup-norm and define the family of closed linear operators $Q(x)$ for all $x \in [0,1]$ by

$$\begin{gathered} D(Q(x)) = \{\varphi \in C^2([0,1]) : a(x)\varphi(0) + b(x)\varphi'(0) = 0,\ \varphi(1) = 0\}, \\ (Q(x)\varphi)(y) = \varphi''(y), \quad y \in (0,1), \end{gathered}$$

from which it is easy to deduce that $A(x) = Q(x)^2$,

$$\begin{gathered} D(A(x)) = \{\varphi \in C^4([0,1]) : a(x)\varphi(0) + b(x)\varphi'(0) = 0, \varphi(1) = 0, \\ a(x)\varphi''(0) + b(x)\varphi'''(0) = 0, \quad \varphi''(1) = 0\}, \\ (A(x)\varphi)(y) = -\varphi^{(iv)}(y), \quad y \in (0,1). \end{gathered}$$

We assume that $a,\, b \in C^1([0,1])$, $a > 0$, $b > 0$ and $\inf_{x\in[0,1]}(a(x) + b(x)) > 0$. Let us define the linear operator H by

$$\begin{gathered} D(H) = \{\varphi \in C^1([0,1]) : \varphi(1) = 0\}, \\ (H\varphi)(y) = \alpha\varphi'(y), \quad y \in (0,1). \end{gathered}$$

Therefore the spectra of $Q(x)$, for every $x \in [0,1]$, is included in $]-\infty, 0]$,

$$\begin{gathered} \overline{D(Q(x))} = \{\varphi \in C^1([0,1]) : \varphi(0) = \varphi(1) = 0\} \quad \text{if } b(x) = 0, \\ \overline{D(Q(x))} = \{\varphi \in C^1([0,1]) : \varphi(0) = 0\} \quad \text{if } b(x) \neq 0, \end{gathered}$$

so $D(Q(x))$ is not dense in E. Let $z \in \mathbb{C} \setminus \mathbb{R}_-$ and $\psi \in E$,

$$Q(x)\varphi - z\varphi = \psi \in E,$$

which implies

$$\begin{gathered} \varphi''(y) - z\varphi(y) = \psi(y) \quad y \in (0,1), \\ a(x)\varphi(0) + b(x)\varphi'(0) = 0, \quad \varphi(1) = 0. \end{gathered}$$

Then we obtain

$$((Q(x) - zI)^{-1}\psi)(y) = \int_0^1 K_\rho(y,x,s)\psi(s)ds,$$

where $\rho = \sqrt{z}$

$$K_\rho(y,x,s) = \begin{cases} \frac{\sinh\rho(1-y)[a(x)\sinh\rho s - b(x)\rho\cosh\rho s]}{\rho[a(x)\sinh\rho - b(x)\rho\cosh\rho]}, & 0 \le s \le y \\ \frac{\sinh\rho(1-s)[a(x)\sinh\rho y - b(x)\rho\cosh\rho y]}{\rho[a(x)\sinh\rho - b(x)\rho\cosh\rho]}, & y \le s \le 1. \end{cases}$$

We consider $Q(x) = -(-A(x))^{1/2}$. One has

$$Q(x)^{-1} = \int_0^1 K_0(y,x,s)\psi(s)ds,$$

where

$$K_0(y,x,s)=\begin{cases}\frac{(1-y)[a(x)s+b(x)]}{a(x)-b(x)}, & 0\le s\le y,\\ \frac{(1-s)[a(x)y-b(x)]}{1+b(x)}, & y\le s\le 1.\end{cases}$$

By direct calculations, one proves (1.3) and (1.4), see [1, p. 52 first example and Proposition 7.1]. Then, all our results apply to the following concrete quasi-elliptic boundary value problem for a large $\omega>0$,

$$\begin{gathered}\frac{\partial^2 u}{\partial x^2}(x,y)-\frac{\partial^4 u}{\partial y^4}(x,y)-\omega u(x,y)=f(x,y),\quad (x,y)\in[0,1]\times[0,1],\\ a(x)u(x,0)+b(x)\frac{\partial u}{\partial y}(x,0)=0,\quad x\in[0,1]\\ \frac{\partial u}{\partial x}(0,y)-\frac{\partial u}{\partial y}(0,y)=d_0(y),\quad u(1,y)=u_1(y),\quad y\in[0,1]\\ a(x)\frac{\partial^2 u}{\partial y^2}(x,0)-b(x)\frac{\partial^3 u}{\partial y^3}(x,0)=0,\quad x\in[0,1]\\ \frac{\partial^2 u}{\partial y^2}(x,1)=u(x,1)=0,\quad x\in[0,1]\\ u(0,y)=\varphi(y),\quad u(1,y)=\psi(y),\quad y\in[0,1].\end{gathered}$$

References

[1] P. Acquistapace and B. Terreni; *A unified approach to abstract linear nonautonomous parabolic equations*, Rendiconti del Seminario Matematico della Università di Padova, tome 78 (1987), p. 47–107.

[2] A. V. Balakrishnan; *Fractional powers of closed operators and the semigroups generated by them*, pacific. J. Math., 10 (1960), 419–437.

[3] F. Boutaous, R. Labbas, B.-K. Sadallah; *Fractional-power approach for solving complete elliptic abstract differential equations with variable-operator coefficients*. Electronic Journal of Differential Equations. Vol. 2012 (2012), No. 05, pp. 1–33.

[4] F. Bouziani, A. Favini, R. Labbas, A. Medeghri; *Study of boundary value and transmission problems governed by a class of variable operators verifying the Labbas-Terreni non commutativity assumption*, Rev Mat Complut. (2011), 24, p. 131–168.

[5] M. Cheggag, A. Favini, R. Labbas, S. Maingot, A. Medeghri; *Abstract differential equations of elliptic type wich general Robin boundary conditions in Hölder spaces*, Applicable Analysis, Vol, 91 No. 8, August 2012, 1453–1475.

[6] G. Dore, S. Yakubov; *Semigroup Estimates and Noncoercive Boundary Value Problems*, Semigroup Forum Vol. 60 (2000), 93-121.

[7] A. Favini, R. Labbas, K. Lemrabet. B.-K. Sadallah; *Study of a complete abstract differential equation of elliptic type with variable operators coefficients (part I)*, Rev. Mat. Complut. 21 (2008), 89–133.

[8] A. Favini, R. Labbas, S. Maingot, H. Tanabe, A. Yagi; *Necessary and sufficient conditions in the study of maximal regularity of elliptic differential equations in Hölder spaces*, Discrete Contin. Dyn. Syst. 22 (2008), pp. 973-987.

[9] P. Grisvard; *Spazi di tracce e applicazioni*, Rend. Mat. (4) Vol. 5, VI (1972), 657–729.

[10] M. Haase; *The functional calculus for sectorial operators and similarity methods*, Thesis, Univarsität Ulm, Germany, 2003.

[11] S. G. Krein; *Linear Differential Equations in Banach Spaces*, Nauka, Moscou, 1967.

[12] T. Kato; *Perturbation Theory for Linear Operators*, Springer-Verlag, Berlin. Heidelberg, New York, 1980.

[13] R. Labbas, B. Terreni; *Sommes d'opérateurs de type elliptique et parabolique, 1ère partie.* Boll. Un. Math. Ital. 1-B, 7 (1987), 545-569.

[14] R. Labbas; *Problèmes aux limites pour une equation différentielle abstraite de type elliptique*, Thèse d'état, Université de Nice (1987).

[15] A. Lunardi; *Analytic Semigroups and Optimal Regularity in Parabolic Problems*, Birkhäuser, Basel, 1995.

[16] C. Martinez Carracedo, M. Sanz Alix; *The Theory of Fractional Powers of Operators*, North-Holland Mathematics Studies 187. New York, Elsevier Science, 2001.

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