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SOLVABILITY OF FRACTIONAL ANALOGUES OF THE NEUMANN PROBLEM FOR A NONHOMOGENEOUS BIHARMONIC EQUATION

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Abstract. In this article we study the solvability of some boundary value problems for inhomogenous biharmobic equations. As a boundary operator we consider the differentiation operator of fractional order in the Miller-Ross sense. This problem is a generalization of the well known Neumann problems.

1. Introduction

Biharmonic equations appear in the study of mathematical models in several real-life processes as, among others, radar imaging [3] or incompressible flows [11]. Omitting a huge amount of works devoted to the study of this kind of equations, we refer some of them regarding to their used methods. Difference schemes and variational methods were used in the works [2, 10]. By using numerical and iterative methods, Dirichlet and Neumann boundary problems for biharmonic equations were studied in the papers [7, 8]. There are some works, for example [12], where a computational method, based on the use of Haar wavelets was used for solving 2D and 3D Poisson and biharmonic equations. We also point out the work made in [9], where regularity of solutions for nonlinear biharmonic equations was investigated. In [4] and the dissertation [13] various problems for complex biharmonic and polyharmonic equations were investigated.

In this article we refer to the domain $\Omega = \{x \in R^n : |x| < 1\}$, as the unit ball. The dimension of the space is $n \geq 3$, and it is denoted $\partial\Omega = \{x \in R^n : |x| = 1\}$ as the unit sphere. The usual Euclidean norm is written as $|x|^2 = x_1^2 + x_2^2 + \cdots + x_n^2$. Now, for any $u : \Omega \to R$ smooth enough function and a given $\alpha > 0$, denoting by $r = |x|$ and $\theta = x/|x|$, the appropriate integral operator of order α in the Riemann-Liouville sense can be defined, in a sense to ([20], p.69), by the following expression

$$J^\alpha[u](x) = \frac{1}{\Gamma(\alpha)} \int_0^r (r-\tau)^{\alpha-1} u(\tau\theta)\, d\tau.$$

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In what follows, we assume that $J^0[u](x) = u(x)$ for all $x \in \Omega$. Let $m-1 < \alpha \leq m$, $m = 1, 2, \dots$. The following expressions

$$ {}_{RL}D^\alpha[u](x) = \frac{d^m}{dr^m} J^{m-\alpha}[u](x), \quad {}_C D^\alpha[u](x) = J^{m-\alpha}[\frac{d^m u}{dr^m}](x), $$

are called, respectively, derivatives of α order in Riemann-Liouville and Caputo sense [20]. Here $\frac{d}{dr}$ is a differentiation operator of the form

$$ \frac{d}{dr} = \sum_{i=1}^n \frac{x_i}{r} \frac{\partial}{\partial x_i}, \quad \frac{d^k}{dr^k} = \frac{d}{dr}(\frac{d^{k-1}}{dr^{k-1}}), \quad k = 2, 3, \dots. $$

Let the parameter j take one of the values, $j = 0, 1, \dots, m$ and consider the set of operators

$$ D_j^\alpha[u](x) = \frac{d^{m-j}}{dr^{m-j}} J^{m-\alpha} \frac{d^j}{dr^j} u(x). $$

If $j \geq 1$ and $D = \frac{d}{dr}$, then

$$ D_j^\alpha = \underbrace{D \cdot D \cdot \dots \cdot D}_{m-j} \cdot {}_C D^{\alpha-j}. $$

This operator is called derivative of α order in Miller-Ross sense [24]. Denote

$$ B_j^\alpha u(x) = r^\alpha D_j^\alpha u(x), $$

$$ B^{-\alpha} u(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^{-\alpha} u(sx) ds. $$

Let $0 < \alpha \leq 2$. Consider the following problems in the domain Ω.

Problem 1.1. Let $0 < \alpha < 2$. Find a function $u(x) \in C^4(\Omega) \cap C(\bar{\Omega})$ such that $B_1^{\alpha+k}[u](x) \in C(\bar{\Omega})$, $k = 0, 1$ satisfying the equation

$$ \Delta^2 u(x) = g(x), \quad x \in \Omega, \tag{1.1} $$

and the boundary value conditions:

$$ D_1^\alpha[u](x) = f_1(x), \quad x \in \partial\Omega, \tag{1.2} $$

$$ D_1^{\alpha+1}[u](x) = f_2(x), x \in \partial\Omega. \tag{1.3} $$

Problem 1.2. Let $1 < \alpha \leq 2$. Find a function $u(x) \in C^4(\Omega) \cap C(\bar{\Omega})$ such that $B_2^{\alpha+k}[u](x) \in C(\bar{\Omega})$, $k = 0, 1$ satisfying equation (1.1) and the boundary value condition:

$$ D_2^\alpha[u](x) = f_1(x), \quad x \in \partial\Omega, \tag{1.4} $$

$$ D_2^{\alpha+1}[u](x) = f_2(x), \quad x \in \partial\Omega. \tag{1.5} $$

Note that the boundary value problems with boundary operators of fractional order for elliptic equations of the second order have been studied in [5, 17, 21, 22, 25, 26, 27, 28, 29, 32, 33]. Moreover, in [6] for the equation (1.1) the boundary-value problem with the conditions

$$ D_0^\alpha[u](x) = f_1(x), D_0^{\alpha+1}[u](x) = f_2(x), \quad x \in \partial\Omega, $$

$0 < \alpha \leq 1$ has been studied.

Note that for all $x \in \partial\Omega$ we have the equality $r\frac{du}{dr} = \frac{du}{dr} = \frac{du}{d\nu}$, where ν is a vector of outward normal to $\partial\Omega$. It is well known (see e.g. [15]) that for all

$x \in \partial\Omega$ the operator $r\frac{d}{dr}(r\frac{d}{dr}-1)\dots(r\frac{d}{dr}-k+1)$ coincides with the operator $\frac{d^k}{d\nu^k}$, $k=1,2,\dots$. Then in the case of $\alpha=1$ for all $x\in\partial\Omega$ we obtain

$$D_1^1 u(x) = \frac{du(x)}{dr} = \frac{du}{d\nu}, r^2 D_j^2 u(x) = r^2\frac{d^2u(x)}{dr^2} = r\frac{d}{dr}\Big(r\frac{d}{dr}-1\Big)u(x).$$

Consequently, for values $\alpha=1$ or $\alpha=2$, problems 1.1 and 1.2 are analogues of the Neumann problem for the equation (1.1). The considered problems in the case of $\alpha=1$ have been studied in [16], and in the case of $\alpha=2$ in [30]. It is proved that in the case of $\alpha=1$ for solvability of the problem the following conditions are necessary and sufficient:

$$\frac{1}{2}\int_\Omega (1-|x|^2)g(x)dx = \int_{\partial\Omega}[f_2(x)-f_1(x)]ds_x, \tag{1.6}$$

and in the case of $\alpha=2$,

$$\frac{1}{2}\int_\Omega (1-|x|^2)\Gamma_3[g(x)]dx = \int_{\partial\Omega} f_2(x)dS_x, \tag{1.7}$$

$$\frac{1}{2}\int_\Omega x_k(1-|x|^2)\Gamma_4[g](x)dx = \int_{\partial\Omega} x_k[f_2(x)-f_1(x)]dS_x, \quad k=1,2,\dots,n, \tag{1.8}$$

where $\Gamma_c[u](x) = (r\frac{\partial}{\partial r}+c)u(x)$, $c>0$.

Note that the Neumann problem in the case of polyharmonic equation was studied in [18, 19, 31].

2. Properties of the operators B_j^α and $B^{-\alpha}$

We assume that the function $u(x)$ is smooth enough in the domain Ω. The following proposition can be proved by direct calculation.

Lemma 2.1. *Let* $v_1(x) = r\frac{du(x)}{dr}, v_2(x) = r\frac{d}{dr}(r\frac{d}{dr}-1)u(x)$. *Then the following equalities hold:*

$$v_1(0) = v_2(0) = 0, \tag{2.1}$$

$$\frac{\partial v_2}{\partial x_k}(0) = 0, \quad k=1,2,\dots,n. \tag{2.2}$$

Similar propositions hold for the function $B_j^\alpha[u](x)$, $j=0,1$.

Lemma 2.2. *Let* $0<\alpha\le 2$. *Then the following equalities hold:*

$$B_1^\alpha[u](0) = 0, \tag{2.3}$$

$$B_2^\alpha[u](0) = 0, \frac{\partial B_2^\alpha[u](0)}{\partial x_i} = 0, \quad i=1,2,\dots,n. \tag{2.4}$$

Proof. Let $0<\alpha<1$. Then by the definition of the operator B_1^α for the function $B_1^\alpha[u](x)$ we have

$$\begin{aligned}
&B_1^\alpha[u](x)\\
&= \frac{r^\alpha}{\Gamma(1-\alpha)}\int_0^r (r-\tau)^{-\alpha}\frac{du}{d\tau}(\tau\theta)d\tau = \frac{r^\alpha}{\Gamma(2-\alpha)}\frac{d}{dr}\int_0^r (r-\tau)^{1-\alpha}\frac{du}{d\tau}(\tau\theta)d\tau\\
&= \frac{r^\alpha}{\Gamma(1-\alpha)}\frac{d}{dr}\Big[\frac{(r-\tau)^{1-\alpha}}{1-\alpha}u(\tau\theta)\Big|_{\tau=0}^{\tau=r} + \int_0^r (r-\tau)^{-\alpha}u(\tau\theta)d\tau\Big]\\
&= \frac{r^\alpha}{\Gamma(1-\alpha)}\frac{d}{dr}\Big[-\frac{r^{1-\alpha}}{1-\alpha}u(0) + r^{1-\alpha}\int_0^1 (1-\xi)^{-\alpha}u(\xi x)d\xi\Big]
\end{aligned}$$

$$= -\frac{u(0)}{\Gamma(1-\alpha)} + (1-\alpha)u_1(x) + r\frac{du_1(x)}{dr},$$

where

$$u_1(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^1 (1-\xi)^{-\alpha}u(\xi x)d\xi.$$

Therefore,

$$B_1^\alpha[u](x) = -\frac{u(0)}{\Gamma(1-\alpha)} + (1-\alpha)u_1(x) + r\frac{du_1(x)}{dr}, \quad x \in \Omega. \tag{2.5}$$

Hence, by equality (2.1) we obtain

$$\begin{aligned}
\lim_{x\to 0} B_1^\alpha[u](x) &= -\frac{u(0)}{\Gamma(1-\alpha)} + (1-\alpha)\lim_{x\to 0} u_1(x) + \lim_{x\to 0} r\frac{du_1(x)}{dr} \\
&= -\frac{u(0)}{\Gamma(1-\alpha)} + \frac{(1-\alpha)u(0)}{\Gamma(1-\alpha)}\int_0^1 (1-\xi)^{-\alpha}d\xi \\
&= -\frac{u(0)}{\Gamma(1-\alpha)} + \frac{(1-\alpha)u(0)}{\Gamma(2-\alpha)} = 0.
\end{aligned}$$

Equality (2.3) is proved for the case $0 < \alpha < 1$.

No let $1 < \alpha < 2$ and $j = 1$. Then by definition of B_1^α we have

$$\begin{aligned}
B_1^\alpha[u](x) &= \frac{r^\alpha}{\Gamma(2-\alpha)}\frac{d}{dr}\int_0^r (r-\tau)^{1-\alpha}\frac{du}{d\tau}(\tau\theta)d\tau \\
&= \frac{r^\alpha}{\Gamma(1-\alpha)}\frac{d^2}{dr^2}\int_0^r \frac{(r-\tau)^{2-\alpha}}{(2-\alpha)}\frac{du}{d\tau}(\tau\theta)d\tau \\
&= \frac{r^\alpha}{\Gamma(2-\alpha)}\frac{d^2}{dr^2}\Big[\frac{(r-\tau)^{2-\alpha}}{2-\alpha}u(\tau\theta)\Big|_{\tau=0}^{\tau=r} + \int_0^r (r-\tau)^{1-\alpha}u(\tau\theta)d\tau\Big] \\
&= \frac{r^\alpha}{\Gamma(2-\alpha)}\frac{d^2}{dr^2}\Big[-\frac{r^{2-\alpha}}{2-\alpha}u(0) + r^{2-\alpha}\int_0^1 (1-\xi)^{1-\alpha}u(\xi x)d\xi\Big] \\
&= -\frac{(1-\alpha)u(0)}{\Gamma(2-\alpha)} + (1-\alpha)(2-\alpha)u_2(x) + 2(2-\alpha)r\frac{du_2(x)}{dr} + r^2\frac{d^2}{dr^2}u_2(x),
\end{aligned}$$

where

$$u_2(x) = \frac{1}{\Gamma(2-\alpha)}\int_0^1 (1-\xi)^{1-\alpha}u(\xi x)d\xi.$$

Therefore,

$$\begin{aligned}
B_1^\alpha[u](x) = &-\frac{(1-\alpha)}{\Gamma(2-\alpha)}u(0) + (1-\alpha)(2-\alpha)u_2(x) \\
&+ 2(2-\alpha)r\frac{du_2(x)}{dr} + r\frac{d}{dr}(r\frac{d}{dr}-1)u_2(x), x \in \Omega.
\end{aligned} \tag{2.6}$$

Then, taking into account the equalities (2.1) and (2.2), we obtain

$$\begin{aligned}
\lim_{x\to 0} B_1^\alpha[u](x) &= -\frac{(1-\alpha)}{\Gamma(2-\alpha)}u(0) + (1-\alpha)(2-\alpha)\lim_{x\to 0} u_2(x) \\
&= -\frac{(1-\alpha)u(0)}{\Gamma(2-\alpha)} + \frac{(1-\alpha)(2-\alpha)u(0)}{\Gamma(2-\alpha)}\int_0^1 (1-\xi)^{1-\alpha}d\xi \\
&= -\frac{(1-\alpha)u(0)}{\Gamma(2-\alpha)} + \frac{(1-\alpha)(2-\alpha)u(0)}{\Gamma(3-\alpha)} = 0.
\end{aligned}$$

Equality (2.3) is proved for the case $1 < \alpha < 2, j = 1$.

Now we turn to the proof of the first equality of (2.4). By definition of B_2^α we have

$$\begin{aligned} B_2^\alpha[u](x) &= \frac{r^\alpha}{\Gamma(2-\alpha)} \int_0^r (r-\tau)^{1-\alpha} \frac{d^2 u}{d\tau^2}(\tau\theta) d\tau \\ &= \frac{r^\alpha}{\Gamma(1-\alpha)} \frac{d^2}{dr^2} \int_0^r \frac{(r-\tau)^{3-\alpha}}{(2-\alpha)(3-\alpha)} \frac{d^2 u}{d\tau^2}(\tau\theta) d\tau \\ &= -\frac{(1-\alpha)u(0)}{\Gamma(2-\alpha)} - \frac{r}{\Gamma(2-\alpha)} \frac{du(0)}{dr} + (1-\alpha)(2-\alpha) u_2(x) \\ &\quad + 2(2-\alpha) r \frac{du_2(x)}{dr} + r^2 \frac{d^2 u_2(x)}{dr^2}. \end{aligned}$$

Thus,

$$\begin{aligned} B_2^\alpha[u](x) = -\frac{(1-\alpha)u(0)}{\Gamma(2-\alpha)} - \frac{r}{\Gamma(2-\alpha)} \frac{du(0)}{dr} + (1-\alpha)(2-\alpha) u_2(x) \\ + 2(2-\alpha) r \frac{du_2(x)}{dr} + r \frac{d}{dr}\Big(r \frac{d}{dr} - 1\Big) u_2(x), \quad x \in \Omega. \end{aligned} \tag{2.7}$$

Equalities (2.1) imply

$$r \frac{du_2(x)}{dr}\Big|_{x=0} = 0, \quad r \frac{d}{dr}\Big(r \frac{d}{dr} - 1\Big) u_2(x)\Big|_{x=0} = 0.$$

Then from representation (2.1) we obtain

$$\lim_{x\to 0} B_2^\alpha[u](x) = -\frac{(1-\alpha)u(0)}{\Gamma(2-\alpha)} + \frac{(1-\alpha)(2-\alpha)u(0)}{\Gamma(2-\alpha)} \frac{\Gamma(2-\alpha)\Gamma(1)}{\Gamma(3-\alpha)} = 0.$$

Further, we denote $y_i = \tau\theta_i$, $i = 1, 2, \dots, n$. Then

$$\frac{du(\tau\theta)}{d\tau} = \sum_{i=1}^n \frac{\partial u(\tau\theta)}{\partial y_i} \frac{dy_i}{d\tau} = \sum_{i=1}^n \theta_i \frac{\partial u(\tau\theta)}{\partial y_i}.$$

Since $\theta = x/r, \theta_i = x_i/r$, it follows that

$$\frac{r}{\Gamma(2-\alpha)} \frac{du(0)}{d\tau} = \frac{r}{\Gamma(2-\alpha)} \sum_{i=1}^n \frac{x_i}{r} \frac{\partial u(\tau\theta)}{\partial y_i}\Big|_{\tau=0} = \frac{1}{\Gamma(2-\alpha)} \sum_{i=1}^n x_i \frac{\partial u(0)}{\partial y_i}.$$

Thus, for any $k = 1, 2, \dots, n$,

$$\frac{\partial}{\partial x_k}\Big[-\frac{r}{\Gamma(2-\alpha)} \frac{du(0)}{d\tau}\Big] = -\frac{1}{\Gamma(2-\alpha)} \frac{\partial u(0)}{\partial y_k}.$$

It is obvious that

$$\frac{\partial}{\partial x_k}\big[-\frac{1-\alpha}{\Gamma(2-\alpha)} u(0)\big] = 0.$$

Further, for any $k = 1, 2, \dots, n$, the equality $\frac{\partial}{\partial x_k} u(\xi x) = \frac{\partial u}{\partial y_k} \frac{\partial y_k}{\partial x_k} = \xi \frac{\partial u}{\partial y_k}$ holds. Hence,

$$\frac{\partial}{\partial x_k} u(\xi x)\big|_{x=0} = \xi \frac{\partial u(0)}{\partial y_k}.$$

Consequently,

$$\frac{\partial}{\partial x_k} u_2(x)\big|_{x=0} = \frac{1}{(3-\alpha)(2-\alpha)\Gamma(2-\alpha)} \frac{\partial u(0)}{\partial y_k}.$$

Further, by the definition of $r\frac{d}{dr}$ we have $r\frac{du_2(x)}{dr} = \sum_{i=1}^n x_i \frac{\partial u_2(x)}{\partial x_i}$. Thus,

$$\frac{\partial}{\partial x_k}\big[r\frac{du_2(x)}{dr}\big] = \sum_{i=1}^n x_i \frac{\partial^2 u_2(x)}{\partial x_k \partial x_i} + \frac{\partial u_2(x)}{\partial x_k}.$$

Therefore,

$$\begin{aligned} \frac{\partial}{\partial x_k}\big[2(2-\alpha)r\frac{du_2(x)}{dr}\big]\big|_{x=0} &= 2(2-\alpha)[\sum_{i=1}^n x_i \frac{\partial^2 u_2(x)}{\partial x_k \partial x_i} + \frac{\partial u_2(x)}{\partial x_k}]\big|_{x=0} \\ &= \frac{2}{(3-\alpha)\Gamma(2-\alpha)}\frac{\partial u(0)}{\partial y_k}. \end{aligned}$$

Further, by (2.2), it follows that

$$\frac{\partial}{\partial x_i}\big[r\frac{d}{dr}(r\frac{d}{dr}-1)u_2(x)\big]\big|_{x=0} = 0.$$

By using all these calculations, from the representation of the function $B_2^\alpha[u](x)$, we obtain

$$\frac{\partial B_2^\alpha[u](0)}{\partial x_k} = \frac{1}{\Gamma(2-\alpha)}\big[-\frac{\partial u(0)}{\partial y_k} + \frac{1-\alpha}{(3-\alpha)}\frac{\partial u(0)}{\partial y_k} + \frac{2}{(3-\alpha)}\frac{\partial u(0)}{\partial y_k}\big] = 0.$$

If $\alpha = 1$ or $\alpha = 2$, then $B_1^1 u(x) = r\frac{du(x)}{dr}, B_1^2 u(x) = r\frac{d}{dr}(r\frac{d}{dr}-1)u(x)$, and for these functions the statement of the lemma follows from the lemma 2.1. $\square$

The following proposition was proved in [27].

Lemma 2.3. *Let $0 < \alpha \le 1$. Then for any $x \in \Omega$ the following equalities hold:*

$$B^{-\alpha}[B_1^\alpha[u]](x) = u(x) - u(0), \tag{2.8}$$

and if $u(0) = 0$, then

$$B_1^\alpha[B^{-\alpha}[u]](x) = u(x). \tag{2.9}$$

A similar statement is true in the case of $1 < \alpha < 2$.

Lemma 2.4. *Let $1 < \alpha < 2, j = 1$. Then equalities* (2.8) *and* (2.9) *hold.*

Proof. Let us prove equality (2.8). Let $x \in \Omega$ and $t \in (0,1]$. Consider the function

$$\Im_t[u](x) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\tau^{-\alpha}B_1^\alpha[u](\tau x)d\tau.$$

By using the definition of B_1^α, we have

$$\Im_t[u](x) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1}\frac{d}{d\tau}J^{2-\alpha}[\frac{d}{d\tau}u](\tau x)d\tau.$$

Integrating the above integral by parts, we obtain

$$\begin{aligned} \Im_t[u](x) &= \frac{\alpha-1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-2}J^{2-\alpha}[\frac{d}{d\tau}u](\tau x)d\tau \\ &= \frac{1}{\Gamma(\alpha-1)}\int_0^t (t-\tau)^{\alpha-2}J^{2-\alpha}[\frac{d}{d\tau}u](\tau x)d\tau = u(tx) - u(0). \end{aligned}$$

If we put $t = 1$, then

$$u(x) = u(0) + \frac{1}{\Gamma(\alpha)}\int_0^1 (1-\tau)^{\alpha-1}\tau^{-\alpha}B_1^\alpha[u](\tau x)d\tau = u(0) + B^{-\alpha}[B_1^\alpha[u]](x).$$

Equality (2.8) is proved.

We turn to the proof of (2.9). Since $u(0) = 0$, then the operator $B^{-\alpha}$ is determined for these functions, and, therefore, applying the operator B_1^α to the function $B^{-\alpha}[u](x)$, we have

$$\begin{aligned} B_1^\alpha[B^{-\alpha}[u]](x) &= r^\alpha \frac{d}{dr} J^{2-\alpha} \frac{d}{dr} B^{-\alpha}[u](x) \\ &= \frac{r^\alpha}{\Gamma(2-\alpha)} \frac{d^2}{dr^2} \int_0^r \frac{(r-\tau)^{2-\alpha}}{2-\alpha} \frac{d}{d\tau} B^{-\alpha}[u](\tau\theta) d\tau. \end{aligned}$$

After the change of variables $\tau s = \xi$, the function

$$B^{-\alpha}[u](\tau\theta) = \frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} s^{-\alpha} u(\tau s\theta) ds$$

will be represented as

$$B^{-\alpha}[u](\tau\theta) = \frac{1}{\Gamma(\alpha)} \int_0^\tau (\tau-\xi)^{\alpha-1} \xi^{-\alpha} u(\xi\theta) d\xi = J^\alpha[\xi^{-\alpha} u].$$

Then integrating by parts, we obtain

$$B_1^\alpha[B^{-\alpha}[u]](x) = r^\alpha \frac{d^2}{dr^2} [J^{2-\alpha}[J^\alpha[\xi^{-\alpha} u]]](x) = r^\alpha \frac{d^2}{dr^2} [J^2[\xi^{-\alpha} u]](x) = u(x).$$

□

Lemma 2.5. *Let $1 < \alpha \le 2$. Then for any $x \in \Omega$ the following equalities hold:*

$$B^{-\alpha}[B_2^\alpha[u]](x) = u(x) - u(0) - \sum_{i=1}^n x_i \frac{\partial u(0)}{\partial x_i}, \tag{2.10}$$

and if $u(0) = 0$ and $\frac{\partial u(0)}{\partial x_i} = 0$ for $i = 1, 2, \dots, n$, then

$$B_2^\alpha[B^{-\alpha}[u]](x) = u(x). \tag{2.11}$$

Proof. Let us prove equality (2.10). As in the proof of (2.8) we consider the function

$$\Im_t[u](x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} \tau^{-\alpha} B_2^\alpha[u](\tau x) d\tau, t \in (0,1].$$

By using the definition of B_2^α, we have

$$\Im_t[u](x) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} J^{2-\alpha}[\frac{d^2}{d\tau^2} u](\tau x) d\tau.$$

But this function by the definition of the fractional order integral has the form

$$\Im_t[u](x) = \frac{\alpha-1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-2} J^{2-\alpha}[\frac{d}{d\tau} u](\tau x) d\tau = J^\alpha\big[J^{2-\alpha}[\frac{d^2}{d\tau^2} u]\big](x).$$

Since $J^\alpha J^{2-\alpha} = J^{\alpha+2-\alpha} = J^2$,

$$\Im_t[u](x) = J^2[\frac{d^2}{d\tau^2} u] = \int_0^t (t-\tau) \frac{d^2}{d\tau^2} u(\tau x) d\tau = -t \frac{d}{d\tau} u(0) + u(tx) - u(0).$$

Further, since

$$\frac{d}{d\tau} u(\tau x) = \sum_{i=1}^n \frac{\partial u(\tau x)}{\partial y_i} \frac{dy_i}{d\tau} = \sum_{i=1}^n x_i \frac{\partial u(\tau x)}{\partial y_i},$$

it follows that

$$\frac{d}{d\tau}u(0)=\sum_{i=1}^{n}x_i\frac{\partial u(0)}{\partial y_i}\equiv\sum_{i=1}^{n}x_i\frac{\partial u(0)}{\partial x_i}.$$

If in the integral $\Im_t[u](x)$ we set $t=1$, then

$$u(x)-u(0)-\sum_{i=1}^{n}x_i\frac{\partial u(0)}{\partial x_i}=\frac{1}{\Gamma(\alpha)}\int_0^1(1-\tau)^{\alpha-1}\tau^{-\alpha}B_2^{\alpha}[u](\tau x)d\tau\equiv B^{-\alpha}[B_2^{\alpha}[u]](x).$$

Equality (2.10) is proved.

If $u(0)=0$ and $\frac{\partial u(0)}{\partial x_i}=0$, $i=1,2,\dots,n$, then the operator $B^{-\alpha}$ is defined on these functions. Applying B_2^{α} we obtain

$$B_2^{\alpha}[B^{-\alpha}[u]](x)=\frac{r^{\alpha}}{\Gamma(2-\alpha)}\int_0^r(r-\tau)^{1-\alpha}\frac{d^2}{d\tau^2}B^{-\alpha}[u](\tau\theta)d\tau.$$

We represent the function $B^{-\alpha}[u](\tau\theta)$ as

$$B^{-\alpha}[u](\tau\theta)=\frac{1}{\Gamma(\alpha)}\int_0^1(1-s)^{\alpha-1}s^{-\alpha}u(s\tau\theta)ds=\frac{1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\xi)^{\alpha-1}\xi^{-\alpha}u(\xi\theta)d\xi\,.$$

Since $\alpha-1>0$, the following equality holds

$$\frac{d}{d\tau}B^{-\alpha}[u](\tau\theta)=\frac{\alpha-1}{\Gamma(\alpha)}\int_0^{\tau}(\tau-\xi)^{\alpha-2}\xi^{-\alpha}u(\xi\theta)d\xi\equiv J^{\alpha-1}[\xi^{-\alpha}u](\tau\theta).$$

It is easy to check the following equalities:

$$\begin{aligned}B_2^{\alpha}[B^{-\alpha}[u]](x)&=\frac{r^{\alpha}}{\Gamma(2-\alpha)}\frac{d}{dr}\int_0^r\frac{(r-\tau)^{2-\alpha}}{2-\alpha}\frac{d}{d\tau}J^{\alpha-1}[\xi^{-\alpha}u](\tau\theta)d\tau\\&=r^{\alpha}\frac{d}{dr}J[\xi^{-\alpha}u](x)=r^{\alpha}\frac{d}{dr}\int_0^r\xi^{-\alpha}u(\xi\theta)d\xi\\&=r^{\alpha}r^{-\alpha}u(x)=u(x).\end{aligned}$$

$\square$

Let $0<\alpha\le 2$, and consider the functions:

$$\begin{aligned}g_{1,\alpha}(x)&=r^{\alpha-5}J^{1-\alpha}[r^4g](x)\\&\equiv\frac{r^{\alpha-5}}{\Gamma(1-\alpha)}\int_0^r(r-\tau)^{-\alpha}\tau^4g(\tau\theta)d\tau,\quad 0<\alpha\le 1.\end{aligned}\tag{2.12}$$

$$\begin{aligned}g_{2,\alpha}(x)&=r^{\alpha-6}J^{2-\alpha}[r^4g](x)\\&\equiv\frac{r^{\alpha-6}}{\Gamma(2-\alpha)}\int_0^r(r-\tau)^{1-\alpha}\tau^4g(\tau\theta)d\tau,\quad 1<\alpha\le 2.\end{aligned}\tag{2.13}$$

Since $J^0[r^4g](x)=r^4g(x)$, it follows that $g_{1,1}(x)=g_{2,2}(x)=g(x)$.

Lemma 2.6. *Let $0<\alpha\le 2$, and $\Delta^2u(x)=g(x)$ for $x\in\Omega$. Then for any $x\in\Omega$ and $j=1,2$ the following statements hold:*

(1) *if $0<\alpha\le 1$, then*

$$\Delta^2B_1^{\alpha}[u](x)=(1-\alpha)g_{1,\alpha}(x)+\Gamma_4[g_{1,\alpha}](x);\tag{2.14}$$

(2) *if $1<\alpha\le 2$, $j=1,2$, then*

$$\Delta^2B_j^{\alpha}[u](x)=(1-\alpha)(2-\alpha)g_{2,\alpha}(x)+2(2-\alpha)\Gamma_4[g_{2,\alpha}](x)+\Gamma_4[\Gamma_3[g_{2,\alpha}]](x).\tag{2.15}$$

Proof. Note that for $u(x)$ the following equality holds:

$$\Delta^2[r\frac{d}{dr}u(x)] = r\frac{d}{dr}\Delta^2 u(x) + 4\Delta^2 u(x) = (r\frac{d}{dr}+4)\Delta^2 u(x) \equiv \Gamma_4[\Delta^2 u](x).$$

Then when $\alpha = 1$ we obtain

$$\Delta^2 B_1^1[u](x) = \Gamma_4[g_{1,1}](x) = \Gamma_4[g](x);$$

and when $\alpha = 2$ we have

$$\Delta^2 B_2^2[u](x) = \Gamma_3[\Gamma_4[g_{1,2}]](x) = \Gamma_4[\Gamma_3[g]](x).$$

Consequently, in these two values of α the equalities (2.14) and (2.15) are proved.

Let $0 < \alpha < 1$. Using the representation of the function $B_1^\alpha[u](x)$ in (2.5), we obtain

$$\Delta^2 B_1^\alpha[u](x) = (1-\alpha)\Delta^2 u_1(x) + \Gamma_4[\Delta^2 u_1](x).$$

Since $\Delta^2 u(x) = g(x)$,

$$\Delta^2 u_1(x) = \frac{1}{\Gamma(1-\alpha)}\int_0^1 (1-\xi)^{-\alpha}\xi^4 g(\xi x)d\xi = \frac{r^{\alpha-5}}{\Gamma(1-\alpha)}\int_0^r (r-\tau)^{-\alpha}\tau^4 g(\tau\theta)d\tau;$$

i.e. $\Delta^2 u_1(x) = g_{1,\alpha}(x)$. Thus, for the functions $B_1^\alpha[u](x)$ we obtain the equality (2.14).

Let $1 < \alpha < 2, j = 1$. Then the representation (2.6) implies:

$$\Delta^2 B_1^\alpha[u](x) = (1-\alpha)(2-\alpha)\Delta^2 u_2(x) + 2(2-\alpha)\Gamma_4[\Delta^2 u_2](x) + \Gamma_4[\Gamma_3[\Delta^2 u_2]](x)(x),$$

Further, taking into account $\Delta^2 u(x) = g(x)$ for $\Delta^2 u_2(x)$, we obtain

$$\begin{aligned}\Delta^2 u_2(x) &= \frac{1}{\Gamma(2-\alpha)}\int_0^1 (1-\xi)^{1-\alpha}\xi^4 g(\xi x)d\xi \\ &= \frac{r^{\alpha-6}}{\Gamma(2-\alpha)}\int_0^r (r-\tau)^{1-\alpha}\tau^4 g(\tau\theta)d\tau = g_{2,\alpha}(x),\end{aligned}$$

i.e. for the functions $\Delta^2 B_1^\alpha[u](x)$ the representation (2.15) holds.

Analogously, to the case $1 < \alpha < 2$ for $j = 2$, the representation (2.7), by the equality

$$\frac{r}{\Gamma(2-\alpha)}\frac{du(0)}{d\tau} = \frac{r}{\Gamma(2-\alpha)}\sum_{i=1}^n \frac{x_i}{r}\frac{\partial u(\tau\theta)}{\partial y_i}\Big|_{\tau=0} = \frac{1}{\Gamma(2-\alpha)}\sum_{i=1}^n x_i\frac{\partial u(0)}{\partial y_i},$$

yields the equality (2.15). □

Lemma 2.7. *Let $0 < \alpha \le 2$ and the functions $g_{1,\alpha}(y)$, $g_{2,\alpha}(y)$ be defined by the equalities* (2.12) *and* (2.13)*, respectively. Then for any $x \in \Omega$ and $j = 1, 2$ the following equalities hold:*

(1) *if $0 < \alpha \le 1$, then*

$$\Delta^2 B_1^\alpha[u](x) = |x|^{-4} B_1^\alpha[|x|^4 g](x); \tag{2.16}$$

(2) *if $1 < \alpha \le 2$, then, for $j = 1, 2$,*

$$\Delta^2 B_j^\alpha[u](x) = |x|^{-4} B_j^\alpha[|x|^4 g](x). \tag{2.17}$$

Proof. Since $r\frac{d}{dr}[|x|^4 g] = |x|^4\Gamma_4[g](x)$, then we have the equality

$$|x|^{-4} r\frac{d}{dr}[|x|^4 g] = \Gamma_4[g](x) = \Gamma_4[\Delta^2 u](x) = \Delta^2\Gamma_0[u](x) \equiv \Delta^2 B_1^1[u](x).$$

Further, if we denote $r\frac{d}{dr}[|x|^4 g](x) = f(x)$, then

$$\begin{aligned}\Delta^2 B_2^2[u] &= \Delta^2(r^2\frac{d^2}{dr^2}[u](x)) = \Delta^2(r\frac{d}{dr}(r\frac{d}{dr}-1)[u](x))\\ &= \Gamma_4[\Gamma_3[\Delta^2 u]](x) = \Gamma_3[\Gamma_4[g]](x) = (r\frac{d}{dr}+3)(|x|^{-4}f)\\ &= r\frac{d}{dr}(|x|^{-4}f) + 3(|x|^{-4}f) = |x|^{-4}(r\frac{d}{dr}-1)f.\end{aligned}$$

Thus

$$\Delta^2 B_2^2[u] = |x|^{-4}(r\frac{d}{dr}-1)(r\frac{d}{dr}[|x|^4 g])(x) = |x|^{-4}r^2\frac{d^2}{dr^2}[|x|^4 g] = |x|^{-4}B_2^2[|x|^4 g].$$

Therefore, equalities (2.16) and (2.17) in the case of integer values of α is proved. In the case of fractional values of α we use the equalities (2.14) and (2.15). To do it we transform the functions $g_{j,\alpha}(x), j = 1, 2$. After changing the variable $\xi = r^{-1}\tau$ the integral, representing the function $g_{1,\alpha}(x)$, can be rewritten in the following form

$$g_{1,\alpha}(x) = \frac{r^{\alpha-5}}{\Gamma(1-\alpha)}\int_0^r (r-\tau)^{-\alpha}\tau^4 g(\tau\theta)d\tau.$$

Then

$$\begin{aligned}(1-\alpha)g_{1,\alpha}(x) + \Gamma_4[g_{1,\alpha}](x) &= \frac{1-\alpha}{\Gamma(1-\alpha)}r^{\alpha-5}\int_0^r (r-\tau)^{-\alpha}\tau^4 g(\tau\theta)d\tau\\ &= r^{-4}\frac{r^\alpha}{\Gamma(1-\alpha)}\frac{d}{dr}\int_0^r (r-\tau)^{-\alpha}\tau^4 g(\tau\theta)d\tau.\end{aligned}$$

We transform the above integral as follows:

$$\begin{aligned}&\frac{r^\alpha}{\Gamma(1-\alpha)}\frac{d}{dr}\int_0^r (r-\tau)^{-\alpha}\tau^4 g(\tau\theta)d\tau\\ &= \frac{r^\alpha}{\Gamma(1-\alpha)}\frac{d}{dr}\int_0^r \tau^4 g(\tau\theta)\frac{d(r-\tau)^{1-\alpha}}{-(1-\alpha)}\\ &= \frac{r^\alpha}{\Gamma(1-\alpha)}\frac{d}{dr}\Big\{\int_0^r \frac{(r-\tau)^{1-\alpha}}{(1-\alpha)}\frac{d}{d\tau}[\tau^4 g(\tau\theta)]d\tau\Big\}\\ &\quad + \frac{r^\alpha}{\Gamma(1-\alpha)}\int_0^r (r-\tau)^{-\alpha}\frac{d}{d\tau}[\tau^4 g(\tau\theta)]d\tau \equiv B_1^\alpha[|x|^4 g](x).\end{aligned}$$

Thus,

$$\Delta^2 B_1^\alpha[u](x) = |x|^{-4}B_1^\alpha[|x|^4 g](x), x \in \Omega.$$

Let $1 < \alpha < 2$ and $j = 1$. Then after changing variables $\xi = r^{-1}\tau$, for the function $g_{2,\alpha}(x)$ we obtain

$$g_{2,\alpha}(x) = \frac{r^{\alpha-6}}{\Gamma(2-\alpha)}\int_0^r (r-\tau)^{1-\alpha}\tau^4 g(\tau\theta)d\tau.$$

Further, if $f(x)$ is a smooth function then

$$r\frac{d}{dr}[r^{\alpha-6}f] = r^{\alpha-6}\big(r\frac{d}{dr} + \alpha - 6\big)f(x).$$

Thus,

$$
\begin{aligned}
&(1-\alpha)(2-\alpha)g_{2,\alpha}(x)+2(2-\alpha)\Gamma_4[g_{2,\alpha}](x)+\Gamma_4[\Gamma_3[g_{2,\alpha}]](x)\\
&=(r\frac{d}{dr}+4)\big[r^{\alpha-6}(r\frac{d}{dr}+3+2(2-\alpha)+\alpha-6)J^{2-\alpha}[\tau^4 g]\big](x)\\
&\quad+r^{\alpha-4}\frac{d^2}{dr^2}\Big[\frac{1}{\Gamma(2-\alpha)}\int_0^r (r-\tau)^{1-\alpha}\tau^4 g(\tau\theta)d\tau\Big].
\end{aligned}
$$

We transform the above integral as follows:

$$
\begin{aligned}
&\frac{1}{\Gamma(2-\alpha)}\int_0^r (r-\tau)^{1-\alpha}\tau^4 g(\tau\theta)d\tau\\
&=\frac{1}{\Gamma(2-\alpha)}\int_0^r \tau^4 g(\tau\theta)\frac{d(r-\tau)^{2-\alpha}}{-(2-\alpha)}\\
&=-\tau^4 g(\tau\theta)\frac{(r-\tau)^{2-\alpha}}{(2-\alpha)\Gamma(2-\alpha)}\Big|_{\tau=0}^{\tau=r}+\frac{1}{\Gamma(2-\alpha)}\int_0^r (r-\tau)^{1-\alpha}\frac{d}{d\tau}[\tau^4 g(\tau\theta)]d\tau\\
&\equiv D_1^{\alpha}[\tau^4 g(\tau\theta)].
\end{aligned}
$$

Hence,

$$
\Delta^2 B_1^{\alpha}[u](x)=|x|^{-4}B_1^{\alpha}[|x|^4 g].
$$

Similarly, we consider the case $1<\alpha<2, j=2$. □

3. Some properties of the solution of the Dirichlet problem

Consider the Dirichlet problem

$$
\begin{gathered}
\Delta^2 v(x)=g_1(x),\quad x\in\Omega\\
v(x)=\varphi_1(x),\frac{dv(x)}{d\nu}=\varphi_2(x),\quad x\in\partial\Omega.
\end{gathered}\tag{3.1}
$$

It is known that (see e.g. [1]), if $g_1(x),\varphi_1(x)$ and $\varphi_2(x)$ are smooth functions, then the solution of (3.1) exists and is unique. The solution of (3.1) is represented as:

$$
v(x)=\int_\Omega G_{2,n}(x,y)g_1(y)dy+w(x),\tag{3.2}
$$

where $G_{2,n}(x,y)$ is the Green function of (3.1), and $w(x)$ is a solution of (3.1) when $g_1(x)=0$; i.e.,

$$
\begin{gathered}
\Delta^2 w(x)=0,\quad x\in\Omega,\\
w(x)=\varphi_1(x),\quad \frac{dw(x)}{d\nu}=\varphi_2(x),\quad x\in\partial\Omega.
\end{gathered}
$$

Denote

$$
v_1(x)=\int_\Omega G_{2,n}(x,y)g_1(y)dy.
$$

The explicit form of the Green's function for the Dirichlet problem is obtained for the cases $n\geq 2$ in [14]. For example, in the case when n is odd or n is even and $n>4$, the Green's function of the problem (3.1) follows from the expression

$$
\begin{aligned}
G_{2,n}(x,y)=d_{2,n}\Big[&|x-y|^{4-n}-\big|x|y|-\frac{y}{|y|}\big|^{4-n}\\
&(2-\frac{n}{2})\big|x|y|-\frac{y}{|y|}\big|^{2-n}(1-|x|^2)(1-|y|^2)\Big],
\end{aligned}
$$

where $d_{2,n}=\frac{1}{\omega_n}\frac{1}{2(n-4)(n-2)}$ and $\omega_n=\frac{2\pi^{n/2}}{\Gamma(n/2)}-$ is area of the unit sphere.

Furthermore, for convenience, we consider only the case when n-odd or n-even and $n>4$. The following proposition was proved in [25].

Lemma 3.1. *Let $\varphi_1(x),\varphi_2(x)$ be smooth functions. Then the following equalities hold:*

$$w(0)=\frac{1}{2\omega_n}\int_{\partial\Omega}[2\varphi_1(y)-\varphi_2(y)]dS_y, \tag{3.3}$$

$$\frac{\partial w(0)}{\partial x_k}=\frac{n}{2\omega_n}\int_{\partial\Omega}y_k[3\varphi_1(y)-\varphi_2(y)]dS_y,\quad k=1,2,\dots,n. \tag{3.4}$$

Lemma 3.2. *Let $g_2(x)$ be a smooth function. Then*

(1) *if $g_1(x)=\Gamma_4[g_2](x)$, then*

$$v_1(0)=\frac{1}{4\omega_n}\int_{\Omega}(1-|y|^2)g_2(y)dy; \tag{3.5}$$

(2) *if $g_1(x)=\Gamma_3[\Gamma_4[g_2]](x)$ then*

$$\frac{\partial v_1(0)}{\partial x_k}=\frac{n}{4\omega_n}\int_{\Omega}y_k(1-|y|^2)\Gamma_4[g](y)dy,\quad k=1,2,\dots,n. \tag{3.6}$$

Now we study the values of $v_1(0)$ and $\frac{\partial v_1(0)}{\partial x_k}$, $k=1,2,\dots,n$, when

$$g_1(x)=(1-\alpha)g_{1,\alpha}(x)+\Gamma_4[g_{1,\alpha}](x),\text{and} \tag{3.7}$$

$$g_1(x)=(1-\alpha)(2-\alpha)g_{2,\alpha}(x)+2(2-\alpha)\Gamma_4[g_{2,\alpha}](x)+\Gamma_4[\Gamma_3[g_{2,\alpha}]](x). \tag{3.8}$$

Lemma 3.3. *Let $0<\alpha\le 2$, $j=1,2,g(x)$ be a smooth function, and $g_{j,\alpha}(x)$ be defined by* (2.12) *or* (2.13). *Then*

(1) if $0<\alpha\le 1$ and $g_1(x)$ is defined by (3.7), *then*

$$\begin{aligned}v_1(0)=&\frac{1}{2\omega_n}\int_{\Omega}\frac{1-|y|^2}{2}g_{1,\alpha}(y)dy+\frac{1-\alpha}{\omega_n2(n-2)(n-4)}\int_{\Omega}\Big[|y|^{4-n}-1\\&+(2-\frac{n}{2})(1-|y|^2)\Big]g_{1,\alpha}(y)dy;\end{aligned} \tag{3.9}$$

(2) if $1<\alpha<2$, $j=1$ and the function $g_1(x)$ is defined by (3.8), *then*

$$\begin{aligned}v_1(0)=&\frac{1}{2\omega_n}\int_{\Omega}\frac{1-|y|^2}{2}\Gamma_3[g_{2,\alpha}](y)dy+\frac{2(2-\alpha)}{2\omega_n}\int_{\Omega}\frac{1-|y|^2}{2}g_{2,\alpha}(y)dy\\&+\frac{(1-\alpha)(2-\alpha)}{\omega_n2(n-2)(n-4)}\int_{\Omega}[|y|^{4-n}-1+(2-\frac{n}{2})(1-|y|^2)]g_{2,\alpha}(y)dy;\end{aligned} \tag{3.10}$$

(3) if $1<\alpha\le 2$, $j=2$ and $g_1(x)$ is defined by (3.8), *then for $v_1(0)$ we have the equality* (3.10), *moreover*

$$\begin{aligned}\frac{\partial v_1(0)}{\partial x_k}=&\frac{n}{4\omega_n}\int_{\Omega}y_k(1-|y|^2)\Gamma_4[g_{2,\alpha}](y)dy\\&+\frac{1}{2\omega_n}\frac{2(2-\alpha)}{(n-2)}\int_{\Omega}y_k\big[|y|^{2-n}-1+\frac{2-n}{2}(1-|y|^2)\big]\Gamma_4[g_{2,\alpha}](y)dy\\&+\frac{(1-\alpha)(2-\alpha)}{2\omega_n(n-2)}\int_{\Omega}y_k[|y|^{2-n}-1+\frac{2-n}{2}(1-|y|^2)]g_{2,\alpha}(y)dy,\end{aligned} \tag{3.11}$$

$k=1,\dots,n$.

Proof. When α is an integer, equalities (3.9) and (3.11) follows from Lemma 3.3. Let $0 < \alpha < 1$, $j = 1$ and $g_1(x)$ be represented in the form (3.7). Then

$$\begin{aligned} v_1(x) &= \int_\Omega G_{2,n}(x,y)g_1(y)dy \\ &= (1-\alpha)\int_\Omega G_{2,n}(x,y)g_{1,\alpha}(y)dy \qquad + \int_\Omega G_{2,n}(x,y)\Gamma_4[g_{1,\alpha}](y)dy. \end{aligned}$$

From the first statement of Lemma 3.2, for the second integral of the above equality we obtain

$$\int_\Omega G_{2,n}(0,y)\Gamma_4[g_{1,\alpha}](y)dy = \frac{1}{2\omega_n}\int_\Omega \frac{1-|y|^2}{2}g_{1,\alpha}(y)dy.$$

For the first integral, using the representation of the functions $G_{2,n}(x,y)$, we have

$$\int_\Omega G_{2,n}(0,y)g_{1,\alpha}(y)dy = d_{2,n}\int_\Omega [|y|^{4-n} - 1 + (2-\frac{n}{2})(1-|y|^2)]g_{1,\alpha}(y)dy.$$

Thus, for $v_1(0)$ we obtain the equality (3.9). Let $1 < \alpha < 2, j = 1$. Then, using the equality (3.8), we obtain

$$\begin{aligned} v_1(x) &= \int_\Omega G_{2,n}(x,y)g_1(y)dy \\ &= (1-\alpha)(2-\alpha)\int_\Omega G_{2,n}(x,y)g_{2,\alpha}(y)dy \\ &\quad + 2(2-\alpha)\int_\Omega G_{2,n}(x,y)\Gamma_4[g_{2,\alpha}](y) + \int_\Omega G_{2,n}(x,y)\Gamma_4[\Gamma_3[g_{2,\alpha}]](y)dy \end{aligned}$$

By (3.5), for the second and third integrals of the last equality we obtain

$$\begin{gathered} \int_\Omega G_{2,n}(0,y)\Gamma_4[g_{2,\alpha}](y)dy = \frac{1}{2\omega_n}\int_\Omega \frac{1-|y|^2}{2}g_{2,\alpha}(y)dy, \\ \int_\Omega G_{2,n}(0,y)\Gamma_4[\Gamma_3[g_{2,\alpha}]](y)dy = \frac{1}{2\omega_n}\int_\Omega \frac{1-|y|^2}{2}\Gamma_3[g_{2,\alpha}](y)dy. \end{gathered}$$

For the first integral we have

$$\int_\Omega G_{2,n}(0,y)g_{1,\alpha}(y)dy = d_{2,n}\int_\Omega [|y|^{4-n} - 1 + (2-\frac{n}{2})(1-|y|^2)]g_{2,\alpha}(y)dy.$$

Therefore, for $v_1(0)$ we obtain the equality (3.10).

No let $1 < \alpha < 2$, $j = 2$. Since in this case $g_1(x)$ has the form (3.8), for $v_1(0)$ again we obtain (3.10). Further, we obtain

$$\begin{gathered} v_{1,1}(x) = (1-\alpha)(2-\alpha)\int_\Omega G_{2,n}(x,y)g_{2,\alpha}(y)dy, \\ v_{1,2}(x) = 2(2-\alpha)\int_\Omega G_{2,n}(x,y)\Gamma_4[g_{2,\alpha}](y)dy, \\ v_{1,3}(x) = \int_\Omega G_{2,n}(x,y)\Gamma_4[\Gamma_3[g_{2,\alpha}]](y)dy. \end{gathered}$$

Using (3.6), for the function $v_{1,3}(x)$ we obtain

$$\frac{\partial v_{1,3}(0)}{\partial x_k} = \frac{n}{4\omega_n}\int_\Omega y_k(1-|y|^2)\Gamma_4[g_{2,\alpha}](y)dy, \quad k = 1,2,\dots,n.$$

Since

$$\frac{\partial}{\partial x_k}|x-y|^{4-n} = \frac{4-n}{2}|x-y|^{2-n}2(x_k-y_k)\big|_{x=0} = -(4-n)|y|^{2-n}y_k,$$

$$\begin{aligned}\frac{\partial}{\partial x_k}\big|x|y|-\frac{y}{|y|}\big|^{4-n} &= \frac{4-n}{2}|x|y|-\frac{y}{|y|}|^{2-n}2(x_k|y|-\frac{y_k}{|y|})|y|\Big|_{x=0}\\ &= -(4-n)y_k,\end{aligned}$$

$$\begin{aligned}&\frac{\partial}{\partial x_k}\big[|x|y|-\frac{y}{|y|}|^{2-n}(1-|x|^2)\big]\\ &= \frac{2-n}{2}\big|x|y|-\frac{y}{|y|}\big|^{-n}2(x_k|y|-\frac{y_k}{|y|})|y|(1-|x|^2)+\big|x|y|-\frac{y}{|y|}\big|^{2-n}(-2x_k)\big|_{x=0}\\ &= -(2-n)y_k,\end{aligned}$$

it follows that for

$$\frac{\partial G_{2,n}(x,y)}{\partial x_k}\Big|_{x=0},$$

we obtain

$$\frac{\partial G_{2,n}(x,y)}{\partial x_k}\Big|_{x=0} = \frac{1}{2\omega_n}\frac{1}{(n-2)}\big[y_k|y|^{2-n}-y_k+\frac{2-n}{2}y_k(1-|y|^2)\big]$$

Then

$$\begin{aligned}\frac{\partial v_{1,1}(0)}{\partial x_k} &= \frac{1}{2\omega_n}\frac{(1-\alpha)(2-\alpha)}{(n-2)}\int_\Omega y_k[|y|^{2-n}-1+\frac{2-n}{2}(1-|y|^2)]g_{2,\alpha}(y)dy,\\ \frac{\partial v_{1,2}(0)}{\partial x_k} &= \frac{1}{2\omega_n}\frac{2(2-\alpha)}{(n-2)}\int_\Omega y_k[|y|^{2-n}-1+\frac{2-n}{2}(1-|y|^2)]\Gamma_4[g_{2,\alpha}](y)dy.\end{aligned}$$

Hence, for $\frac{\partial v_1(0)}{\partial x_k}$ we obtain (3.11). $\square$

4. Main results

Let $g_{1,\alpha}(x)$ and $g_{2,\alpha}(x), x\in R^n$ be defined by (2.12) and (2.13), and let n be odd, or n be even with $n>4$.

Theorem 4.1. *Let $0<\alpha<2$, $g(x)$, $f_1(x)$ and $f_2(x)$ be smooth functions.*

(1) If $0<\alpha\le 1$ and $j=1$, then problem 1.1 is solvable if and only if

$$\begin{aligned}&\int_{\partial\Omega}[f_2(y)+(\alpha-2)f_1(y)]dS_y\\ &= \int_\Omega \frac{1-|y|^2}{2}g_{1,\alpha}(y)dy\\ &\quad + \frac{1-\alpha}{(n-2)(n-4)}\int_\Omega[|y|^{4-n}-1+(2-\frac{n}{2})(1-|y|^2)]g_{1,\alpha}(y)dy\end{aligned}\tag{4.1}$$

(2) If $1<\alpha<2$ and $j=1$, then problem 1.1 is solvable if and only if

$$\begin{aligned}&\int_{\partial\Omega}[f_2(y)+(\alpha-2)f_1(y)]dS_y\\ &= \int_\Omega\frac{1-|y|^2}{2}\Gamma_3[g_{2,\alpha}](y)dy+2(2-\alpha)\int_\Omega\frac{1-|y|^2}{2}g_{2,\alpha}(y)dy\\ &\quad + \frac{(1-\alpha)(2-\alpha)}{(n-2)(n-4)}\int_\Omega[|y|^{4-n}-1+(2-\frac{n}{2})(1-|y|^2)]g_{2,\alpha}(y)dy\,.\end{aligned}\tag{4.2}$$

(3) If the solution of the problem 1.1 exists then it is unique up to a constant term and can be represented as
$$u(x) = C + B^{-\alpha}[v](x), \tag{4.3}$$
where $v(x)$ is a solution of (3.1)*, satisfying the condition $v(0) = 0$, with the functions*
$$\varphi_1(x) = f_1(x), \quad \varphi_2(x) = f_2(x) + \alpha f_1(x) \tag{4.4}$$
$$g_1(x) = |x|^{-4} B_j^{\alpha}[|x|^4 g](x). \tag{4.5}$$

Proof. Let $u(x)$ be a solution of problem 1.1. Apply the operator B_1^{α} to the function $u(x)$, and denote $v(x) = B_1^{\alpha}[u](x)$. Then in the case $0 < \alpha \le 1$, using (2.16) from lemma 2.7, we obtain
$$\Delta^2 v(x) = \Delta^2 B_1^{\alpha}[u](x) = |x|^{-4} B_1^{\alpha}[|x|^4 g](x) \equiv g_1(x), \quad 0 < \alpha \le 1.$$
and if $1 < \alpha < 2$, then by (2.17), we have
$$\Delta^2 v(x) = \Delta^2 B_1^{\alpha}[u](x) = |x|^{-4} B_1^{\alpha}[|x|^4 g](x) \equiv g_1(x), \quad 1 < \alpha < 2.$$
Then by assumption, $B_1^{\alpha}[u](x) \in C(\bar{\Omega})$. Therefore, $v(x) \in C(\bar{\Omega})$ and
$$v(x)\big|_{\partial\Omega} = f_1(x) \equiv \varphi_1(x).$$
Further, if $0 < \alpha \le 1$, then by the definition of $B_1^{\alpha+1}$,
$$\begin{aligned} B_1^{\alpha+1}[u](x) &= r^{\alpha+1} \frac{d}{dr} J^{2-(\alpha+1)}[\frac{d}{dr} u](x) = r^{\alpha+1} \frac{d}{dr} J^{1-\alpha}[\frac{d}{dr} u](x) \\ &= r^{\alpha+1} \frac{d}{dr} [r^{-\alpha} \cdot B_1^{\alpha}[u]](x) = r \frac{d}{dr} B_1^{\alpha}[u](x) - \alpha B_1^{\alpha}[u](x). \end{aligned}$$
Therefore, the boundary condition (2.3) of the problem 1.1 implies the condition
$$\frac{\partial v(x)}{\partial \nu}\big|_{\partial\Omega} = f_2(x) + \alpha f_1(x) \equiv \varphi_2(x).$$
Similarly, in the case $1 < \alpha < 2$, $j = 1$ from definition of $B_1^{\alpha+1}$, we have
$$\begin{aligned} B_1^{\alpha+1}[u](x) &= r^{\alpha+1} \frac{d^2}{dr^2} J^{3-(\alpha+1)}[\frac{d}{dr} u](x) = r^{\alpha+1} \frac{d}{dr} [\frac{d}{dr} J^{2-\alpha}[\frac{d}{dr} u]](x) \\ &= r^{\alpha+1} \frac{d}{dr} [r^{-\alpha} B_1^{\alpha}[u]](x) = r \frac{d}{dr} B_1^{\alpha}[u](x) - \alpha B_1^{\alpha}[u](x). \end{aligned}$$
Consequently, in this case,
$$\frac{\partial v(x)}{\partial \nu}\big|_{\partial\Omega} = f_2(x) + \alpha f_1(x) \equiv \varphi_2(x).$$
Thus, if $u(x)$ is a solution of problem 1.1, then for the function $v(x) = B_1^{\alpha}[u](x)$ we obtain the problem (3.1) with the functions (4.4) and (4.5).

By (2.3), the additional condition $v(0) = 0$ holds. For smooth enough functions $g_1(x), \varphi_1(x)$ and $\varphi_2(x)$ the solution of (3.1) exists, is unique and can be represented as in (3.2).

Let $0 < \alpha \le 1$. Then, using the representation of the function $g_1(x)$ as (2.14), and from (3.3) and (3.9), we obtain
$$\begin{aligned} v(0) &= \frac{1}{2\omega_n} \int_{\partial\Omega} [2\varphi_1(y) - \varphi_2(y)] dS_y + \frac{1}{2\omega_n} \int_{\Omega} \frac{1 - |y|^2}{2} g_{1,\alpha}(y) dy \\ &\quad + \frac{1-\alpha}{\omega_n 2(n-2)(n-4)} \int_{\Omega} [|y|^{4-n} - 1 + (2 - \frac{n}{2})(1 - |y|^2)] g_{1,\alpha}(y) dy. \end{aligned}$$

Hence, the condition $v(0) = 0$ holds if

$$\begin{aligned}
&-\int_{\partial\Omega} [2\varphi_1(y) - \varphi_2(y)] dS_y \\
&= \int_\Omega \frac{1-|y|^2}{2} g_{1,\alpha}(y) dy \\
&\quad + \frac{1-\alpha}{(n-2)(n-4)} \int_\Omega [|y|^{4-n} - 1 + (2 - \frac{n}{2})(1-|y|^2)] g_{1,\alpha}(y) dy.
\end{aligned}$$

Since

$$2\varphi_1(y) - \varphi_2(y) = 2f_1(y) - f_2(y) - \alpha f_1(y) = -[f_2(y) + (\alpha-2) f_1(y)],$$

this condition can be rewritten as (4.1). Therefore, necessity of condition (4.1) is proved.

Applying the equality $v(x) = B_1^\alpha[u](x)$, the operator $B^{-\alpha}$, by (2.8), yields

$$B^{-\alpha}[v](x) = B^{-\alpha}[B_1^\alpha[u]](x) = u(x) - u(0),$$

i.e. if the solution of problem 1.1 exists, and can be represented as in (4.2). Now we show that condition (4.1) is also sufficient for the existence of solutions of problem 1.1. Indeed, if condition (4.1) holds, then for solutions of problem (3.1) with functions (4.4) and (4.5), condition $v(0) = 0$ holds. Then for such functions the operator $B^{-\alpha}$ is defined and we can consider the function $u(x) = C + B^{-\alpha}[v](x)$. This function satisfies all conditions of the problem 1.1. Indeed, since $\Delta^2 v(x) = g_1(x)$ and $g_1(x) = (1-\alpha) g_{1,\alpha}(x) + \Gamma_4[g_{1,\alpha}](x)$, then, using (2.16) we can write the equalities

$$\begin{aligned}
\Delta^2 u(x) &= \Delta^2[C + B^{-\alpha}[v](x)] \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (t-\tau)^{\alpha-1} \tau^{-\alpha} \Delta^2 v(\tau x) d\tau \\
&= \frac{1}{\Gamma(\alpha)} \int_0^1 (t-\tau)^{\alpha-1} \tau^{4-\alpha} [|\tau x|^{-4} B_1^\alpha[\tau^4 g]](\tau x) d\tau \\
&= \frac{|x|^{-4}}{\Gamma(\alpha)} \int_0^1 (t-\tau)^{\alpha-1} \tau^{-\alpha} B_1^\alpha[\tau^4 g](\tau x) d\tau \\
&= |x|^{-4} B^{-\alpha}[B_1^\alpha[|x|^4 g]](x) = |x|^{-4}|x|^4 g(x) = g(x).
\end{aligned}$$

Using (2.9), we obtain

$$\begin{aligned}
D_1^\alpha[u](x)\big|_{\partial\Omega} &= B_1^\alpha[u](x)\big|_{\partial\Omega} = B_1^\alpha[C + B^{-\alpha}[v]](x)\big|_{\partial\Omega} \\
&= v(x)\big|_{\partial\Omega} = \varphi_1(x) = f_1(x),
\end{aligned}$$

$$\begin{aligned}
D_1^{\alpha+1}[u](x)\big|_{\partial\Omega} &= B_1^{\alpha+1}[u](x)\big|_{\partial\Omega} = r\frac{\partial}{\partial r} B_1^\alpha[u](x) - \alpha B_1^\alpha[u](x)\big|_{\partial\Omega} \\
&= r\frac{\partial}{\partial r} v(x) - \alpha v(x)\big|_{\partial\Omega} = \varphi_2(x) - \alpha\varphi_1(x) \\
&= f_2(x) + \alpha f_1(x) - \alpha f_1(x) = f_2(x).
\end{aligned}$$

Consequently, the function $u(x) = C + B^{-\alpha}[v](x)$ satisfies all conditions of the problem 1.1.

Let $1 < \alpha < 2$, $j = 1$. In this case $v(x) = B_1^\alpha[u](x)$ will be a solution of problem (3.1) with functions $\varphi_1(x) = f_1(x)$, $\varphi_2(x) = f_2(x) + \alpha f_1(x)$ and

$$g_1(x) = |x|^{-4} B_1^\alpha[|x|^4 g](x)$$

$$\equiv (1-\alpha)(2-\alpha)g_{2,\alpha}(x) + 2(2-\alpha)\Gamma_4[g_{2,\alpha}](x) + \Gamma_4[\Gamma_3[g_{2,\alpha}]](x).$$

By (2.6), the condition $v(0) = 0$, holds additionally. Then, using (3.3) and (3.10), we have

$$\begin{aligned} v(0) &= \frac{1}{2\omega_n}\int_{\partial\Omega}[2\varphi_1(y) - \varphi_2(y)]dS_y + \frac{2(2-\alpha)}{2\omega_n}\int_\Omega \frac{1-|y|^2}{2} g_{1,\alpha}(y)dy \\ &\quad + \frac{1}{2\omega_n}\int_\Omega \frac{1-|y|^2}{2}\Gamma_3[g_{2,\alpha}](y)dy \\ &\quad + \frac{(1-\alpha)(2-\alpha)}{\omega_n 2(n-2)(n-4)}\int_\Omega [|y|^{4-n} - 1 + (2-\frac{n}{2})(1-|y|^2)]g_{2,\alpha}(y)dy. \end{aligned}$$

Thus, for the condition $v(0) = 0$, the following equality is necessary

$$\begin{aligned} &-\int_{\partial\Omega}[2\varphi_1(y) - \varphi_2(y)]dS_y \\ &= 2(2-\alpha)\int_\Omega \frac{1-|y|^2}{2} g_{2,\alpha}(y)dy + \int_\Omega \frac{1-|y|^2}{2}\Gamma_3[g_{2,\alpha}](y)dy \\ &\quad + \frac{(1-\alpha)(2-\alpha)}{(n-2)(n-4)}\int_\Omega [|y|^{4-n} - 1 + (2-\frac{n}{2})(1-|y|^2)]g_{2,\alpha}(y)dy. \end{aligned}$$

Since $2\varphi_1(y) - \varphi_2(y) = -[f_2(y) + (\alpha-2)f_1(y)]$, this condition can be rewritten as (4.3). Therefore, necessity of the condition (4.3) is proved. Further, by repetition of the argument in the case $0 < \alpha < 1$, one can show the rest of the theorem. □

Theorem 4.2. *Let $1 < \alpha \le 2$, $j = 2$, $g(x)$, $f_1(x)$ and $f_2(x)$ be smooth functions. Then problem 1.2 is solvable if and only if:*

$$\begin{aligned} &\int_{\partial\Omega}[f_2(y) + (\alpha-2)f_1(y)]dS_y \\ &= 2(2-\alpha)\int_\Omega \frac{1-|y|^2}{2} g_{2,\alpha}(y)dy + \int_\Omega \frac{1-|y|^2}{2}\Gamma_3[g_{2,\alpha}](y)dy \\ &\quad + \frac{(1-\alpha)(2-\alpha)}{(n-2)(n-4)}\int_\Omega [|y|^{4-n} - 1 + (2-\frac{n}{2})(1-|y|^2)]g_{2,\alpha}(y)dy, \end{aligned} \tag{4.6}$$

and

$$\begin{aligned} &\int_{\partial\Omega} y_k[f_2(y) + (\alpha-3)f_1(y)]dS_y \\ &= \frac{1}{2}\int_\Omega y_k(1-|y|^2)\Gamma_4[g_{2,\alpha}](y)dy \\ &\quad + \frac{2(2-\alpha)}{n(n-2)}\int_\Omega y_k[|y|^{2-n} - 1 + \frac{2-n}{2}(1-|y|^2)]\Gamma_4[g_{2,\alpha}](y)dy \\ &\quad + \frac{(1-\alpha)(2-\alpha)}{n(n-2)}\int_\Omega y_k[|y|^{2-n} - 1 + \frac{2-n}{2}(1-|y|^2)]g_{2,\alpha}(y)dy, \end{aligned} \tag{4.7}$$

for $k = 1, \dots, n$.

If a solution of the problem 1.2 exists, then it is unique up to a first order polynomial and can be represented as

$$u(x) = c_0 + \sum_{i=1}^n c_i x_i + B^{-\alpha}[v](x), \tag{4.8}$$

where c_i, $i = 0, 1, \dots, n$ are arbitrary constants, and $v(x)$ is a solution of the problem (3.1) with functions $g_1(x) = |x|^{-4}B_2^{\alpha}[|x|^4 g](x), \varphi_1(x) = f_1(x)$ and $\varphi_2(x) = f_2(x) + \alpha f_1(x)$, and which satisfies conditions $v(0) = 0$, $\frac{\partial v(0)}{\partial x_i} = 0$, $i = 1, 2, \dots, n$.

Proof. Let $u(x)$ be a solution of problem 1.2. Apply to the function $u(x)$ the operator B_2^{α}, and denote it by $v(x) = B_2^{\alpha}[u](x)$. Then (2.12) and

$$\begin{aligned} B_2^{\alpha+1}[u](x) &= r^{\alpha+1}\frac{d}{dr}J^{3-(\alpha+1)}[\frac{d^2}{dr^2}u](x) = r^{\alpha+1}\frac{d}{dr}J^{2-\alpha}[\frac{d^2}{dr^2}u](x) \\ &= r^{\alpha+1}\frac{d}{dr}[r^{-\alpha}\cdot B_2^{\alpha}[u]](x) = r\frac{d}{dr}B_2^{\alpha}[u](x) - \alpha B_2^{\alpha}[u](x). \end{aligned}$$

imply that the function $v(x)$ is a solution of the problem (3.1) with functions $g_1(x) = |x|^{-4}B_2^{\alpha}[|x|^4 g](x)$, $\varphi_1(x) = f_1(x)$, $\varphi_2(x) = f_2(x) + \alpha f_1(x)$.

Moreover, by lemma 2.2, the function $v(x) = B_2^{\alpha}[u](x)$ should satisfy conditions $v(0) = 0$, $\frac{\partial v(0)}{\partial x_k} = 0$, $k = 1, 2, \dots, n$.

For enough smooth functions $g_1(x)$, $\varphi_1(x)$ and $\varphi_2(x)$ the solution of problem (3.1) exists, is unique and can be represented as (3.2).

Further, using the representation of the function $g_1(x)$ in the form (2.15), by similar arguments, as in the case $1 < \alpha < 2, j = 1$, one can show that the equality $v(0) = 0$ holds if the condition (4.4) holds.

Now we check that the equalities $\frac{\partial v(0)}{\partial x_k} = 0$, $k = 1, 2, \dots, n$ hold if condition (4.5) holds. To do it we use the representation of the function $v(x)$ in the form (3.2) and the lemma 3.3. Since the function $g_1(x) = |x|^{-4}B_2^{\alpha}[|x|^4 g](x)$ can be represented as (3.8), then by (3.4) and (3.11), we obtain

$$\begin{aligned} \frac{\partial v(0)}{\partial x_k} &= \frac{n}{2\omega_n}\int_{\partial\Omega} y_k[3\varphi_1(y) - \varphi_2(y)]dS_y + \frac{n}{4\omega_n}\int_{\Omega} y_k(1-|y|^2)\Gamma_4[g_{2,\alpha}](y)dy \\ &\quad + \frac{1}{2\omega_n}\frac{2(2-\alpha)}{(n-2)}\int_{\Omega} y_k[|y|^{2-n} - 1 + \frac{2-n}{2}(1-|y|^2)]\Gamma_4[g_{2,\alpha}](y)dy \\ &\quad + \frac{1}{2\omega_n}\frac{(1-\alpha)(2-\alpha)}{(n-2)}\int_{\Omega} y_k[|y|^{2-n} - 1 + \frac{2-n}{2}(1-|y|^2)]g_{2,\alpha}(y)dy, \end{aligned}$$

for $k = 1, \dots, n$. Consequently, equalities $\frac{\partial v(0)}{\partial x_k} = 0$, $k = 1, 2, \dots, n$ hold if

$$\begin{aligned} &\int_{\partial\Omega} y_k[\varphi_2(y) - 3\varphi_1(y)]dS_y \\ &= \frac{1}{2}\int_{\Omega} y_k(1-|y|^2)\Gamma_4[g_{2,\alpha}](y)dy \\ &\quad + \frac{2(2-\alpha)}{n(n-2)}\int_{\Omega} y_k[|y|^{2-n} - 1 + \frac{2-n}{2}(1-|y|^2)]\Gamma_4[g_{2,\alpha}](y)dy \\ &\quad + \frac{(1-\alpha)(2-\alpha)}{n(n-2)}\int_{\Omega} y_k[|y|^{2-n} - 1 + \frac{2-n}{2}(1-|y|^2)]g_{2,\alpha}(y)dy, \end{aligned}$$

for $k = 1, \dots, n$.

Since $\varphi_2(y) - 3\varphi_1(y) = f_2(x) + \alpha f_1(x) - 3f_1(x) = f_2(x) + (\alpha - 3)f_1(x)$, the above condition can be rewritten as (4.7).

Applying the operator $B^{-\alpha}$ to the equality $v(x) = B_1^{\alpha}[u](x)$, by (2.10), we obtain

$$B^{-\alpha}[v](x) = B^{-\alpha}[B_1^{\alpha}[u]](x) = u(x) - u(0) - \sum_{i=1}^{n} x_i\frac{\partial u(0)}{\partial x_i}.$$

Denoting

$$c_0 = u(0), \quad c_i = \frac{\partial u(0)}{\partial x_i}, \quad i = 1, 2, \dots, n,$$

we obtain the representation (4.8). Therefore, if solution of the problem 1.2 exists, then it can be represented as (4.6).

Now we show that conditions (4.6) and (4.7) are also sufficient for existence of a solution of the problem 1.2. Indeed, if conditions (4.6) and (4.7) hold, then for a solution of the problem (3.1) with functions

$$g_1(x) = |x|^{-4} B_1^\alpha[|x|^4 g](x), \quad \varphi_1(x) = f_1(x), \quad \varphi_2(x) = f_2(x) + \alpha f_1(x)$$

the conditions

$$v(0) = 0, \quad \frac{\partial v(0)}{\partial x_i} = 0, \quad i = 1, 2, \dots, n,$$

hold. Then in the class of such functions the operator $B^{-\alpha}$ is defined, and we can consider the function

$$u(x) = c_0 + \sum_{i=1}^n c_i x_i + B^{-\alpha}[v](x).$$

We show that this function satisfies all conditions of the problem 1.1. Indeed, since

$$\Delta^2 v(x) = g_1(x) \equiv |x|^{-4} B_1^\alpha[|x|^4 g](x),$$

it follows that

$$\begin{aligned}
\Delta^2 u(x) &= \Delta^2 \Big[c_0 + \sum_{i=1}^n c_i x_i + B^{-\alpha}[v](x)\Big] = \frac{1}{\Gamma(\alpha)} \int_0^1 (t-\tau)^{\alpha-1} \tau^{-\alpha} \Delta^2 v(\tau x) d\tau \\
&= |x|^{-4} B^{-\alpha}[B_2^\alpha[|x|^4 g]](x).
\end{aligned}$$

The above expression, by (2.11), equals to $g(x)$. Further, using (2.11), we obtain

$$\begin{aligned}
D_2^\alpha[u](x)\big|_{\partial\Omega} &= B_2^\alpha[u](x)\big|_{\partial\Omega} = B_2^\alpha[c_0 + \sum_{i=1}^n c_i x_i + B^{-\alpha}[v]](x)\big|_{\partial\Omega} \\
&= v(x)\big|_{\partial\Omega} = \varphi_1(x) = f_1(x),
\end{aligned}$$

$$\begin{aligned}
&D_2^{\alpha+1}[u](x)\big|_{\partial\Omega} \\
&= B_2^{\alpha+1}[u](x)\big|_{\partial\Omega} = r^{\alpha+1} \frac{d}{dr} J^{3-(\alpha+1)} \frac{d^2}{dr^2} u(x) \\
&= r^{\alpha+1} \frac{d}{dr} J^{2-\alpha} \frac{d^2}{dr^2} u(x) = r \frac{d}{dr} B_1^\alpha[u](x) - \alpha B_1^\alpha[u](x)\big|_{\partial\Omega} \\
&= r \frac{d}{dr} B_1^\alpha[c_0 + \sum_{i=1}^n c_i x_i + B^{-\alpha}[v]](x) - \alpha B_1^\alpha[c_0 + \sum_{i=1}^n c_i x_i + B^{-\alpha}[v]](x)\big|_{\partial\Omega} \\
&= r \frac{d}{dr} B_1^\alpha[u](x) - \alpha B_1^\alpha[u](x)\big|_{\partial\Omega} = r \frac{dv(x)}{dr} - \alpha v(x)|_{\partial\Omega} \\
&= \varphi_2(x) - \alpha \varphi_1(x) = f_2(x) + \alpha f_1(x) - \alpha f_1(x) = f_2(x).
\end{aligned}$$

Consequently, the function $c_0 + \sum_{i=1}^n c_i x_i + B^{-\alpha}[v]$ satisfies all conditions of the problem 1.2. $\square$

Remark 4.3. If in (4.1) $\alpha = 1$, then condition on solvability of the problem 1.1 coincides with the condition (1.6). Similarly, in the case $\alpha = 2$ condition on solvability of the problem 1.2 coincides with the conditions (1.7) and (1.8).

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