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GROUND STATES FOR A MODIFIED CAPILLARY SURFACE EQUATION IN WEIGHTED ORLICZ-SOBOLEV SPACE

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ABSTRACT. In this article, we prove a compact embedding theorem for the weighted Orlicz-Sobolev space of radially symmetric functions. Using the embedding theorem and critical points theory, we prove the existence of multiple radial solutions and radial ground states for the following modified capillary surface equation

$$-\operatorname{div}\Big(\frac{|\nabla u|^{2p-2}\nabla u}{\sqrt{1+|\nabla u|^{2p}}}\Big)+T(|x|)|u|^{\alpha-2}u=K(|x|)|u|^{s-2}u,\quad u>0,\ x\in\mathbb{R}^N,$$
$$u(|x|)\to 0,\quad \text{as } |x|\to\infty,$$

where $N\geq 3$, $1<\alpha<p<2p<N$, s satisfies some suitable conditions, $K(|x|)$ and $T(|x|)$ are continuous, nonnegative functions.

1. Introduction

In this article, we study the following modified capillary surface equation in a weighted Orlicz-Sobolev space,

$$\begin{gathered}-\operatorname{div}\Big(\frac{|\nabla u|^{2p-2}\nabla u}{\sqrt{1+|\nabla u|^{2p}}}\Big)+T(|x|)|u|^{\alpha-2}u=K(|x|)|u|^{s-2}u,\quad u>0,\ x\in\mathbb{R}^N,\\ u(|x|)\to 0,\quad \text{as } |x|\to\infty,\end{gathered}\tag{1.1}$$

where $N\geq 3$, $1<\alpha<p<2p<N$, s satisfies some suitable conditions, ∇u denotes the gradient of u, T and K are continuous, nonnegative and measurable functions, i.e., $T,K:(0,+\infty)\to[0,+\infty]$ and may be unbounded, decaying and vanishing.

Recently, these type equations have attracted much attention. As $p=1$, the problem (1.1) becomes known as the prescribed mean curvature equation or the capillary surface equation. Peletier and Serrin [15] studied the following problem

$$\begin{gathered}-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^{2}}}\Big)=-\lambda u+u^q,\quad x\in\mathbb{R}^N,\\ u(x)\to 0,\quad \text{as } x\to\infty,\end{gathered}\tag{1.2}$$

where $\lambda>0, q>1$ and obtained the existence of radial ground states. As $\lambda=0$, Ni and Serrin [12,13] established that if $1<q\leq\frac{N}{N-2}$, no positive solutions exist,

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on the contrary, if $q \geq \frac{N+2}{N-2}$, there is a continuum of solutions. del Pino and Guerra [6] proved the existence of large finite number of ground states, provided that q lies below but close enough to the critical exponent $\frac{N+2}{N-2}$. Moreover, existence, nonexistence and multiplicity of solutions decaying to zero at infinity have been proved by [3,4,7,8,17].

As $p > 1$, using minimization sequence method and Mountain Pass Lemma, Narukawa and Suzuki [11] discussed the existence of nonzero solutions for the modified capillary surface equation

$$\begin{gathered} -\operatorname{div}\Big(\frac{|\nabla u|^{2p-2}\nabla u}{\sqrt{1+|\nabla u|^{2p}}}\Big) = \lambda f(x,u), \quad u \geq 0,\ x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, λ is a positive parameter; Liang [9] investigated the following modified capillary equation

$$\begin{gathered} -\operatorname{div}\Big(\frac{|\nabla u|^{2p-2}\nabla u}{\sqrt{1+|\nabla u|^{2p}}}\Big) = f(x,u), \quad x \in \Omega, \\ u = 0, \quad x \in \partial\Omega, \end{gathered} \tag{1.4}$$

and obtained a negative and a positive solution by variational methods. In particular, Azzollini, d'Avenia and Pomponio [1] studied the quasilinear elliptic problems

$$\begin{gathered} -\nabla[\phi'(|\nabla u|^2)\nabla u] + |u|^{\alpha-2}u = |u|^{s-2}u, \quad x \in \mathbb{R}^N, \\ u(x) \to 0, \quad \text{as } |x| \to \infty, \end{gathered} \tag{1.5}$$

where $\phi(t)$ behaves like $t^{\frac{q}{2}}$ for small t and $t^{\frac{p}{2}}$ for large t, $1 < p < q < N$, and obtained some existence results in Orlicz-Sobolev space by using critical points theory.

On the other hand, some authors studied the semilinear (quasilinear) elliptic equations with unbounded or decaying radial potentials. Su, Wang and Willem [18,19] proved some embedding results for the weighted Sobolev spaces of radially symmetric functions. Zhang [20] obtained some Strauss-type decay estimates and obtained some continuous and compact embedding theorems.

In this article, we prove the existence of multiple radial solutions and radial ground states for the problem (1.1). Firstly, we obtain a compact embedding theorem for the weighted Orlicz-Sobolev space of radially symmetric functions. Secondly, we obtain the existence of radial ground states for the problem (1.1) with unbounded or decaying radial potentials by using this compact embedding theorem and critical points theory.

Consider the functional

$$J(u) = \frac{1}{p}\int_{\mathbb{R}^N}(\sqrt{1+|\nabla u|^{2p}}-1)dx + \frac{1}{\alpha}\int_{\mathbb{R}^N} T(|x|)|u|^{\alpha}dx - \frac{1}{s}\int_{\mathbb{R}^N} K(|x|)|u|^s dx, \tag{1.6}$$

where

$$\sqrt{1+|\nabla u|^{2p}}-1 \sim \begin{cases} |\nabla u|^p, & \text{as } |\nabla u| \to \infty, \\ \frac{1}{2}|\nabla u|^{2p}, & \text{as } |\nabla u| \to 0. \end{cases} \tag{1.7}$$

Solutions of (1.1) are, at least formally, critical points of the functional $J(u)$. By (1.7), we obtain that this different growth at zero and at infinity of the function $\sqrt{1+|\nabla u|^{2p}}-1$ and the whole space $\mathbb{R}^N$ suggest us not to use classical Sobolev spaces. Hence, we should define a class of weighted Orlicz-Sobolev space with respect to the functional (1.6) is well defined and C^1. For dealing with the compact properties of the functional $J(u)$, we would like to get compactness lies in the fact that the group of translation constitutes an obstruction to compact embedding in $\mathbb{R}^N$, and examine the affects of the unbounded or decaying potentials $T(|x|)$ and $K(|x|)$. Hence, we restrict the domain of the functional $J(u)$ to the suitable Orlicz-Sobolev space.

Now we state our main theorems in this paper. Let $|x| = r$, $T(|x|)$, $K(|x|)$ be continuous nonnegative functions in $(0, \infty)$, and

- (T1) There exist real number a and a_0, such that $\liminf_{r\to\infty} T(r)/r^a > 0$, and $\liminf_{r\to 0} T(r)/r^{a_0} > 0$;
- (K1) There exist real number b and b_0, such that $\limsup_{r\to\infty} K(r)/r^b < \infty$, and $\limsup_{r\to 0} K(r/r^{b_0} < \infty$, $K(r) > 0$.

The existence and embedding results depend on the potentials T, K near 0 and ∞. We define the following relations between $p, 2p$, and a, b or a_0, b_0:

$$s_* = \begin{cases} \frac{(2p)\alpha(N-1+b)-a\alpha}{2p(N-1)+a(2p-1)}, & b \geq a > -p, \\ \frac{2p(N+b)}{(N-2p)}, & b \geq -p, a \leq -p, \\ \alpha, & b \leq \max\{a, -p\}, \end{cases} \tag{1.8}$$

and

$$s^* = \begin{cases} \frac{2p(N+b_0)}{(N-2p)}, & b_0 \geq -p,\ a_0 \geq -p, \\ \frac{(2p)a(N-1+b_0)-a_0\alpha}{2p(N-1)+a_0(2p-1)}, & -p > a_0 > -\frac{(N-1)}{(2p-1)}2p,\ b_0 \geq a_0, \\ \infty, & a_0 \leq -\frac{(N-1)}{(p-1)}p,\ b_0 \geq a_0. \end{cases} \tag{1.9}$$

Remark 1.1. The idea which for establishing conditions (1.8) and (1.9) comes from Su, Wang and Willem [18, 19]. In this article, we not only develop the methods in [18,19,20] to the modified capillary surface equation, but also improve and extend the results in classical Sobolev space to the Orlicz-Sobolev space.

Theorem 1.2 (Multiplicity Result)**.** *Assume that* (T1) *and* (K1) *hold,* $1 < \alpha < p < 2p < N$*,* $s_* < s < s^*$*, then there exist infinitely many radially symmetric solutions for* (1.1).

Theorem 1.3 (Ground States)**.** *Assume that* (T1) *and* (K1) *hold,* $1 < \alpha < p < 2p < N$*,* $s_* < s < s^*$*, then there exists a radial ground states for* (1.1).

This article is organized as follows. In Section 2, we introduce a weighted Orlicz-Sobolev space of radially symmetric function and recall some important lemmas. In Section 3, we prove some inequalities with radial functions, extending some inequalities in classic Sobolev space to the Orlicz-Sobolev space, and establish a new compact embedding theorem (i.e. Theorem 3.1). Section 4 is devoted to the proof of Theorems 1.2 and Theorem 1.3.

2. Weighted Orlicz-Sobolev spaces

As a first step, we recall some well known facts on the sum of Lebesgue spaces and introduce some notation of function space.

Definition 2.1 ([2]). Let $1 < p < q$ and $\Omega \subset \mathbb{R}^N$. We denote with $L^p(\Omega) + L^q(\Omega)$ the completion of $\mathcal{C}_c^\infty(\Omega, \mathbb{R}^N)$ in the norm

$$\|u\|_{L^p(\Omega)+L^q(\Omega)} = \inf\big\{\|v\|_p + \|w\|_q : v \in L^p(\Omega), w \in L^q(\Omega), u = v + w\big\}. \tag{2.1}$$

In this article, we set $q = 2p$ and $\|u\|_{p,2p} = \|u\|_{L^p(\Omega)+L^{2p}(\Omega)}$. Moreover, from [2], we obtain that $L^p(\Omega) + L^{2p}(\Omega)$ are Orlicz spaces.

For $\alpha > 1, s > 1$, we define

$$L^\alpha(\mathbb{R}^N; T) = \big\{u : \mathbb{R}^N \to \mathbb{R} : u \text{ is Lebesgue measurable}, \int_{\mathbb{R}^N} T(|x|)|u|^\alpha dx < \infty\big\},$$

and

$$L^s(\mathbb{R}^N; K) = \big\{u : \mathbb{R}^N \to \mathbb{R} : u \text{ is Lebesgue measurable}, \int_{\mathbb{R}^N} K(|x|)|u|^s dx < \infty\big\}.$$

The corresponding norms in $L^\alpha(\mathbb{R}^N; T)$ and $L^s(\mathbb{R}^N; K)$ are respectively

$$\begin{aligned} \|u\|_{L_T^\alpha(\mathbb{R}^N)} &= \Big(\int_{\mathbb{R}^N} T(|x|)|u|^\alpha dx\Big)^{1/\alpha}, \\ \|u\|_{L_K^s(\mathbb{R}^N)} &= \Big(\int_{\mathbb{R}^N} K(|x|)|u|^s dx\Big)^{1/s}. \end{aligned} \tag{2.2}$$

From [2], we have a list of properties of the Orlicz spaces $L^p(\Omega) + L^{2p}(\Omega)$.

Proposition 2.2 ([2]). *Let $\Omega \subset \mathbb{R}^N$, $u \in L^p(\Omega) + L^{2p}(\Omega)$ and $\Lambda_u = \{x \in \Omega | \ |u(x)| > 1\}$. We have*

(i) *if $\Omega' \subset \Omega$ is such that $|\Omega'| < +\infty$, then $u \in L^p(\Omega')$;*
(ii) *if $\Omega' \subset \Omega$ is such that $u \in L^\infty(\Omega')$, then $u \in L^{2p}(\Omega')$;*
(iii) *$|\Lambda_u| < +\infty$;*
(iv) *$u \in L^p(\Lambda_u) \cap L^{2p}(\Lambda_u^c)$;*
(v) *the infimum in (2.1) is attained;*
(vi) *$L^p(\Omega) + L^{2p}(\Omega)$ is reflexive and $(L^p(\Omega) + L^{2p}(\Omega))' = L^{p'}(\Omega) \cap L^{(2p)'}(\Omega)$;*
(vii) *$\|u\|_{L^p(\Omega)+L^{2p}(\Omega)} \leq \max\{\|u\|_{L^p(\Lambda_u)}, \|u\|_{L^{2p}(\Lambda_u^c)}\}$;*
(viii) *if $B \subset \Omega$, then $\|u\|_{L^p(\Omega)+L^{2p}(\Omega)} \leq \|u\|_{L^p(B)+L^{2p}(B)} + \|u\|_{L^p(\Omega\setminus B)+L^{2p}(\Omega\setminus B)}$.*

Let $\mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R})$ denote the collection of smooth functions with compact support and

$$(\mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}))_{\rm rad} = \{u \in \mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}) : u \text{ is radial}\}.$$

Definition 2.3. Let $\alpha > 1$, $\mathcal{W}$ be the completion of $\mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R})$ in the norm

$$\|u\|_{\mathcal{W}} = \|u\|_{L_T^\alpha(\mathbb{R}^N)} + \|\nabla u\|_{p,2p}, \tag{2.3}$$

$\mathcal{W}_{\rm rad}$ be the completion of $(\mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}))_{\rm rad}$ in the norm $\|\cdot\|$, namely

$$\mathcal{W}_{\rm rad} = \overline{(\mathcal{C}_c^\infty(\mathbb{R}^N, \mathbb{R}))_{\rm rad}}^{\|\cdot\|}.$$

Lemma 2.4. *The space $(\mathcal{W}_{\rm rad}, \|\cdot\|)$ is a reflexive Banach space.*

Proof. Firstly, we prove that $(\mathcal{W}_{\rm rad}, \|\cdot\|)$ is a Banach space. In fact, since $L^\alpha(\mathbb{R}^N; T)$ and $L^p(\mathbb{R}^N) + L^{2p}(\mathbb{R}^N)$ are completed. Let $\{u_n\}_n$ be a Cauchy sequence in $\mathcal{W}_{\rm rad}$, then $\{u_n\}_n$ is a Cauchy sequence in $L^\alpha(\mathbb{R}^N; T)$, and there exists $u \in L^\alpha(\mathbb{R}^N; T)$, such that $\|u_n - u\|_{L_T^\alpha(\mathbb{R}^N)} \to 0$, as $n \to \infty$. Also $\{\nabla u_n\}_n$ is a Cauchy sequence in $L^p(\mathbb{R}^N) + L^{2p}(\mathbb{R}^N)$, there exists $\delta \in L^p(\mathbb{R}^N) + L^{2p}(\mathbb{R}^N)$, such that $\|\nabla u_n - \delta\|_{p,2p} \to 0$, as $n \to \infty$.

Sufficiently, for every $\xi \in \mathcal{C}_c^\infty(\mathbb{R}^N)$, $n \in \mathbb{N}$, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} T(|x|)u_n \nabla\xi dx = \int_{\mathbb{R}^N} T(|x|)u\nabla\xi dx, \quad \lim_{n\to\infty} \int_{\mathbb{R}^N} \xi\nabla u_n dx = \int_{\mathbb{R}^N} \xi\delta dx.$$

In fact, by Hölder inequality and Proposition 2.2 (v), by considering $(\mathbf{v}_n, \mathbf{w}_n)$ in $inL^p(\mathbb{R}^N) \times L^{2p}(\mathbb{R}^N)$ such that

$$\nabla u_n - \delta = \mathbf{v}_n + \mathbf{w}_n\,, \quad \|\nabla u_n - \delta\|_{p,2p} = \|\mathbf{v}_n\|_p + \|\mathbf{w}_n\|_{2p},$$

we have

$$\Big|\int_{\mathbb{R}^N} T(|x|)(u_n - u)\nabla\xi dx\Big| \le \|\nabla\xi\|_{\alpha'}\|u_n - u\|_{L^\alpha_T(\mathbb{R}^N)} \to 0,$$

and

$$\begin{aligned} \Big|\int_{\mathbb{R}^N} \xi(\nabla u_n - \delta)dx\Big| &= \Big|\int_{\mathbb{R}^N} \xi\mathbf{v}_n dx + \int_{\mathbb{R}^N} \xi\mathbf{w}_n dx\Big| \\ &\le \|\xi\|_{p'}\|\mathbf{v}_n\|_p + \|\xi\|_{(2p)'}\|\mathbf{w}_n\|_{2p} \to 0. \end{aligned}$$

Obviously, by the definition of weak derivatives, we have

$$\int_{\mathbb{R}^N} T(|x|)u_n\nabla\xi dx = -\int_{\mathbb{R}^N} T(|x|)\xi\nabla u_n dx.$$

Hence, we obtain

$$\int_{\mathbb{R}^N} T(|x|)u\nabla\xi dx = -\int_{\mathbb{R}^N} T(|x|)\xi\delta dx;$$

that is, $\nabla u = \delta$.

Secondly, we prove that $(\mathcal{W}_{\rm rad}, \|\cdot\|)$ is reflexive. Indeed, we consider the norm

$$\|u\|^*_{p,2p} = \inf\{(\|v\|_p^2 + \|w\|_{2p}^2)^{\frac{1}{2}} \,|\, v \in L^p(\mathbb{R}^N), w \in L^{2p}(\mathbb{R}^N), u = v + w\},$$

and then, on $\mathcal{W}_{\rm rad}$, the norm

$$\|u\|^*_{\mathcal{W}_{\rm rad}} = \|u\|_{L^\alpha_T(\mathbb{R}^N)} + \|\nabla u\|^*_{p,2p},$$

is equivalent to the norm $\|u\|_{\mathcal{W}_{\rm rad}}$. Moreover, by [2, Proposition 2.6], the norm $\|u\|_{L^\alpha_T(\mathbb{R}^N)}$ and the norm $\|\cdot\|^*$ are uniformly convex. So, on $\mathcal{W}_{\rm rad}$, we consider uniformly convex norm $\|\nabla.\|^*_{p,2p}$ and the norm $\|\cdot\|_{L^\alpha_T(\mathbb{R}^N)}$. By a well known result, also the norm

$$\|\cdot\|^\sharp_{\mathcal{W}_{\rm rad}} = \sqrt{\|\cdot\|^2_{L^\alpha_T(\mathbb{R}^N)} + (\|\nabla.\|^*_{p,2p})^2},$$

is uniformly convex and then $(\mathcal{W}_{\rm rad}, \|\cdot\|^\sharp)$ is reflexive. Hence the norm $\|\cdot\|^\sharp_{\mathcal{W}_{\rm rad}}$ is equivalent to $\|\cdot\|_{\mathcal{W}_{\rm rad}}$. Then, we obtain that $(\mathcal{W}_{\rm rad}, \|\cdot\|)$ is also reflexive. □

Remark 2.5. Similar to [1, Theorem 2.8], we obtain that $\mathcal{W}_{\rm rad}$ coincides with the set of radial functions of $\mathcal{W}$. Hence, using the principle of symmetric criticality in [14], we only consider the functional $J(u)$ in (1.6) restricted to the weighted Orlicz-Sobolev space $\mathcal{W}_{\rm rad}$.

3. Embedding theorem

To obtain the compactness of the functional $J(u)$, we prove a compact embedding theorem (Theorem 3.1). Denote by B_r the ball in $\mathbb{R}^N$ centered at 0 with radius r.

Theorem 3.1. *Let $1 < \alpha < p < 2p < N$. Assume* (T1) *and* (K1) *hold, then we have the continuous embedding*
$$\mathcal{W}_{\rm rad} \hookrightarrow L^s(\mathbb{R}^N; K)$$
for $s_* \leq s \leq s^*$ when $s^* < \infty$, and for $s_* \leq s < \infty$ when $s^* = \infty$. Furthermore, the embedding is compact for $s_* < s < s^*$.

Firstly, we prove some inequalities on radial functions which are interesting.

Lemma 3.2. *If $1 < p < 2p < N$, there exists $\widehat{M} > 0$ such that for every $u \in \mathcal{W}_{\rm rad}$,*
$$|u(x)| \leq \begin{cases} \widehat{M}|x|^{-(\frac{N-2p}{2p})}\|\nabla u\|_{p,2p}, & \text{for } |x| \geq 1,\\ \widehat{M}|x|^{-(\frac{N-p}{p})}\|\nabla u\|_{p,2p}, & \text{for } 0 < |x| < 1. \end{cases} \tag{3.1}$$

The proof of the above lemma is similar to that of [1, Lemma 2.13] and of [19, Lemma 1].

Lemma 3.3. *Let $1 < p < 2p < N$. Assume $2p < s < \infty$ and write $s = \frac{2p(N+c)}{(N-2p)}$, for some $-p \leq c < \infty$. Then there exists $\widetilde{M} > 0$ such that for all $u \in \mathcal{W}_{\rm rad}$*
$$\Big(\int_{\mathbb{R}^N} |x|^c|u|^s dx\Big)^{1/s} \leq \widetilde{M} \max\big(\|\nabla u\|_{p,2p}, \|\nabla u\|^2_{p,2p}\big). \tag{3.2}$$

Proof. By denseness, it is sufficient to prove that $u \in (\mathcal{C}_c^\infty(\mathbb{R}^N,\mathbb{R}))_{\rm rad}, (\mathbf{v},\mathbf{w}) \in L^p(\mathbb{R}^N) \times L^{2p}(\mathbb{R}^N)$, such that $\nabla u = \mathbf{v} + \mathbf{w}$. By using Lemma 3.2, and $s = \frac{2p(N+c)}{(N-2p)}$, we have
$$\begin{aligned}
&\int_{\mathbb{R}^N} |x|^c|u|^s dx\\
&= \omega_N \int_0^\infty r^{(N-1+c)}|u(r)|^s dr\\
&= -\frac{s\omega_N}{(N+c)} \int_0^\infty r^{(N+c)}|u(r)|^{(s-2)}u(r)u'(r)dr\\
&\leq \frac{(2p)\omega_N}{(N-2p)} \int_0^\infty r^{(N+c)}|u(r)|^{(s-1)}|u'(r)|dr\\
&= \frac{2p}{(N-2p)} \int_{\mathbb{R}^N} |x|^{(c+1)}|u|^{(s-1)}|\nabla u|dx\\
&\leq \frac{2p}{(N-2p)}\Big(\int_{\mathbb{R}^N} |x|^{(c+1)}|u|^{(s-1)}|\mathbf{v}|dx + \int_{\mathbb{R}^N} |x|^{(c+1)}|u|^{(s-1)}|\mathbf{w}|dx\Big)\\
&\leq \frac{2p}{(N-2p)}\Big[\Big(\int_{\mathbb{R}^N} |\mathbf{v}|^p dx\Big)^{1/p}\Big(\int_{\mathbb{R}^N} |x|^c|u|^s|x|^{\frac{(p+c)}{(p-1)}}|u|^{\frac{(s-p)}{(p-1)}}dx\Big)^{(\frac{p-1}{p})}\\
&\quad + \Big(\int_{\mathbb{R}^N} |\mathbf{w}|^{2p}dx\Big)^{\frac{1}{2p}}\Big(\int_{\mathbb{R}^N} |x|^c|u|^s|x|^{\frac{(2p+c)}{(p-1)}}|u|^{\frac{(s-2p)}{(2p-1)}}dx\Big)^{(\frac{2p-1}{2p})}\Big]\\
&\leq M'\frac{2p}{(N-2p)}\Big[\|\mathbf{v}\|_{L^p(\mathbb{R}^N)}\|\nabla u\|_{p,2p}^{(\frac{s-p}{p})}\Big(\int_{\mathbb{R}^N} |x|^c|u|^s dx\Big)^{(\frac{p-1}{p})}
\end{aligned}$$

$$
\begin{aligned}
&\quad + \|\mathbf{w}\|_{L^{2p}(\mathbb{R}^N)} \|\nabla u\|_{p,2p}^{(\frac{s-2p}{2p})} \Big(\int_{\mathbb{R}^N} |x|^c |u|^s dx \Big)^{(\frac{2p-1}{2p})} \Big] \\
&\le M' \frac{2p}{(N-2p)} \max \Big(\|\nabla u\|_{p,2p}^{(\frac{s-p}{p})}, \|\nabla u\|_{p,2p}^{(\frac{s-2p}{2p})} \Big) \Big[\|\mathbf{v}\|_{L^p(\mathbb{R}^N)} \Big(\int_{\mathbb{R}^N} |x|^c |u|^s dx \Big)^{(\frac{2p-1}{2p})} \\
&\quad + \|\mathbf{w}\|_{L^{2p}(\mathbb{R}^N)} \Big(\int_{\mathbb{R}^N} |x|^c |u|^s dx \Big)^{(\frac{2p-1}{2p})} \Big] \\
&\le M' \frac{2p}{(N-2p)} \max \Big(\|\nabla u\|_{p,2p}^{(\frac{s-p}{p})}, \|\nabla u\|_{p,2p}^{(\frac{s-2p}{2p})} \Big) \|\nabla u\|_{p,2p} \Big(\int_{\mathbb{R}^N} |x|^c |u|^s dx \Big)^{(\frac{2p-1}{2p})} \\
&\le \widetilde{M} \max \Big(\|\nabla u\|_{p,2p}^{\frac{s}{p}}, \|\nabla u\|_{p,2p}^{\frac{s}{2p}} \Big) \Big(\int_{\mathbb{R}^N} |x|^c |u|^s dx \Big)^{(\frac{2p-1}{2p})},
\end{aligned}
$$

where ω_N is the volume of the unit sphere in $\mathbb{R}^N$. It follows that

$$
\Big(\int_{\mathbb{R}^N} |x|^c |u|^s dx \Big)^{1/s} \le \widetilde{M} \max \Big(\|\nabla u\|_{p,2p}, \|\nabla u\|_{p,2p}^2 \Big).
$$

$\square$

Lemma 3.4. *Assume* (T1) *holds,* $1 < \alpha < p < 2p < N$, *and* $a > -\frac{(N-1)}{(2p-1)} 2p$. *Then there exists* $\widehat{M_0} > 0$ *such that for all* $u \in \mathcal{W}_{\rm rad}$,

$$
|u(x)| \le \widehat{M_0} |x|^{-(\frac{2p(N-1)+a(2p-1)}{\alpha(2p)})} \|u\|_{\mathcal{W}_{\rm rad}}, \quad \text{for } |x| \gg 1. \tag{3.3}
$$

Proof. By assumption (T1), there exists $R > 1$ such that for some $M_0 > 0$,

$$
T(|x|) \ge M_0 |x|^a, \quad |x| > R > 1.
$$

For $u \in \mathcal{W}_{\rm rad}$, as $\theta > -(N-1)$, we have

$$
\begin{aligned}
\frac{d}{dr} (r^{(\theta+N-1)} |u|^\alpha) &= \alpha r^{(\theta+N-1)} |u|^{(\alpha-2)} u \frac{du}{dr} + (\theta+N-1) |u|^\alpha r^{(\theta+N-2)} \\
&\ge \alpha r^{(\theta+N-1)} |u|^{(\alpha-2)} u \frac{du}{dr}.
\end{aligned} \tag{3.4}
$$

Next we only consider $|u| \ge 1$, when $|u| \le 1$, set $|u'| = \frac{1}{|u|}$, then $|u'| \ge 1$. For all $u \in \mathcal{W}_{\rm rad}$, $(\mathbf{v}, \mathbf{w}) \in L^p(\mathbb{R}^N) \times L^{2p}(\mathbb{R}^N)$, such that $\nabla u = \mathbf{v} + \mathbf{w}$. Since, $a > -\frac{(N-1)}{(2p-1)} 2p$, so take $\theta = \min\{\frac{a(p-1)}{p}, \frac{a(2p-1)}{2p}\}$, then $\theta > -(N-1)$. For $r > R$, $1 < \alpha < p < 2p < N$, we have

$$
\begin{aligned}
|u|^\alpha r^{(\theta+N-1)} &\le \alpha \int_r^\infty |u|^{(\alpha-1)} t^{(\theta+N-1)} |u'(t)| dt \\
&= \frac{\alpha}{\omega_N} \int_{B_r^c} |x|^\theta |u|^{(\alpha-1)} |\nabla u| dx \\
&\le \frac{\alpha}{\omega_N} \Big(\int_{B_r^c} |x|^\theta |u|^{(\alpha-1)} |\mathbf{v}| dx + \int_{B_r^c} |x|^\theta |u|^{(\alpha-1)} |\mathbf{w}| dx \Big) \\
&\le \frac{\alpha}{\omega_N} \Big[\|\mathbf{v}\|_{L^p(B_r^c)} \Big(\int_{B_r^c} |x|^{\frac{\theta p}{(p-1)}} |u|^{(\frac{(\alpha-1)p}{p-1})} dx \Big)^{(\frac{p-1}{p})} \\
&\quad + \|\mathbf{w}\|_{L^{2p}(B_r^c)} \Big(\int_{B_r^c} |x|^{\frac{\theta(2p)}{(2p-1)}} |u|^{(\frac{(\alpha-1)(2p)}{(2p-1)})} dx \Big)^{(\frac{2p-1}{2p})} \Big] \\
&\le \frac{\alpha}{\omega_N} \Big[\|\mathbf{v}\|_{L^p(B_r^c)} \Big(\int_{B_r^c} |x|^a |u|^{(\frac{(\alpha-1)p}{p-1})} dx \Big)^{(\frac{p-1}{p})}
\end{aligned}
$$

$$
\begin{aligned}
&\quad + \|\mathbf{w}\|_{L^{2p}(B_r^c)}\Big(\int_{B_r^c}|x|^a|u|^{(\frac{(\alpha-1)2p}{(2p-1)})}dx\Big)^{(\frac{2p-1}{2p})}\Big]\\
&\le M_{(\alpha,p,N)}\Big[\|\mathbf{v}\|_{L^p(B_r^c)}\Big(\int_{B_r^c}T(|x|)|u|^{(\frac{(\alpha-1)p}{p-1})}dx\Big)^{(\frac{p-1}{p(\alpha-1)}(\alpha-1))}\\
&\quad + \|\mathbf{w}\|_{L^{2p}(B_r^c)}\Big(\int_{B_r^c}T(|x|)|u|^{(\frac{(\alpha-1)2p}{2p-1})}dx\Big)^{(\frac{2p-1}{2p(\alpha-1)}(\alpha-1))}\\
&\le M_{(\alpha,p,N)}\|\nabla u\|_{p,2p}\|u\|^{(\alpha-1)}_{L_T^{\frac{(\alpha-1)p}{(p-1)}}(\mathbb{R}^N)}\\
&\le M_{(\alpha,p,N)}\|\nabla u\|_{p,2p}\|u\|^{(\alpha-1)}_{L_T^{\alpha}(\mathbb{R}^N)}.
\end{aligned}
$$

By Young inequality, and $|x|=r$, we obtain

$$
\begin{aligned}
|u|\,|x|^{(\frac{\theta+N-1}{\alpha})} &\le M^{1/\alpha}_{(\alpha,p,N)}\|\nabla u\|^{1/\alpha}_{p,2p}\|u\|^{(\frac{\alpha-1}{\alpha})}_{L_T^\alpha(\mathbb{R}^N)}\\
&\le \widehat{M}_0(\|\nabla u\|_{p,2p}+\|u\|_{L_T^\alpha(\mathbb{R}^N)});
\end{aligned}
$$

i.e.,

$$
|u| \le \widehat{M}_0|x|^{-(\frac{2p(N-1)+a(2p-1)}{\alpha(2p)})}\|u\|_{\mathcal{W}_{\rm rad}},
$$

where the constant $\widehat{M}_0=\widehat{M}_{0(\alpha,p,N)}$. □

Lemma 3.5. *Assume* (T1) *holds,* $1<\alpha<p<2p<N$*. Then there exist* $1>r_0>0$ *and* $\widetilde{M}_0>0$ *such that for all* $u\in\mathcal{W}_{\rm rad}$*,*

$$
|u(x)| \le \widetilde{M}_0|x|^{-(\frac{2p(N-1)+a_0(2p-1)}{\alpha(2p)})}\|u\|_{\mathcal{W}_{\rm rad}}, \quad \text{for } 0<|x|\le r_0<1, \tag{3.5}
$$

where $\widetilde{M}_0=\widetilde{M}_0(a_0,r_0,\alpha,N)$.

Proof. By assumption (T1), there exists $1>r_0>0$ such that for some constant $M_0>0$,

$$
T(|x|)\ge M_0|x|^{a_0}, \quad 0<|x|\le r_0<1.
$$

For $u\in\mathcal{W}_{\rm rad}$, we have

$$
\frac{d}{dr}(r^{(\beta+N-1)}|u|^\alpha)=\alpha r^{(\beta+N-1)}|u|^{(\alpha-2)}u\frac{du}{dr}+(\beta+N-1)|u|^\alpha r^{(\beta+N-2)}.
$$

Thus, for $0<r\le r_0<1$,

$$
r^{(\beta+N-1)}|u|^\alpha \le \alpha\int_r^{r_0}|u|^{(\alpha-1)}t^{(\beta+N-1)}|u'(t)|dt+(\beta+N-1)\int_r^{r_0}|u|^\alpha t^{(\beta+N-2)}dt. \tag{3.6}
$$

As $\beta\ge a_0+1$, we have

$$
\begin{aligned}
\int_r^{r_0}|u|^\alpha t^{(\beta+N-2)}dt &= \int_r^{r_0}t^{(a_0+N-1)}|u|^\alpha t^{(\beta-a_0-1)}dt\\
&\le \omega_N^{-1}r_0^{(\beta-a_0-1)}\int_{B_{r_0}(0)\setminus B_r(0)}|x|^{a_0}|u|^\alpha dx\\
&\le \omega_N^{-1}M_0^{-1}r_0^{(\beta-a_0-1)}\int_{B_{r_0}(0)\setminus B_r(0)}T(|x|)|u|^\alpha dx\\
&\le \omega_N^{-1}M_0^{-1}r_0^{(\beta-a_0-1)}\|u\|^\alpha_{L_T^\alpha}.
\end{aligned} \tag{3.7}
$$

Let $\beta = \max\{\frac{(2p-1)}{2p}a_0, \frac{(p-1)}{p}a_0\}$, for all $u \in \mathcal{W}_{\rm rad}$, $(\mathbf{v}, \mathbf{w}) \in L^p(\mathbb{R}^N) \times L^{2p}(\mathbb{R}^N)$, such that $\nabla u = \mathbf{v} + \mathbf{w}$, we only consider $|u| \geq 1$. If $|u| \leq 1$, set $|u'| = 1/|u|$, then we have $|u'| \geq 1$. Hence, we have

$$
\begin{aligned}
&\int_r^{r_0} |u|^{(\alpha-1)} t^{(\beta+N-1)} |u'(t)| dt \\
&= \omega_N^{-1} \int_{B_{r_0}(0)\setminus B_r(0)} |x|^{\beta} |u|^{(\alpha-1)} |\nabla u| dx \\
&\leq \omega_N^{-1} \Big(\int_{B_{r_0}(0)\setminus B_r(0)} |x|^{\beta} |u|^{(\alpha-1)} |\mathbf{v}| dx + \int_{B_{r_0}(0)\setminus B_r(0)} |x|^{\beta} |u|^{(\alpha-1)} |\mathbf{w}| dx \Big) \\
&\leq \omega_N^{-1} \Big[\|\mathbf{v}\|_{L^p(B_{r_0}(0)\setminus B_r(0))} \Big(\int_{B_{r_0}(0)\setminus B_r(0)} |x|^{\frac{\beta p}{(p-1)}} |u|^{\frac{(\alpha-1)p}{(p-1)}} dx \Big)^{\frac{(p-1)}{p}} \\
&\quad + \|\mathbf{w}\|_{L^{2p}(B_{r_0}(0)\setminus B_r(0))} \Big(\int_{B_{r_0}(0)\setminus B_r(0)} |x|^{\frac{\beta(2p)}{(2p-1)}} |u|^{\frac{(\alpha-1)(2p)}{(2p-1)}} dx \Big)^{\frac{(2p-1)}{2p}} \Big] \\
&\leq \omega_N^{-1} \|\mathbf{v}\|_{L^p(B_{r_0}(0)\setminus B_r(0))} \Big(\int_{B_{r_0}(0)\setminus B_r(0)} |x|^{a_0} |u|^{(\frac{(\alpha-1)p}{p-1})} dx \Big)^{(\frac{p-1}{p(\alpha-1)}(\alpha-1))} \\
&\quad + \omega_N^{-1} \|\mathbf{w}\|_{L^{2p}(B_{r_0}(0)\setminus B_r(0))} \Big(\int_{B_{r_0}(0)\setminus B_r(0)} |x|^{a_0} |u|^{(\frac{2p(\alpha-1)}{2p-1})} dx \Big)^{(\frac{2p-1}{2p(\alpha-1)}(\alpha-1))} \\
&\leq \omega_N^{-1} \|\nabla u\|_{p,2p(B_{r_0}(0)\setminus B_r(0))} \Big(\int_{B_{r_0}(0)\setminus B_r(0)} |x|^{a_0} |u|^{(\frac{(\alpha-1)p}{p-1})} dx \Big)^{(\frac{p-1}{p(\alpha-1)}(\alpha-1))} \\
&\leq \omega_N^{-1} M_0^{-(\frac{p-1}{p})} \|\nabla u\|_{p,2p(B_{r_0}(0)\setminus B_r(0))} \\
&\quad \times \Big(\int_{B_{r_0}(0)\setminus B_r(0)} T(|x|) |u|^{(\frac{(\alpha-1)p}{p-1})} dx \Big)^{(\frac{p-1}{p(\alpha-1)}(\alpha-1))} \\
&\leq \omega_N^{-1} M_1 \|\nabla u\|_{p,2p(B_{r_0}(0)\setminus B_r(0))} \Big(\int_{B_{r_0}(0)\setminus B_r(0)} T(|x|) |u|^{\alpha} dx \Big)^{(\frac{\alpha-1}{\alpha})} \\
&= \omega_N^{-1} M_1 \|\nabla u\|_{p,2p} \|u\|^{(\alpha-1)}_{L^{\alpha}_T(B_{r_0}(0)\setminus B_r(0))}.
\end{aligned}
$$

Since

$$
\beta + N - 1 \geq 0 \iff a_0 > -\frac{(N-1)}{(2p-1)} 2p.
$$

It follows that $\beta + N - 1 \leq 0$ implies $\beta - a_0 - 1 \geq (\frac{N-p}{p-1})$. Hence, from the above arguments, we have

$$
|u(x)| \leq \widetilde{M}_0 |x|^{-(\frac{2p(N-1)+a_0(2p-1)}{\alpha(2p)})} \|u\|_{\mathcal{W}_{\rm rad}}, \quad 0 < |x| \leq r_0 < 1,
$$

where the constant $\widetilde{M}_0 = \widetilde{M}_0(a_0, r_0, \alpha, N)$. □

Lemma 3.6. *Let $1 < \alpha < p < 2p < N$, $2p < s \leq \infty$. Then for any $0 < r < 1 < R < \infty$, the following embedding is compact*

$$
\mathcal{W}_{\rm rad}(B_R \setminus B_r) \hookrightarrow L^s(B_R \setminus B_r; K).
$$

The proof of the above lemma is similar to [19, Lemma 6].

Proof of Theorem 3.1. First we prove that the embedding is continuous. It is sufficient to show

$$S_{\rm rad}(T,K)=\inf_{u\in\mathcal{W}_{\rm rad}(\mathbb{R}^N)}\frac{\|\nabla u\|_{p,2p}+\|u\|_{L^\alpha_T(\mathbb{R}^N)}}{\|u\|_{L^s_K(\mathbb{R}^N)}}>0. \tag{3.8}$$

If not, assume that there exists $\{u_n\}\subset\mathcal{W}_{\rm rad}$ such that

$$\|\nabla u\|_{p,2p}+\|u\|_{L^\alpha_T(\mathbb{R}^N)}=o(1),\quad \text{as } n\to\infty, \tag{3.9}$$

$$\|u\|_{L^s_K(\mathbb{R}^N)}=1,\quad \text{for all } n\in\mathbb{N}. \tag{3.10}$$

It is a contradiction, if we have

$$\|u\|_{L^s_K(\mathbb{R}^N)}=0. \tag{3.11}$$

By (T1) and (K1), there exist $R_0>1>r_0>0$, for some M_0,

$$\begin{gathered} K(|x|)\le M_0|x|^b,\quad T(|x|)\ge M_0|x|^a,\quad \text{for } |x|\ge R_0,\\ K(|x|)\le M_0|x|^{b_0},\quad T(|x|)\ge M_0|x|^{a_0},\quad \text{for } 0<|x|\le r_0. \end{gathered} \tag{3.12}$$

For $R>R_0$ and $0<r<r_0$, we estimate the integrals $\big(\int_{B_r}K(|x|)|u_n|^s dx\big)^{1/s}$ and $\big(\int_{B^c_R}K(|x|)|u_n|^s dx\big)^{1/s}$ in different cases according to the definitions of s^* and s_*, B^c_R denotes the complement of B_R.

Firstly, we estimate the term $\Big(\int_{B_r}K(|x|)|u_n|^s dx\Big)^{1/s}$.

Case 1.1: For $a_0\ge -p$, $b_0\ge -p$. Let $s=\frac{2p(N+c)}{(N-2p)}$, by $s\le s^*$, we obtain $\eta_1=b_0-c\ge 0$. Hence by Lemma 3.3 and (3.9), we have

$$\begin{aligned} \Big(\int_{B_r}K(|x|)|u_n|^s dx\Big)^{1/s} &\le M_0^{1/s}\Big(\int_{B_r}|x|^{b_0}|u_n|^s dx\Big)^{1/s}\\ &\le M_0^{1/s}r^{(\frac{b_0-c}{s})}\Big(\int_{B_r}|x|^{c}|u_n|^s dx\Big)^{1/s}\\ &\le M_0^{1/s}r^{(\frac{b_0-c}{s})}\max\Big(\|\nabla u_n\|_{p,2p},\|\nabla u_n\|^2_{p,2p}\Big)\\ &= r^{(\frac{b_0-c}{s})}o(1),\quad \text{as } n\to\infty. \end{aligned} \tag{3.13}$$

Case 1.2: For $-p>a_0>-\frac{(N-1)}{(2p-1)}2p, b_0\ge a_0$. From $s\le s^*$, we obtain

$$\eta_2=b_0-a_0-(s-\alpha)\frac{2p(N-1)+a_0(2p-1)}{\alpha(2p)}\ge 0.$$

We choose a cut-off function ϕ such that $\phi=1$ for $0\le|x|\le\frac{r_0}{2}$, and $\phi=0$ for $|x|\ge r_0$. Then by Lemma 3.5, for $r<\frac{r_0}{2}$, we have

$$\begin{aligned} &\Big(\int_{B_r}K(|x|)|u_n|^s dx\Big)^{1/s}\\ &\le M_0^{1/s}\Big(\int_{B_r}|x|^{b_0}|\phi u_n|^s dx\Big)^{1/s}\\ &= M_0^{1/s}\Big(\int_{B_r}|x|^{(b_0-a_0)}|\phi u_n|^{(s-\alpha)}|x|^{a_0}|\phi u_n|^{\alpha}dx\Big)^{1/s}\\ &\le M_2\|\phi u_n\|_{\mathcal{W}}^{(\frac{s-\alpha}{s})}\Big(\int_{B_r}|x|^{(b_0-a_0-(s-\alpha)\frac{2p(N-1)+a_0(2p-1)}{\alpha(2p)})}T(|x|)|u_n|^{\alpha}dx\Big)^{1/s} \end{aligned}$$

$$
\begin{aligned}
&\le M_3 r^{(\frac{b_0-a_0}{s}-(s-\alpha)\frac{2p(N-1)+a_0(2p-1)}{s\alpha(2p)})}\|u_n\|_{\mathcal{W}_{\rm rad}}^{(\frac{s-\alpha}{s})}\|u_n\|_{L_T^\alpha(B_r)}^{\alpha/s}\\
&\le M_3' r^{(\frac{b_0-a_0}{s}-(s-\alpha)\frac{2p(N-1)+a_0(2p-1)}{s\alpha(2p)})}\|u_n\|_{\mathcal{W}_{\rm rad}}\\
&= r^{\left(\frac{b_0-a_0}{s}-(s-\alpha)\frac{2p(N-1)+a_0(2p-1)}{s\alpha(2p)}\right)}o(1), \quad \text{as } n\to\infty.
\end{aligned} \tag{3.14}
$$

Case 1.3: For $a_0 \le -\frac{(N-1)}{(p-1)}p$, $b_0 \ge a_0$, in the case $s^* = \infty$. For $\infty > s > \alpha$, it holds

$$
\eta_3 = b_0 - a_0 - (s-\alpha)\frac{2p(N-1)+a_0(2p-1)}{\alpha(2p)} \ge 0.
$$

With the same function ϕ given in Case 1.2, and $r < \frac{r_0}{2}$, by Lemma 3.5, we have

$$
\begin{aligned}
&\Big(\int_{B_r} K(|x|)|u_n|^s dx\Big)^{1/s}\\
&\le M_0^{1/s}\Big(\int_{B_r}|x|^{b_0}|\phi u_n|^s dx\Big)^{1/s}\\
&= M_0^{1/s}\Big(\int_{B_r}|x|^{(b_0-a_0)}|\phi u_n|^{(s-\alpha)}|x|^{a_0}|\phi u_n|^\alpha dx\Big)^{1/s}\\
&\le M_4\|\phi u_n\|_{\mathcal{W}_{\rm rad}}^{(\frac{s-\alpha}{s})}\Big(\int_{B_r}|x|^{(b_0-a_0-(s-\alpha)\frac{2p(N-1)+a_0(2p-1)}{\alpha(2p)})}T(|x|)|u_n|^\alpha dx\Big)^{1/s}\\
&\le M_5 r^{(\frac{b_0-a_0}{s}-(s-\alpha)\frac{2p(N-1)+a_0(2p-1)}{s\alpha(2p)})}\|u_n\|_{\mathcal{W}_{\rm rad}}^{(\frac{s-\alpha}{s})}\|u_n\|_{L_T^\alpha(B_r)}^{\frac{\alpha}{s}}\\
&\le M_5' r^{(\frac{b_0-a_0}{s}-(s-\alpha)\frac{2p(N-1)+a_0(2p-1)}{s\alpha(2p)})}\|u_n\|_{\mathcal{W}_{\rm rad}}\\
&= r^{(\frac{b_0-a_0}{s}-(s-\alpha)\frac{2p(N-1)+a_0(2p-1)}{s\alpha(2p)})}o(1), \quad \text{as } n\to\infty.
\end{aligned} \tag{3.15}
$$

Secondly, we estimate the term $\Big(\int_{B_R^c} K(|x|)|u_n|^s dx\Big)^{1/s}$.

Case 2.1: For $-p < a \le b$, by $s \ge s_*$, we obtain

$$
\lambda_1 = b - a - (s-\alpha)\frac{2p(N-1)+a(2p-1)}{\alpha(2p)} \le 0.
$$

Hence by Lemma 3.4 and (3.9), for $R > R_0 > 1$, we have

$$
\begin{aligned}
&\Big(\int_{B_R^c} K(|x|)|u_n|^s dx\Big)^{1/s}\\
&\le M_0^{1/s}\Big(\int_{B_R^c}|x|^{b}|u_n|^s dx\Big)^{1/s}\\
&= M_0^{1/s}\Big(\int_{B_R^c}|x|^{(b-a)}|u_n|^{(s-\alpha)}|x|^{a}|u_n|^\alpha dx\Big)^{1/s}\\
&\le M_6\|u_n\|_{\mathcal{W}_{\rm rad}}^{(\frac{s-\alpha}{s})}\Big(\int_{B_R^c}|x|^{(b-a-(s-\alpha)\frac{2p(N-1)+a(2p-1)}{\alpha(2p)})}T(|x|)|u_n|^\alpha dx\Big)^{1/s}\\
&\le M_7 R^{\frac{1}{s}(b-a-(s-\alpha)\frac{2p(N-1)+a(2p-1)}{\alpha(2p)})}\|u_n\|_{\mathcal{W}_{\rm rad}}^{(\frac{s-\alpha}{s})}\|u_n\|_{L_T^\alpha(B_r^c)}^{\frac{\alpha}{s}}\\
&\le M_7' R^{\frac{1}{s}(b-a-(s-\alpha)\frac{2p(N-1)+a(2p-1)}{\alpha(2p)})}\|u_n\|_{\mathcal{W}_{\rm rad}}\\
&= R^{\frac{1}{s}(b-a-(s-\alpha)\frac{2p(N-1)+a(2p-1)}{\alpha(2p)})}o(1), \quad \text{as } n\to\infty.
\end{aligned} \tag{3.16}
$$

Case 2.2: For $b \geq -p, a \leq -p$, let $s = \frac{2p(N+c)}{(N-2p)}$, by $s \geq s_*$, we obtain $\lambda_2 = b-c \leq 0$. Hence by Lemma 3.3, for $R > R_0 > 1$, we have

$$\begin{aligned} \Big(\int_{B_R^c} K(|x|)|u_n|^s dx\Big)^{1/s} &\leq M_0^{1/s}\Big(\int_{B_R^c} |x|^{(b-c)}|x|^c|u_n|^s dx\Big)^{1/s} \\ &\leq M_8 R^{(\frac{b-c}{s})} \max\big(\|\nabla u_n\|_{p,2p}, \|\nabla u_n\|_{p,2p}^2\big) \\ &\leq M_8' R^{(\frac{b-c}{s})}\|u_n\|_{\mathcal{W}_{\rm rad}} = R^{(\frac{b-c}{s})} o(1), \quad \text{as } n \to \infty. \end{aligned} \tag{3.17}$$

Case 2.3: For $b \leq \max\{a, -p\}$, $s > \alpha = s_*$. As for $R > R_0 > 1$, when $a > -p, b \leq a$, it always holds

$$\lambda_3 = b - a - (s-\alpha)\frac{2p(N-1) + a(2p-1)}{\alpha(2p)} < 0,$$

so similar to Case 2.1, we have

$$\begin{aligned} \Big(\int_{B_R^c} K(|x|)|u_n|^s dx\Big)^{1/s} &\leq M_7' R^{\frac{1}{s}(b-a-(s-\alpha)\frac{2p(N-1)+a(2p-1)}{\alpha(2p)})}\|u_n\|_{\mathcal{W}_{\rm rad}} \\ &= R^{\frac{1}{s}(b-a-(s-\alpha)\frac{2p(N-1)+a(2p-1)}{\alpha(2p)})} o(1), \quad \text{as } n \to \infty. \end{aligned} \tag{3.18}$$

and when $a \leq -p, b \leq -p \leq c$, let $s = \frac{2p(N+c)}{(N-2p)}$, we obtain $(b-c) \leq 0$, we have similar to Case 2.2 that

$$\Big(\int_{B_R^c} K(|x|)|u_n|^s dx\Big)^{1/s} \leq M_8' R^{(\frac{b-c}{s})}\|u_n\|_{\mathcal{W}_{\rm rad}} = R^{(\frac{b-c}{s})} o(1), \quad \text{as } n \to \infty. \tag{3.19}$$

Now we write

$$\begin{aligned} \int_{\mathbb{R}^N} K(|x|)|u_n|^s dx = \int_{B_r} K(|x|)|u_n|^s dx + \int_{B_R^c} K(|x|)|u_n|^s dx \\ + \int_{B_R \setminus B_r} K(|x|)|u_n|^s dx. \end{aligned}$$

As s^* is finite and $s_* \leq s \leq s^*$, by (3.13), (3.14), (3.16), (3.17), (3.18) and Lemma 3.6, we obtain that (3.11) holds. As s^* is infinite and $s_* \leq s < \infty$, by (3.15), (3.16), (3.17), (3.18) and Lemma 3.6, we obtain that (3.11) holds. Therefore the embedding is continuous in each case.

Now we show that the embedding obtained above is compact. Let $\{u_n\} \subset \mathcal{W}_{\rm rad}$ be such that

$$\|u_n\|_{\mathcal{W}_{\rm rad}} = \|\nabla u_n\|_{p,2p} + \|u_n\|_{L_T^\alpha(\mathbb{R}^N)} \leq M. \tag{3.20}$$

Without loss of generality, we consider

$$u_n \rightharpoonup 0, \quad \text{in } \mathcal{W}_{\rm rad} \text{ as } n \to \infty. \tag{3.21}$$

To obtain the compactness, we only need to show that

$$\lim_{n\to\infty} \Big(\int_{\mathbb{R}^N} K(|x|)|u_n|^s dx\Big)^{1/s} = 0. \tag{3.22}$$

As $s_* < s < s^*$, the exponents η_i of r in the estimates (3.13), (3.14), (3.15) are strictly positive, and the exponents λ_j of R in the estimates (3.16), (3.17), (3.18) are strictly negative, we obtain the following estimates by similar arguments as above

$$\Big(\int_{B_r} K(|x|)|u_n|^s dx\Big)^{1/s} \leq M r^{\eta_i}\|u_n\|_{\mathcal{W}_{\rm rad}}, \quad i = 1, 2, 3, \tag{3.23}$$

$$\Big(\int_{B_R^c} K(|x|)|u_n|^s dx\Big)^{1/s} \le MR^{\lambda_j}\|u_n\|_{\mathcal{W}_{\rm rad}}, \quad j=1,2,3, \tag{3.24}$$

By (3.20), (3.23), (3.24) and Lemma 3.6, we obtain (3.22). Hence the embedding is compact in each case. In conclusion, the proof of Theorem 3.1 is complete. □

4. Proof of Theorems 1.2 and 1.3

In this section, we prove our main theorems. Now, let us define the functional $J:\mathcal{W}_{\rm rad}\to\mathbb{R}$ as:

$$J(u)=\frac{1}{p}\int_{\mathbb{R}^N}(\sqrt{1+|\nabla u|^{2p}}-1)dx+\frac{1}{\alpha}\int_{\mathbb{R}^N}T(|x|)|u|^\alpha dx-\frac{1}{s}\int_{\mathbb{R}^N}K(|x|)|u|^s dx. \tag{4.1}$$

Obviously, by [5, Lemma 2.2], the functional J is well defined and it is of class $\mathcal{C}^1$. We obtain that solutions of (1.1) are critical points of the functional J. By Remark 2.5 and using the standard Palais' result [14], we infer that $\mathcal{W}_{\rm rad}$ is a natural constraint for the functional J.

In the following propositions and lemmas, we show that the functional J satisfies the geometrical assumptions $\mathbb{Z}_2$-symmetric version of the Mountain Pass Lemma [16]. More precisely, we have the following result.

Proposition 4.1. *The functional J satisfies the following properties:*

(i) $J(0)=0$;
(ii) *there exist $\rho,\bar c$ such that $J(u)\ge\bar c$, for any $u\in\mathcal{W}_{\rm rad}$ with $\|u\|_{W_{\rm rad}}=\rho$;*
(iii) *there exists $\bar u\in\mathcal{W}_{\rm rad}$ such that $J(\bar u)\le 0$.*

Proof. (i) Trivially, $J(0)=0$. (ii) As there exists a positive constant c such that

$$c|\nabla u|^p\le\sqrt{1+|\nabla u|^{2p}}-1,\quad \text{if } |\nabla u|\ge 1,$$
$$c|\nabla u|^{2p}\le\sqrt{1+|\nabla u|^{2p}}-1,\quad \text{if } 0\le|\nabla u|\le 1.$$

Then, if $\|u\|_{\mathcal{W}_{\rm rad}}$ is sufficiently small, by $\alpha<p<2p<s$, Proposition 2.2 (iv), and since $\mathcal{W}_{\rm rad}\hookrightarrow L^s(\mathbb{R}^N;K)$, we have that

$$\begin{aligned}
J(u)&\ge c_1\int_{\Lambda^c_{\nabla u}}|\nabla u|^{2p}dx+c_2\int_{\Lambda_{\nabla u}}|\nabla u|^p dx+\frac{1}{\alpha}\int_{\mathbb{R}^N}T(|x|)|u|^\alpha dx\\
&\quad-\frac{1}{s}\int_{\mathbb{R}^N}K(|x|)|u|^s dx\\
&\ge c\max\Big(\int_{\Lambda^c_{\nabla u}}|\nabla u|^{2p}dx,\int_{\Lambda_{\nabla u}}|\nabla u|^p dx\Big)+\frac{1}{\alpha}\int_{\mathbb{R}^N}T(|x|)|u|^\alpha dx\\
&\quad-\frac{1}{s}\int_{\mathbb{R}^N}K(|x|)|u|^s dx\\
&\ge c\big[\|\nabla u\|^{2p}_{p,2p}+\|u\|^\alpha_{L^\alpha_T(\mathbb{R}^N)}-\|u\|^s_{L^s_K(\mathbb{R}^N)}\big]\\
&\ge c\big[\|u\|^{2p}_{\mathcal{W}_{\rm rad}}-\|u\|^s_{\mathcal{W}_{\rm rad}}\big]\ge\bar c.
\end{aligned}$$

(iii) Let $u\in C_c^\infty(\mathbb{R}^N,\mathbb{R})$, as there exists a positive constant C such that

$$\begin{cases}(\sqrt{1+|\nabla u|^{2p}}-1)\le C|\nabla u|^p, & \text{if } |\nabla u|\ge 1,\\ C|\nabla u|^{2p}, & \text{if } 0\le|\nabla u|\le 1;\end{cases}$$

then for all $t > 0$, we obtain

$$\begin{aligned} J(tu) &\le C_1 \int_{\Lambda^c_{\nabla(tu)}} |\nabla(tu)|^{2p} dx + C_2 \int_{\Lambda_{\nabla(tu)}} |\nabla(tu)|^p dx \\ &\quad + \frac{1}{\alpha} \int_{\mathbb{R}^N} T(|x|)|tu|^\alpha dx - \frac{1}{s} \int_{\mathbb{R}^N} K(|x|)|tu|^s dx \\ &\le C\Big[t^{2p} \int_{\mathbb{R}^N} |\nabla u|^{2p} dx + t^p \int_{\mathbb{R}^N} |\nabla u|^p dx \\ &\quad + t^\alpha \int_{\mathbb{R}^N} T(|x|)|u|^\alpha dx - t^s \int_{\mathbb{R}^N} K(|x|)|u|^s dx\Big]. \end{aligned}$$

Therefore, for t sufficiently large, there exists $u_0 = tu$ such that $J(u_0) = J(tu) < 0$. ☐

Proposition 4.2. *The functional $J|_{\mathcal{W}_{\rm rad}}$ satisfies the (PS) condition.*

Proof. Let $\{u_n\}_n \subset \mathcal{W}_{\rm rad}$ be a (PS)-sequence for the J, namely for a suitable $\overline{c} \in \mathbb{R}$

$$J(u_n) \to \overline{c} \quad \text{and} \quad J'(u_n) \to 0 \quad \text{in } \mathcal{W}'_{\rm rad}.$$

Let us check that $\{u_n\}_n$ is bounded. In fact, as there exists $0 < \mu < 1$ such that

$$\frac{|\nabla u|^{2p}}{\sqrt{1 + |\nabla u|^{2p}}} \le \frac{s\mu}{2}(\sqrt{1 + |\nabla u|^{2p}} - 1), \quad \text{for all } t \ge 0,$$

then we have

$$\overline{c} + o_n(1)\|u_n\| = J(u_n) - \frac{1}{s} J'(u_n)u_n;$$

i.e.,

$$\begin{aligned} &\overline{c} + o_n(1)\|u_n\| \\ &= \int_{\mathbb{R}^N} \big[\frac{1}{p}(\sqrt{1 + |\nabla u_n|^{2p}} - 1) - \frac{1}{s} \frac{|\nabla u|^{2p}}{\sqrt{1 + |\nabla u|^{2p}}}\big] dx + \big(\frac{1}{\alpha} - \frac{1}{s}\big) \int_{\mathbb{R}^N} T(|x|)|u_n|^\alpha dx \\ &\ge \frac{(2 - \mu p)}{2p} \int_{\mathbb{R}^N} (\sqrt{1 + |\nabla u_n|^{2p}} - 1) dx + \big(\frac{1}{\alpha} - \frac{1}{s}\big) \int_{\mathbb{R}^N} T(|x|)|u_n|^\alpha dx \\ &\ge c\big[\min\big(\|\nabla u_n\|^{2p}_{p,2p}, \|\nabla u_n\|^p_{p,2p}\big) + \|u_n\|^\alpha_{L^\alpha_T(\mathbb{R}^N)}\big]. \end{aligned}$$

Therefore, by Theorem 3.1, there exists $u_0 \in \mathcal{W}_{\rm rad}$ such that

$$u_n \rightharpoonup u_0, \text{ weakly in } \mathcal{W}_{\rm rad}, \tag{4.2}$$

$$u_n \to u_0, \quad \text{strongly in } L^s(\mathbb{R}^N; K), \tag{4.3}$$

$$u_n \to u_0, \quad \text{a.e. in } \mathbb{R}^N. \tag{4.4}$$

Inspired by [11], we write $J(u) = A(u) - B(u)$, where $A(u) = A_1(u) + A_2(u)$ and

$$A_1(u) = \frac{1}{p} \int_{\mathbb{R}^N} (\sqrt{1 + |\nabla u|^{2p}} - 1) dx, \quad A_2(u) = \frac{1}{\alpha} \int_{\mathbb{R}^N} T(|x|)|u|^\alpha dx,$$

$$B(u) = \frac{1}{s} \int_{\mathbb{R}^N} K(|x|)|u|^s dx.$$

Then, we have

$$A(u_n) - B(u_n) \to \overline{c}, \quad A'(u_n) - B'(u_n) \to 0, \quad \text{in } \mathcal{W}'_{\rm rad}.$$

By (4.3), we infer that

$$B(u_n) \to B(u_0), \quad B'(u_n) \to B'(u_0), \quad \text{in } \mathcal{W}'_{\rm rad}.$$

Therefore,

$$A'(u_n) \to B'(u_0) \quad \text{in } \mathcal{W}'_{\rm rad}. \tag{4.5}$$

Since $A_1(u)$ and $A_2(u)$ are convex, so $A(u)$ is convex, we have

$$A(u_0) \geq A(u_n) + A'(u_n)(u_0 - u_n),$$

namely

$$A(u_n) \leq A(u_0) + A'(u_n)(u_n - u_0).$$

So, by (4.2) and (4.5), we obtain $\limsup_{n\to\infty} A(u_n) \leq A(u_0)$. Since A is convex and continuous, we obtain A is lower weak semicontinuity

$$A(u_0) \leq \liminf_{n\to\infty} A(u_n);$$

therefore,

$$A(u_n) \to A(u_0), \quad \text{as } n \to \infty. \tag{4.6}$$

By (4.2) and arguing as in [10, page 208], we have

$$\nabla u_n \rightharpoonup \nabla u_0, \quad \text{weakly in } L^p(\mathbb{R}^N) + L^{2p}(\mathbb{R}^N), \tag{4.7}$$

$$u_n \rightharpoonup u_0, \quad \text{weakly in } L^\alpha(\mathbb{R}^N; T), \tag{4.8}$$

and A_1 and A_2 are lower weak semicontinuity, we have

$$A_1(u_0) \leq \liminf_{n\to\infty} A_1(u_n), \quad A_2(u_0) \leq \liminf_{n\to\infty} A_2(u_n).$$

Thus, together with (4.6), we obtain

$$A_1(u_0) = \liminf_{n\to\infty} A_1(u_n), \tag{4.9}$$

$$A_2(u_0) = \liminf_{n\to\infty} A_2(u_n). \tag{4.10}$$

Then (4.8) and (4.10), imply

$$u_n \to u_0, \quad \text{in } L^\alpha(\mathbb{R}^N; T).$$

Moreover, by (4.7) and (4.9) and by [5, Lemma 2.3], we have

$$\nabla u_n \to \nabla u_0, \quad \text{in } L^p(\mathbb{R}^N) + L^{2p}(\mathbb{R}^N).$$

Therefore, $u_n \to u_0$ in $\mathcal{W}_{\rm rad}$. □

Proof of Theorem 1.2. By the $\mathbb{Z}_2$-symmetric version of the Mountain Pass Lemma, we only need to prove that there exist $\{V_n\}_n$, a sequence of finite dimensional subspaces of $\mathcal{W}_{\rm rad}$ with $\dim V_n = n$ and $V_n \subset V_{n+1}$, and $\{R_n\}_n$, a sequence of positive numbers, such that $J(u) \leq 0$ for all $u \in V_n \backslash B_{R_n}$.

Let $\{\phi_n\}_n$ be a sequence of radially symmetric test functions such that, for any $n \geq 1$, the functions $\phi_1, \phi_2, \dots, \phi_n$ are linearly independent. Denote by $V_n = \operatorname{span}\{\phi_1, \phi_2, \dots, \phi_n\} \subset (C_c^\infty(\mathbb{R}^N, \mathbb{R}))_{\rm rad} \subset \mathcal{W}_{\rm rad}$.

By the proof of Proposition 4.1 (iii), and since V_n is a finite dimensional space of test functions, so the norms in V_n are equivalent, and we conclude observing that, if $u \in V_n \backslash B_{R_n}$ and R_n is sufficiently large,

$$\begin{aligned}
J(u) &\leq C\big[\|\nabla u\|_{p,2p}^{2p} + \|T(|x|)u\|_\alpha^\alpha - \|K(|x|)u\|_s^s\big] \\
&\leq C\big[\|u\|_{\mathcal{W}_{\rm rad}}^{2p} + \|u\|_{\mathcal{W}_{\rm rad}}^\alpha - \|u\|_{\mathcal{W}_{\rm rad}}^s\big] \\
&\leq C[R_n^{2p} + R_n^\alpha - R_n^s] \leq 0.
\end{aligned}$$

So J satisfies the $\mathbb{Z}_2$-symmetric version of the Mountain Pass Lemma [16], and problem (1.1) has infinitely many radially symmetric solutions. $\square$

To obtain a ground state solution in $\mathcal{W}_{\rm rad}$, we need the following lemmas.

Let us denote with $\mathcal{M}$ the set of all nontrivial solutions of (1.1) in $\mathcal{W}_{\rm rad}$, namely

$$\mathcal{M} = \{u \in \mathcal{W}_{\rm rad}\backslash\{0\} | J'(u) = 0\}.$$

Obviously, we know that $\mathcal{M} \neq \emptyset$.

Lemma 4.3. *There exists a positive constant $\bar{c} > 0$, such that $\|u\| \geq \bar{c}$, for all $u \in \mathcal{M}$.*

Proof. As $J'(u) = 0$, namely

$$\int_{\mathbb{R}^N} \frac{|\nabla u|^{2p}}{\sqrt{1 + |\nabla u|^{2p}}} dx + \int_{\mathbb{R}^N} T(|x|)|u|^\alpha dx - \int_{\mathbb{R}^N} K(|x|)|u|^s dx = 0.$$

Since there exists a positive constant c such that

$$c|\nabla u|^{(p-2)} \leq \begin{cases} \frac{|\nabla u|^{(2p-2)}}{\sqrt{1+|\nabla u|^{2p}}}, & \text{if } |\nabla u| \geq 1, \\ \frac{|\nabla u|^{(2p-2)}}{\sqrt{1+|\nabla u|^{2p}}}, & \text{if } 0 \leq |\nabla u| \leq 1; \end{cases}$$

we have

$$\begin{aligned} \|u\|^s_{L^s_K(\mathbb{R}^N)} &= \int_{\mathbb{R}^N} \frac{|\nabla u|^{2p}}{\sqrt{1 + |\nabla u|^{2p}}} dx + \int_{\mathbb{R}^N} T(|x|)|u|^\alpha dx \\ &\geq c\ \max\Big(\int_{\Lambda^c_{\nabla u}} |\nabla u|^{2p} dx, \int_{\Lambda_{\nabla u}} |\nabla u|^p dx\Big) + \int_{\mathbb{R}^N} T(|x|)|u|^\alpha dx \\ &\geq c\big[\|\nabla u\|^{2p}_{p,2p} + \|u\|^\alpha_{L^\alpha_T(\mathbb{R}^N)}\big] \\ &\geq c\|u\|^{2p}_{W_{\rm rad}} \geq c\|u\|^{2p}_{L^s_K(\mathbb{R}^N)}. \end{aligned}$$

$\square$

Lemma 4.4. *There exists a positive constant $\bar{c} > 0$, such that $J(u) \geq \bar{c}$, for all $u \in \mathcal{M}$*

Proof. Let $u \in \mathcal{M}$. Repeating the arguments of the proof of Proposition 4.2 and by Lemma 4.3, we have

$$J(u) = J(u) - \frac{1}{s} J'(u)u \geq c\big[\min(\|\nabla u\|^{2p}_{p,2p}, \|\nabla u\|^p_{p,2p}) + \|u\|^\alpha_{L^\alpha_T(\mathbb{R}^N)}\big] \geq \bar{c}.$$

$\square$

Remark 4.5. By Lemma 4.4, we infer that

$$\tau = \inf_{u\in\mathcal{M}} J(u) > 0,$$

and by Theorem 1.3, we obtain that this infimum is achieved.

Proof of Theorem 1.3. Let $\{u_n\}_n \subset \mathcal{M}$ be a minimizing sequence, namely

$$J(u_n) \to \tau \quad \text{and} \quad J'(u_n) = 0.$$

Then $\{u_n\}_n$ is a (PS)-sequence for the functional J and we obtain the result by means of Proposition 4.2. $\square$

Remark 4.6. As special case, our result can be applied to mean curvature equation or the capillary equation

$$-\operatorname{div}\Big(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\Big)+T(|x|)|u|^{\alpha-2}u=K(|x|)|u|^{s-2}u,\quad u>0,\ x\in\mathbb{R}^N,$$
$$u(|x|)\to 0,\quad \text{as } |x|\to\infty.$$

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