Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 81, pp. 1–23.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

MULTIPLE POSITIVE SOLUTIONS FOR SINGULAR SEMIPOSITONE NONLINEAR INTEGRAL BOUNDARY-VALUE PROBLEMS ON INFINITE INTERVALS

YING WANG

Abstract. In this article, we study the existence of multiple positive solutions for singular semipositone boundary-value problem (BVP) with integral boundary conditions on infinite intervals. By using the properties of the Green's function and the Guo-Krasnosel'skii fixed point theorem, we obtain the existence of multiple positive solutions under conditions concerning the nonlinear functions. The method in this article can be used for a large number of problems. We illustrate the validity of our results with an example in the last section.

1. Introduction

In the Cahn-Hillard theory used in hydrodynamics for studying the behavior of nonhomogeneous fluids, the following system of partial differential equation was derived:

$$\rho_t + \operatorname{div}(\rho v) = 0, \quad \frac{dv}{dt} + \nabla(\mu(\rho) - \gamma\Delta\rho) = 0,$$

with density ρ and velocity v of the fluid, μ is its chemical potential, γ is a constant. In the simplest model, this system can be reduced into the boundary value problem for the ordinary differential equation of the second order [9, 11],

$$(t^k u')' = 4\lambda^2 t^k (u+1)u(u-\xi), \quad u'(0) = 0, \quad u(\infty) = \xi,$$

where $k \in \mathbb{N}$, $\xi \in (0,1)$, $\lambda \in (0,+\infty)$ is a parameter. The function $u(t) \equiv \xi$ is a solution of this problem and it corresponds to the case of homogeneous fluid (without bubbles). The solution itself has a great physical significance and the numerical treatment was done in [9, 11].

In this article, we study the generalized problem

$$\begin{gathered}
(p(t)x'(t))' + f(t,x(t)) = 0, \quad t \in (0,+\infty), \\
\alpha_1 x(0) - \beta_1 \lim_{t\to 0^+} p(t)x'(t) = \int_0^\infty g(t)x(t)dt, \\
\alpha_2 \lim_{t\to+\infty} x(t) + \beta_2 \lim_{t\to+\infty} p(t)x'(t) = \int_0^\infty h(t)x(t)dt,
\end{gathered} \tag{1.1}$$

2000 *Mathematics Subject Classification.* 34B16, 34B18, 34B40.
Key words and phrases. Multiple positive solutions; singular semipositone BVP; Guo-Krasnosel'skii fixed point theorem; integral boundary condition.

Submitted January 16, 2015. Published March 31, 2015.

where $\lambda > 0$ is a parameter, $\alpha_1, \alpha_2 \geq 0$, $\beta_1, \beta_2 > 0$, $g, h \in L[0,+\infty)$, with $\int_0^\infty g(t)dt < +\infty$, $\int_0^\infty h(t)dt < +\infty$, $p \in C[0,+\infty) \cap C^1(0,+\infty)$ with $p > 0$ on $(0,+\infty)$, $\int_0^\infty \frac{1}{p(s)}ds < +\infty$, $\rho = \alpha_2\beta_1 + \alpha_1\beta_2 + \alpha_1\alpha_2 B(0,\infty) > 0$ in which $B(t,s) = \int_t^s \frac{1}{p(v)}dv$, $f : (0,+\infty) \times (0,+\infty) \to (-\infty,+\infty)$ is a continuous function and $f(t,u)$ may be singular at $t = 0$ and $u = 0$.

The study of BVP on infinite intervals was initiated in the early 1950s. Since then, great efforts have been devoted to nonlinear BVP due to their theoretical challenge and great application potential. Many results on the existence of (positive) solutions for BVP on infinite intervals have been obtained, and for more details the reader is referred to [1, 2, 3, 4, 5, 6, 7, 8, 10, 12, 13, 15, 16] and the references therein. Chen and Zhang [5], obtained some sufficient and necessary conditions for the existence of positive solutions for

$$\begin{gathered} x''(t)) + f(t, x(t)) = 0, \quad t \in (0,+\infty), \\ x(0) = r \geq 0, \quad \lim_{t\to+\infty} x(t) = \text{const.}, \\ \text{or } x(0) = r \geq 0, \quad \lim_{t\to+\infty} x'(t) = l \geq 0, \end{gathered}$$

where $f : (0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a continuous function, $f(t,1) \not\equiv 0$. Liu et al [12] established the existence of positive solutions for the following equation on infinite intervals by applying the fixed point theorem of cone map

$$\begin{gathered} (p(t)x'(t))' + m(t)f(t, x(t)) = 0, \; t \in (0,+\infty), \\ \alpha_1 x(0) - \beta_1 \lim_{t\to0^+} p(t)x'(t) = 0, \\ \alpha_2 \lim_{t\to+\infty} x(t) + \beta_2 \lim_{t\to+\infty} p(t)x'(t) = 0, \end{gathered}$$

in which $f : [0,+\infty) \times [0,+\infty) \to [0,+\infty)$ is a continuous function, $m : (0,+\infty) \to [0,+\infty)$ is a Lebesgue integrable function and may be singular at $t = 0$.

Motivated by the above works, we shall study the existence of multiple positive solutions for (1.1). We should address here that our work presented in this paper has various new features. Firstly, the boundary conditions are more general; that is, (1.1) includes two-point, three-point and multi-point boundary value problems as special cases. Secondly, we study the BVP on infinite intervals, which expands the domain of definition of t from finite interval to infinite interval, since we can not use the Ascoli-Arzela theorem in $[0,+\infty)$, some modification of the compactness criterion in $[0,+\infty)$ (see Lemma 2.5) can help to resolve this problem. Thirdly, the nonlinear term f in (1.1) is more complicated, we require f has singularity on $t = 0$ and $u = 0$, in addition, we do not need require f be positive, but the solution we obtain in (1.1) is a positive solution, where $x \in C[0,+\infty)$ is said to be a positive solution of (1.1) if and only if x satisfies (1.1) and $x(t) > 0$ for any $t \in [0,+\infty)$.

2. Preliminaries

For convenience of notation, we let

$$\begin{gathered} a(t) = \beta_1 + \alpha_1 B(0,t), \quad b(t) = \beta_2 + \alpha_2 B(t,\infty), \\ a(\infty) = \lim_{t\to+\infty} a(t) = \beta_1 + \alpha_1 B(0,\infty) < +\infty, \quad a(0) = \lim_{t\to0} a(t) = \beta_1, \\ b(\infty) = \lim_{t\to+\infty} b(t) = \beta_2, \quad b(0) = \lim_{t\to0} b(t) = \beta_2 + \alpha_2 B(0,\infty) < +\infty, \end{gathered}$$

$$\Delta = \begin{vmatrix} \rho - \int_0^\infty g(t)b(t)dt & \int_0^\infty g(t)a(t)dt \\ \int_0^\infty h(t)b(t)dt & \rho - \int_0^\infty h(t)a(t)dt \end{vmatrix}.$$

It is obvious that $a(t)$ is increasing and $b(t)$ is decreasing on $[0, +\infty)$. Define

$$G(t,s) = \frac{1}{\rho} \begin{cases} a(s)b(t), & 0 \le s \le t < +\infty, \\ a(t)b(s), & 0 \le t \le s < +\infty. \end{cases} \tag{2.1}$$

Denote $\tau(t) = a(t)b(t)$, then for any $0 \le t, s < +\infty$, we obtain

$$\begin{gathered} 0 \le G(t,s) \le G(s,s) \le \frac{b(0)a(s)}{\rho}, \quad 0 \le G(t,s) \le \frac{\tau(t)}{\rho}, \\ \overline{G}(s) = \lim_{t\to+\infty} G(t,s) = \frac{\beta_2 a(s)}{\rho} \le G(s,s) < +\infty. \end{gathered} \tag{2.2}$$

Lemma 2.1. *Suppose $\theta = 1/(a(\infty)b(0))$, then $G(t,s) \ge \theta\tau(t)G(s,s)$, $0 \le t, s < +\infty$.*

Proof. From (2.2) and the properties of $a(t)$, $b(t)$, for $0 \le t, s < +\infty$, we have

$$\frac{G(t,s)}{G(s,s)} = \begin{cases} \frac{b(t)}{b(s)}, & s \le t, \\ \frac{a(t)}{a(s)}, & t \le s, \end{cases} = \begin{cases} \frac{a(t)b(t)}{a(t)b(s)}, & s \le t, \\ \frac{a(t)b(t)}{a(s)b(t)}, & t \le s, \end{cases} \ge \frac{\tau(t)}{a(\infty)b(0)}.$$

Therefore, $G(t,s) \ge \theta\tau(t)G(s,s)$ for $0 \le t, s < +\infty$. ∎

In this article, we assume the following conditions:

- (H1) $\Delta > 0$, $\rho - \int_0^\infty g(t)b(t)dt > 0$, $\rho - \int_0^\infty h(t)a(t)dt > 0$.
- (H2) $f : (0,+\infty) \times (0,+\infty) \to (-\infty,+\infty)$ is a continuous function and
$$-\psi(t) \le f(t,u) \le \phi(t)(g(u) + h(u)), \quad (t,u) \in (0,+\infty) \times (0,+\infty),$$
where $\psi, \phi : (0,+\infty) \to [0,+\infty)$ is continuous and singular at $t = 0$, $\psi(t), \phi(t) \not\equiv 0$ on $[0,+\infty)$, $g : (0,+\infty) \to [0,+\infty)$ is continuous and non-increasing, $h : [0,+\infty) \to [0,+\infty)$ is continuous, g and h are bounded in any bounded set of $[0,+\infty)$.
- (H3) $0 < \int_0^\infty \psi(s)ds < +\infty$, $0 < \int_0^\infty G(s,s)(\psi(s) + \phi(s))ds < +\infty$.

Lemma 2.2. *Suppose* (H1) *holds, $\int_0^\infty \frac{1}{p(s)}ds < +\infty$, $\rho > 0$, then the BVP*

$$\begin{gathered} (p(t)\omega'(t))' + \psi(t) = 0, \quad t \in (0,+\infty), \\ \alpha_1\omega(0) - \beta_1 \lim_{t\to0^+} p(t)\omega'(t) = \int_0^\infty g(t)\omega(t)dt, \\ \alpha_2 \lim_{t\to+\infty} \omega(t) + \beta_2 \lim_{t\to+\infty} p(t)\omega'(t) = \int_0^\infty h(t)\omega(t)dt \end{gathered}$$

has a unique solution for any $\psi \in L(0,+\infty)$. Moreover, this unique solution can be expressed in the form

$$\omega(t) = \Big(\int_0^\infty G(t,s)\psi(s)ds + A(\psi)a(t) + B(\psi)b(t)\Big), \tag{2.3}$$

where $G(t,s)$ is defined by (2.1) *and*

$$A(\psi) = \frac{1}{\Delta} \begin{vmatrix} \int_0^\infty g(t)\int_0^\infty G(t,s)\psi(s)\,ds\,dt & \rho - \int_0^\infty g(t)b(t)dt \\ -\int_0^\infty h(t)\int_0^\infty G(t,s)\psi(s)\,ds\,dt & \int_0^\infty h(t)b(t)dt \end{vmatrix},$$

$$B(\psi)=\frac{1}{\Delta}\begin{vmatrix}\int_0^\infty h(t)\int_0^\infty G(t,s)\psi(s)\,ds\,dt & \rho-\int_0^\infty h(t)a(t)dt\\ -\int_0^\infty g(t)\int_0^\infty G(t,s)\psi(s)\,ds\,dt & \int_0^\infty g(t)a(t)dt\end{vmatrix}.$$

The proof of the above lemma is similar to [13], so we omit it. Let

$$A=\frac{1}{\Delta}\begin{vmatrix}\int_0^\infty g(t)\tau(t)dt & \rho-\int_0^\infty g(t)b(t)dt\\ -\int_0^\infty h(t)\tau(t)dt & \int_0^\infty h(t)b(t)dt\end{vmatrix},$$

$$B=\frac{1}{\Delta}\begin{vmatrix}\int_0^\infty h(t)\tau(t)dt & \rho-\int_0^\infty h(t)a(t)dt\\ -\int_0^\infty g(t)\tau(t)dt & \int_0^\infty g(t)a(t)dt\end{vmatrix}.$$

We choose a constant $\overline{d}$, such that $\overline{d}\ge a(\infty)b(0)+Aa(\infty)+Bb(0)$, and denote

$$\zeta(t)=\frac{\tau(t)+Aa(t)+Bb(t)}{\overline{d}}, \tag{2.4}$$

then $\overline{d}\theta\ge 1,\ 0<\zeta(t)\le 1$.

Lemma 2.3. *The solution defined by* (2.3) *satisfies* $\omega(t)\le\eta\zeta(t)$*, where* $\eta=\frac{\overline{d}}{\rho}\int_0^\infty\psi(s)ds$.

Proof. Since $\omega(t)$ is the unique solution of (2.3). By (2.2)-(2.4), we have

$$\begin{aligned}\omega(t)&\le\Big(\int_0^\infty\frac{\tau(t)\psi(s)}{\rho}ds+Aa(t)\int_0^\infty\frac{\psi(s)}{\rho}ds+Bb(t)\int_0^\infty\frac{\psi(s)}{\rho}ds\Big)\\&\le(\tau(t)+Aa(t)+Bb(t))\int_0^\infty\frac{\psi(s)}{\rho}ds\\&=\frac{\overline{d}\zeta(t)}{\rho}\int_0^\infty\psi(s)ds=\eta\zeta(t).\end{aligned}$$

□

To study (1.1) we use the space

$$X=\big\{x\in C[0,+\infty):\lim_{t\to+\infty}x(t)\text{ exists}\big\}. \tag{2.5}$$

Clearly $(X,\|\cdot\|)$ is a Banach space with the norm $\|x\|=\sup_{t\in[0,+\infty)}|x(t)|$, see [16]. Let

$$K=\big\{x\in X:x(t)\ge\frac{\gamma\zeta(t)}{2}\|x\|,\ t\in[0,+\infty)\big\},$$

where $0<\gamma=\min\{1,\frac{\beta_1}{a(\infty)},\frac{\beta_2}{b(0)}\}\le 1,\zeta(t)$ is defined by (2.4). It is easy to see that K is a cone in X.

Next we consider the singular nonlinear boundary value problem

$$\begin{gathered}(p(t)x'(t))'+(f(t,[x(t)-\omega(t)]^*)+\psi(t))=0,\quad t\in(0,+\infty),\\ \alpha_1x(0)-\beta_1\lim_{t\to0^+}p(t)x'(t)=\int_0^\infty g(t)x(t)dt,\\ \alpha_2\lim_{t\to+\infty}x(t)+\beta_2\lim_{t\to+\infty}p(t)x'(t)=\int_0^\infty h(t)x(t)dt,\end{gathered} \tag{2.6}$$

where $\omega(t)$ is defined in Lemma 2.2, $[z(t)]^*=\max\{z(t),0\}$.

Lemma 2.4. *If x is a solution of* (2.6) *with $x(t)>\omega(t)$ for any $t\in[0,+\infty)$, then $x(t)-\omega(t)$ is a positive solution of* (1.1).

Proof. If x is a positive solution of (2.6) such that $x(t) > \omega(t)$ for any $t \in [0, +\infty)$, then from (2.6) and the definition of $[z(t)]^*$, we have

$$
\begin{gathered}
(p(t)x'(t))' + (f(t, x(t) - \omega(t)) + \psi(t)) = 0,\ t \in (0, +\infty),\\
\alpha_1 x(0) - \beta_1 \lim_{t\to 0^+} p(t)x'(t) = \int_0^\infty g(t)x(t)dt,\\
\alpha_2 \lim_{t\to +\infty} x(t) + \beta_2 \lim_{t\to +\infty} p(t)x'(t) = \int_0^\infty h(t)x(t)dt.
\end{gathered}
\tag{2.7}
$$

Let $u(t) = x(t) - \omega(t)$, $t \in [0, +\infty)$, then $(p(t)x'(t))' = (p(t)u'(t))' + (p(t)\omega'(t))'$. Thus, (2.7) becomes

$$
\begin{gathered}
(p(t)u'(t))' + f(t, u(t)) = 0, \quad t \in (0, +\infty),\\
\alpha_1 u(0) - \beta_1 \lim_{t\to 0^+} p(t)u'(t) = \int_0^\infty g(t)x(t)dt,\\
\alpha_2 \lim_{t\to +\infty} u(t) + \beta_2 \lim_{t\to +\infty} p(t)u'(t) = \int_0^\infty h(t)x(t)dt.
\end{gathered}
$$

Then $u(t) = x(t) - \omega(t)$ is a positive solution of (1.1). □

To overcome the singularity, we consider the approximate problem

$$
\begin{gathered}
(p(t)x'(t))' + \Big(f\big(t, [x(t) - \omega(t)]^* + \frac{1}{n}\big) + \psi(t)\Big) = 0, \quad t \in (0, +\infty),\\
\alpha_1 x(0) - \beta_1 \lim_{t\to 0^+} p(t)x'(t) = \int_0^\infty g(t)x(t)dt,\\
\alpha_2 \lim_{t\to +\infty} x(t) + \beta_2 \lim_{t\to +\infty} p(t)x'(t) = \int_0^\infty h(t)x(t)dt,
\end{gathered}
\tag{2.8}
$$

where n is a positive integer. Under the assumptions (H1)–(H3), for any $n \in \mathbb{N}$, where $\mathbb{N}$ is a natural number set, we define a nonlinear integral operator $T_n : K \to X$ by

$$
\begin{aligned}
(T_n x)(t) = &\int_0^\infty G(t, s)\Big(f\big(s, [x(s) - \omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds\\
&+ A(f_n + \psi)a(t) + B(f_n + \psi)b(t), \quad t \in [0, +\infty),
\end{aligned}
\tag{2.9}
$$

where

$$
A(f_n + \psi) = \frac{1}{\Delta}\begin{vmatrix} \int_0^\infty g(t)\int_0^\infty G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)\,ds\,dt & \rho-\int_0^\infty g(t)b(t)dt\\ -\int_0^\infty h(t)\int_0^\infty G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)\,ds\,dt & \int_0^\infty h(t)b(t)dt \end{vmatrix},
$$

$$
B(f_n + \psi) = \frac{1}{\Delta}\begin{vmatrix} \int_0^\infty h(t)\int_0^\infty G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)\,ds\,dt & \rho-\int_0^\infty h(t)a(t)dt\\ -\int_0^\infty g(t)\int_0^\infty G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)\,ds\,dt & \int_0^\infty g(t)a(t)dt \end{vmatrix}.
$$

Obviously, the existence of solutions to (2.8) is equivalent to the existence of solutions in K for operator equation $T_n x = x$ defined by (2.9).

We list the following lemmas which are needed in our arguments.

Lemma 2.5 ([4]). *Let X be defined by (2.5) and $M \subset X$. Then M is relatively compact in X if the following conditions hold:*

1. *M is uniformly bounded in X;*

(2) *the functions from M are equicontinuous on any compact subinterval of* $[0, +\infty)$*;*
(3) *the functions from M are equiconvergent, that is, for any given $\varepsilon > 0$, there exists a $T = T(\varepsilon) > 0$ such that $|x(t) - x(+\infty)| < \varepsilon$, for any $t > T$, $x \in M$.*

Lemma 2.6 ([7]). *Let P be a positive cone in Banach space E, Ω_1, Ω_2 are bounded open sets in E, $\theta \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, $A : P \cap \overline{\Omega}_2 \backslash \Omega_1 \to P$ is a completely continuous operator. If the following conditions are satisfied:*

$$\|Ax\| \le \|x\|, \ \forall x \in P \cap \partial\Omega_1, \quad \|Ax\| \ge \|x\|, \ \forall x \in P \cap \partial\Omega_2,$$

or

$$\|Ax\| \ge \|x\|, \ \forall x \in P \cap \partial\Omega_1, \quad \|Ax\| \le \|x\|, \ \forall x \in P \cap \partial\Omega_2,$$

then A has at least one fixed point in $P \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

3. Main results

Lemma 3.1. *Assume that* (H1)–(H3) *hold. Then $T_n : K \to K$ is a completely continuous operator for any fixed $n \in \mathbb{N}$.*

Proof. (1) we show $T_n : K \to X$ is well defined. For $x \in K$, there exists $r > 0$ such that $|x(t)| \le r$, for $t \in [0, +\infty)$, also $|[x(t) - \omega(t)]^*| \le |x(t)| \le r$, $t \in [0, +\infty)$. From (H2) and the definition of g and h, we have

$$S_{r,n} := \sup\left\{g(u) + h(u) : \frac{1}{n} \le u \le r + 1\right\} < +\infty.$$

Thus, by (H2) and (H3), for any $t \in [0, +\infty)$, we have

$$\begin{aligned}
&\int_0^\infty G(t,s)\Big(f\big(s, [x(s) - \omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds \\
&\le \int_0^\infty G(s,s)\Big(\phi(s)\Big(g\big([x(s) - \omega(s)]^* + \frac{1}{n}\big) \\
&\quad + h\big([x(s) - \omega(s)]^* + \frac{1}{n}\big)\Big) + \psi(s)\Big)ds \\
&\le \int_0^\infty G(s,s)(\phi(s)S_{r,n} + \psi(s))ds \\
&\le (S_{r,n} + 1)\int_0^\infty G(s,s)(\phi(s) + \psi(s))ds < +\infty.
\end{aligned} \tag{3.1}$$

By (H2) and (H3), for any $t \in [0, +\infty)$, we also have

$$\begin{aligned}
&A((f_n + \psi))a(t) \\
&= \frac{a(t)}{\Delta}\begin{vmatrix} \int_0^\infty g(t)\int_0^\infty G(t,s)\Big(f\Big(s,[x(s)-\lambda\omega(s)]^*+\frac{1}{n}\Big)+\psi(s)\Big)\,ds\,dt & \rho-\int_0^\infty g(t)b(t)dt \\ -\int_0^\infty h(t)\int_0^\infty G(t,s)\Big(f\Big(s,[x(s)-\lambda\omega(s)]^*+\frac{1}{n}\Big)+\psi(s)\Big)\,ds\,dt & \int_0^\infty h(t)b(t)dt \end{vmatrix} \\
&\le \frac{a(\infty)}{\Delta}\begin{vmatrix} \int_0^\infty g(t)(S_{r,n}+1)\int_0^\infty G(s,s)(\phi(s)+\psi(s))\,ds\,dt & \rho-\int_0^\infty g(t)b(t)dt \\ -\int_0^\infty h(t)(S_{r,n}+1)\int_0^\infty G(s,s)(\phi(s)+\psi(s))\,ds\,dt & \int_0^\infty h(t)b(t)dt \end{vmatrix} \\
&= \frac{a(\infty)}{\Delta}\begin{vmatrix} \int_0^\infty g(t)dt & \rho-\int_0^\infty g(t)b(t)dt \\ -\int_0^\infty h(t)dt & \int_0^\infty h(t)b(t)dt \end{vmatrix}(S_{r,n} + 1)\int_0^\infty G(s,s)(\phi(s) + \psi(s))ds \\
&= \overline{A}a(\infty)(S_{r,n} + 1)\int_0^\infty G(s,s)(\phi(s) + \psi(s))ds < +\infty,
\end{aligned} \tag{3.2}$$

where

$$\overline{A}=\frac{1}{\Delta}\begin{vmatrix}\int_0^\infty g(t)dt & \rho-\int_0^\infty g(t)b(t)dt\\ -\int_0^\infty h(t)dt & \int_0^\infty h(t)b(t)dt\end{vmatrix}. \tag{3.3}$$

In the same way, for any $t\in[0,+\infty)$, we obtain

$$B((f_n+\psi))b(t)\le \overline{B}b(0)(S_{r,n}+1)\int_0^\infty G(s,s)(\phi(s)+\psi(s))ds<+\infty, \tag{3.4}$$

where

$$\overline{B}=\frac{1}{\Delta}\begin{vmatrix}\int_0^\infty h(t)dt & \rho-\int_0^\infty h(t)a(t)dt\\ -\int_0^\infty g(t)dt & \int_0^\infty g(t)a(t)dt\end{vmatrix}. \tag{3.5}$$

Hence, by (3.1), (3.2) and (3.4), we can see that

$$\begin{aligned}(T_nx)(t)&=\int_0^\infty G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\\ &\quad +A(f_n+\psi)a(t)+B(f_n+\psi)b(t)\\ &\le (1+\overline{A}a(\infty)+\overline{B}b(0))(S_{r,n}+1)\int_0^\infty G(s,s)(\phi(s)+\psi(s))ds\\ &<+\infty,\quad t\in[0,+\infty).\end{aligned} \tag{3.6}$$

Then, from (3.6), T_nx is well defined for any $x\in K$.

On the other hand, for any $t,t_j\in[0,+\infty)$, $t_j\to t$, by the continuity of $G(t,s)$, we obtain

$$\begin{aligned}&G(t_j,s)\Big(f\big(s,[x(s)-\lambda\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)\\ &\to G(t,s)\Big(f\big(s,[x(s)-\lambda\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big),\end{aligned} \tag{3.7}$$

for $s\in[0,+\infty)$ as $j\to+\infty$. By (2.2), we have

$$\begin{aligned}&\int_0^\infty G(t_j,s)\Big(f\big(s,[x(s)-\lambda\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\\ &\le (S_{r,n}+1)\int_0^\infty G(s,s)(\phi(s)+\psi(s))ds<+\infty,\\ &\int_0^\infty G(t,s)\Big(f\big(s,[x(s)-\lambda\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\\ &\le (S_{r,n}+1)\int_0^\infty G(s,s)(\phi(s)+\psi(s))ds<+\infty.\end{aligned} \tag{3.8}$$

So, by (H3), (3.7), (3.8) and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}&\lim_{t_j\to t}\int_0^\infty G(t_j,s)\Big(f\big(s,[x(s)-\lambda\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\\ &=\int_0^\infty \lim_{t_j\to t}G(t_j,s)\Big(f\big(s,[x(s)-\lambda\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\\ &=\int_0^\infty G(t,s)\Big(f\big(s,[x(s)-\lambda\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds.\end{aligned}$$

Consequently, together with the continuity of $a(t)$ and $b(t)$, we have

$$\begin{aligned}&|T_nx(t_j)-T_nx(t)|\\ &=\Big|\int_0^\infty G(t_j,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\end{aligned}$$

$$
\begin{aligned}
&+ A(f_n+\psi)a(t_j) + B(f_n+\psi)b(t_j) \\
&- \int_0^\infty G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds \\
&- A(f_n+\psi)a(t) - B(f_n+\psi)b(t)\Big| \\
\le& \Big|\int_0^\infty (G(t_j,s)-G(t,s))\Big(f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds\Big| \\
&+ A(f_n+\psi))|a(t_j)-a(t)| + B(f_n+\psi))|b(t_j)-b(t)| \\
\to& 0, \text{ as } j\to+\infty.
\end{aligned}
$$

Therefore, $T_n x \in C[0,+\infty)$. In what follows, for any $\bar{t}_j \in [0,+\infty)$, $\bar{t}_j \to +\infty$, by (2.2), we have

$$
\begin{aligned}
&G(\bar{t}_j,s)\Big(f\big(s,[x(s)-\lambda\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big) \\
&\to \overline{G}(s)\Big(f\big(s,[x(s)-\lambda\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big),
\end{aligned}
$$

for $s\in[0,+\infty)$ as $j\to+\infty$. Then by the Lebesgue dominated convergence theorem and the property of $a(t)$, $b(t)$, we also have

$$
\begin{aligned}
\lim_{j\to+\infty}(T_n x)(\bar{t}_j) =& \int_0^\infty \overline{G}(s)\Big(f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds \\
&+ A(f_n+\psi)a(\infty) + B(f_n+\psi)\beta_2 < +\infty.
\end{aligned}
$$

So, for any $x\in K$, we obtain $T_n x\in X$, which implies that T_n maps K to X.

(2) we show $T_n(K)\subseteq K$. For any $x\in K$, from the definition of $\|\cdot\|$ and (2.2), we have

$$
\begin{aligned}
\|T_n x\| \le& \int_0^\infty G(s,s)\Big(f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds \\
&+ A(f_n+\psi)a(\infty) + B(f_n+\psi)b(0).
\end{aligned} \tag{3.9}
$$

By Lemma 2.1, (2.4) and the monotonicity of $a(t)$, $b(t)$, we obtain

$$
\begin{aligned}
&(T_n x)(t) \\
&\ge \theta\tau(t)\int_0^\infty G(s,s)\Big(f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds \\
&\quad + A(f_n+\psi)a(t) + B(f_n+\psi)b(t) \\
&\ge \frac{\theta\tau(t)}{2}\int_0^\infty G(s,s)\Big(f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds \\
&\quad + A(f_n+\psi)a(t) + B(f_n+\psi)b(t) \\
&= \frac{1}{2}\Big(\theta\tau(t)\int_0^\infty G(s,s)\Big(f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds \\
&\quad + A(f_n+\psi)a(t) + B(f_n+\psi)b(t)\Big) + \frac{1}{2}\left(A(f_n+\psi)a(t) + B(f_n+\psi)b(t)\right) \\
&\ge \frac{\theta}{2}\left(\tau(t) + Aa(t) + Bb(t)\right)\int_0^\infty G(s,s)\Big(f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds \\
&\quad + \frac{1}{2}\left(A(f_n+\psi)a(t) + B(f_n+\psi)b(t)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\theta \bar{d}\zeta(t)}{2}\int_0^\infty G(s,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds \\
&\quad + \frac{1}{2}\left(A(f_n+\psi)a(t)+B(f_n+\psi)b(t)\right) \\
&\ge \frac{\zeta(t)}{2}\int_0^\infty G(s,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds \\
&\quad + \frac{1}{2}\zeta(t)\left(A(f_n+\psi)a(t)+B(f_n+\psi)b(t)\right) \\
&= \frac{\zeta(t)}{2}\Big(\int_0^\infty G(s,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds \\
&\quad + A(f_n+\psi)a(t)+B(f_n+\psi)b(t)\Big) \\
&\ge \frac{\zeta(t)}{2}\Big(\int_0^\infty G(s,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds \\
&\quad + \frac{\beta_1}{a(\infty)}A(f_n+\psi)a(\infty)+\frac{\beta_2}{b(0)}B(f_n+\psi)b(0)\Big) \\
&\ge \frac{\gamma\zeta(t)}{2}\Big(\int_0^\infty G(s,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds \\
&\quad + A(f_n+\psi)a(\infty)+B(f_n+\psi)b(0)\Big) \qquad (3.10)
\end{aligned}
$$

Combining (3.9) and (3.10), we have $(T_nx)(t) \ge \frac{\gamma\zeta(t)}{2}\|T_nx\|$, for any $t \in [0,+\infty)$. Therefore, $T_n(K) \subseteq K$.

(3) for any positive integers $n,\ k \in \mathbb{N}$, we define an operator $T_{n,k}: K \to X$ by

$$
\begin{aligned}
(T_{n,k}x)(t) &= \int_{1/k}^\infty G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds \\
&\quad + A_k(f_n+\psi)a(t)+B_k(f_n+\psi)b(t),\ t\in[0,+\infty),
\end{aligned} \qquad (3.11)
$$

where

$$
\begin{aligned}
&A_k(f_n+\psi) \\
&= \frac{1}{\Delta}\begin{vmatrix} \int_0^\infty g(t)\int_{1/k}^\infty G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)\,ds\,dt & \rho-\int_0^\infty g(t)b(t)dt \\ -\int_0^\infty h(t)\int_{1/k}^\infty G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)\,ds\,dt & \int_0^\infty h(t)b(t)dt \end{vmatrix}, \\
&B_k(f_n+\psi) \\
&= \frac{1}{\Delta}\begin{vmatrix} \int_0^\infty h(t)\int_{1/k}^\infty G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)\,ds\,dt & \rho-\int_0^\infty h(t)a(t)dt \\ -\int_0^\infty g(t)\int_{1/k}^\infty G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)\,ds\,dt & \int_0^\infty g(t)a(t)dt \end{vmatrix}.
\end{aligned}
$$

Using the similar method as the discussion in (1) and (2), we obtain $T_{n,k}: K \to X$ is well defined and $T_{n,k}(K) \subseteq K$. In what follows, we will prove that $T_{n,k}: K \to K$ is completely continuous, for each $k \ge 1$.

(i) we show $T_{n,k}: K \to K$ is continuous for any natural numbers n,k. Let $x_\upsilon, x \in K$ are such that $\|x_\upsilon - x\| \to 0$ as $\upsilon \to +\infty$. By (3.11) and (H3), we know

$$
\Big|\int_{1/k}^\infty G(t,s)\Big(f\big(s,[x_\upsilon(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds
$$

$$
\begin{aligned}
&- \int_{1/k}^{\infty} G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds\Big| \\
&\le \int_{1/k}^{\infty} G(s,s)\Big(f\big(s,[x_\upsilon(s)-\omega(s)]^* + \frac{1}{n}\big) + f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + 2\psi(s)\Big)ds \\
&\le \int_{1/k}^{\infty} G(s,s)\Big(\phi(s)\Big(g\big([x_\upsilon(s)-\omega(s)]^* + \frac{1}{n}\big) + h\big([x_\upsilon(s)-\omega(s)]^* + \frac{1}{n}\big)\Big) \\
&\quad + \phi(s)\Big(g\left([x(s)-\omega(s)]^* + \frac{1}{n}\right) + h\big([x(s)-\omega(s)]^* + \frac{1}{n}\big)\Big) + 2\psi(s)\Big)ds \\
&\le 2(S_{r',n}+1)\int_0^{\infty} G(s,s)(\phi(s)+\psi(s))ds < +\infty, \quad t\in[0,+\infty)
\end{aligned}
\tag{3.12}
$$

where $S_{r',n} := \sup\{g(u)+h(u) : \frac{1}{n} \le u \le r'+1\} < +\infty$ (by (H2)), r' is a real number such that $r' \ge \max_{\upsilon\in\mathbb{N}}\{\|x\|, \|x_\upsilon\|\}$. Denote

$$
\begin{aligned}
&A_{k,\upsilon}(f_n+\psi) \\
&= \frac{1}{\Delta}\begin{vmatrix} \int_0^{\infty} g(t)\int_{1/k}^{\infty} G(t,s)\Big(f\Big(s,[x_\upsilon(s)-\omega(s)]^*+\frac{1}{n}\Big)+\psi(s)\Big)\,ds\,dt & \rho-\int_0^{\infty} g(t)b(t)dt \\ -\int_0^{\infty} h(t)\int_{1/k}^{\infty} G(t,s)\Big(f\Big(s,[x_\upsilon(s)-\omega(s)]^*+\frac{1}{n}\Big)+\psi(s)\Big)\,ds\,dt & \int_0^{\infty} h(t)b(t)dt \end{vmatrix}, \\
&B_{k,\upsilon}(f_n+\psi) \\
&= \frac{1}{\Delta}\begin{vmatrix} \int_0^{\infty} h(t)\int_{1/k}^{\infty} G(t,s)\Big(f\Big(s,[x_\upsilon(s)-\omega(s)]^*+\frac{1}{n}\Big)+\psi(s)\Big)\,ds\,dt & \rho-\int_0^{\infty} h(t)a(t)dt \\ -\int_0^{\infty} g(t)\int_{1/k}^{\infty} G(t,s)\Big(f\Big(s,[x_\upsilon(s)-\omega(s)]^*+\frac{1}{n}\Big)+\psi(s)\Big)\,ds\,dt & \int_0^{\infty} g(t)a(t)dt \end{vmatrix}.
\end{aligned}
$$

Through calculation, we obtain

$$
\begin{aligned}
&|A_{k,\upsilon}(f_n+\psi) - A_k(f_n+\psi)|a(t) \\
&\le \frac{a(\infty)}{\Delta}\begin{vmatrix} \int_0^{\infty} g(t)dt & \rho-\int_0^{\infty} g(t)b(t)dt \\ -\int_0^{\infty} h(t)dt & \int_0^{\infty} h(t)b(t)dt \end{vmatrix} \\
&\quad \times \int_{1/k}^{\infty} G(s,s)\Big(f\big(s,[x_\upsilon(s)-\omega(s)]^* + \frac{1}{n}\big) + f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + 2\psi(s)\Big)ds \\
&\le \overline{A}\int_{1/k}^{\infty} G(s,s)\Big(\phi(s)\Big(g\big([x_\upsilon(s)-\omega(s)]^* + \frac{1}{n}\big) + h\big([x_\upsilon(s)-\omega(s)]^* + \frac{1}{n}\big)\Big) + \psi(s) \\
&\quad + \phi(s)\Big(g\big([x(s)-\omega(s)]^* + \frac{1}{n}\big) + h\big([x(s)-\omega(s)]^* + \frac{1}{n}\big)\Big) + \psi(s)\Big)ds \\
&\le 2\overline{A}a(\infty)(S_{r',n}+1)\int_0^{\infty} G(s,s)(\phi(s)+\psi(s))ds < +\infty, \quad t\in[0,+\infty).
\end{aligned}
\tag{3.13}
$$

In the same way, we obtain

$$
\begin{aligned}
&|B_{k,\upsilon}(f_n+\psi) - B_k(f_n+\psi)|b(t) \\
&\le 2\overline{B}b(0)(S_{r',n}+1)\int_0^{\infty} G(s,s)(\phi(s)+\psi(s))ds < +\infty, \quad t\in[0,+\infty).
\end{aligned}
\tag{3.14}
$$

From (3.12)-(3.14), for any $\varepsilon > 0$, by (H3), there exists a sufficiently large A_0 ($A_0 > 1/k$), such that

$$
\max\{1,\ a(\infty)\overline{A},\ b(0)\overline{B}\}(S_{r',n}+1)\int_{A_0}^{\infty} G(s,s)(\phi(s)+\psi(s))ds < \frac{\varepsilon}{12}. \tag{3.15}
$$

On the other hand, by the continuity of $f\left(s, u+\frac{1}{n}\right)$ on $[1/k, A_0]\times[0,r']$, for the above $\varepsilon>0$, there exists a $\delta>0$ such that for any $s\in[1/k,A_0]$ and $u,\upsilon\in[0,r']$, when $|u-\upsilon|=\left|\left(u+\frac{1}{n}\right)-\left(\upsilon+\frac{1}{n}\right)\right|<\delta$, we have

$$\left|f\big(s,u+\frac{1}{n}\big)-f\big(s,\upsilon+\frac{1}{n}\big)\right|<\frac{\varepsilon}{6}\Big(\max\big\{1,\ a(\infty)\overline{A},\ b(0)\overline{B}\big\}\int_{1/k}^{A_0}G(s,s)ds\Big)^{-1}. \tag{3.16}$$

From $\|x_\upsilon-x\|\to 0$ $(n\to+\infty)$ and the definition of the norm $\|\cdot\|$ in the space X, for the above $\delta>0$, there exists a sufficiently large nature number V_0, such that when $\upsilon>V_0$, for all $s\in[1/k,A_0]$, we have

$$\begin{aligned}
&\left|\big([x_\upsilon(s)-\omega(s)]^*+\frac{1}{n}\big)-\big([x(s)-\omega(s)]^*+\frac{1}{n}\big)\right| \\
&\le \Big|\frac{|x_\upsilon(s)-\omega(s)|+x_\upsilon(s)-\omega(s)}{2}-\frac{|x(s)-\omega(s)|+x(s)-\omega(s)}{2}\Big| \\
&=\Big|\frac{|x_\upsilon(s)-\omega(s)|-|x(s)-\omega(s)|}{2}+\frac{x_\upsilon(s)-x(s)}{2}\Big| \\
&\le |x_\upsilon(s)-x(s)|\le\|x_\upsilon-x\|<\delta.
\end{aligned}\tag{3.17}$$

Hence, by (3.15)–(3.17), when $\upsilon>V_0$, $t\in[0,+\infty)$, we have the inequality

$$\begin{aligned}
&|A_{k,\upsilon}(f_n+\psi)-A_k(f_n+\psi)|a(t)\\
&\le |A_{k,\upsilon}(f_n+\psi)-A_k(f_n+\psi)|a(\infty)\\
&=\frac{a(\infty)}{\Delta}\left|\begin{vmatrix}F_1 & \rho-\int_0^\infty g(t)b(t)dt\\ F_2 & \int_0^\infty h(t)b(t)dt\end{vmatrix}-\begin{vmatrix}H_1 & \rho-\int_0^\infty g(t)b(t)dt\\ H_2 & \int_0^\infty h(t)b(t)dt\end{vmatrix}\right|\\
&\le a(\infty)\overline{A}\int_{1/k}^{A_0}G(s,s)\left|f\big(s,[x_\upsilon(s)-\omega(s)]^*+\frac{1}{n}\big)-f\big(s,[x-\omega(s)]^*+\frac{1}{n}\big)\right|ds\\
&\quad+2a(\infty)\overline{A}(S_{r',n}+1)\int_{A_0}^\infty G(s,s)(\phi(s)+\psi(s))ds\le\frac{\varepsilon}{3},
\end{aligned}\tag{3.18}$$

where

$$\begin{aligned}
F_1&=\int_0^\infty g(t)\Big(\int_{1/k}^{A_0}+\int_{A_0}^\infty\Big)G(t,s)\Big(f\big(s,[x_\upsilon(s)-\lambda\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)\,ds\,dt,\\
F_2&=-\int_0^\infty h(t)\Big(\int_{1/k}^{A_0}+\int_{A_0}^\infty\Big)G(t,s)\Big(f\big(s,[x_\upsilon(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)\,ds\,dt,\\
H_1&=\int_0^\infty g(t)\Big(\int_{1/k}^{A_0}+\int_{A_0}^\infty\Big)G(t,s)\Big(f\big(s,[x-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)\,ds\,dt,\\
H_2&=-\int_0^\infty h(t)\Big(\int_{1/k}^{A_0}+\int_{A_0}^\infty\Big)G(t,s)\Big(f\big(s,[x-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)\,ds\,dt.
\end{aligned}$$

Using the same method as (3.18), when $\upsilon>V_0$, $t\in[0,+\infty)$, we obtain

$$|B_{k,\upsilon}(f_n+\psi)-B_k(f_n+\psi)|b(t)\le\frac{\varepsilon}{3}. \tag{3.19}$$

Then, by (3.18), (3.19) and the above discussion, when $\upsilon>V_0$, $t\in[0,+\infty)$, we obtain

$$|(T_{n,k}x_\upsilon)(t)-(T_{n,k}x)(t)|$$

$$
\begin{aligned}
&= \Big| \int_{1/k}^{\infty} G(t,s)\Big(f\big(s,[x_\upsilon(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big) ds \\
&\quad + A_{k,\upsilon}(f_n+\psi)a(t) + B_{k,\upsilon}(f_n+\psi)b(t) \\
&- \int_{1/k}^{\infty} G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big) ds \\
&\quad - A_k(f_n+\psi)a(t) - B_k(f_n+\psi)b(t)\Big| \\
&\le \int_{1/k}^{A_0} G(s,s)\big| f\big(s,[x_\upsilon(s)-\omega(s)]^* + \frac{1}{n}\big) - f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big)\big| ds \\
&\quad + |A_{k,\upsilon}(f_n+\psi) - A_k(f_n+\psi)|a(\infty) + |B_{k,\upsilon}(f_n+\psi) - B_k(f_n+\psi)|b(0) \\
&\quad + \int_{A_0}^{\infty} G(s,s)\Big(f\big(s,[x_\upsilon(s)-\omega(s)]^* + \frac{1}{n}\big) + f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + 2\psi(s)\Big) ds \\
&\le \frac{5\varepsilon}{6} + 2(S_{r',n}+1)\int_0^{\infty} G(s,s)(\phi(s)+\psi(s))ds < \varepsilon.
\end{aligned}
$$

This implies that the operator $T_{n,k} : K \to K$ is continuous for any natural numbers n, k.

(ii) we show $T_{n,k} : K \to K$ is a compact operator for natural numbers n, k. First of all, let M be any bounded subset of K. Then there exists a constant $R > 0$ such that $\|x\| \le R$ for any $x \in M$. By (3.11), (H2) and (H3), for any $x \in M$, $t \in [0, +\infty)$, we have

$$
\begin{aligned}
&\Big| \int_{1/k}^{\infty} G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big) ds \Big| \\
&\le \int_{1/k}^{\infty} G(s,s)\Big(\phi(s)\Big(g\big([x(s)-\omega(s)]^* + \frac{1}{n}\big) + h\big([x(s)-\omega(s)]^* + \frac{1}{n}\big)\Big) + \psi(s)\Big) ds \\
&\le \int_{1/k}^{\infty} G(s,s)(\phi(s)S_{R,n} + \psi(s))ds \\
&\le (S_{R,n}+1)\int_0^{\infty} G(s,s)(\phi(s)+\psi(s))ds < +\infty,
\end{aligned} \tag{3.20}
$$

where $S_{R,n} := \sup\{g(u) + h(u) : \frac{1}{n} \le u \le R+1\}$. By proof similar to (3.2), (3.4), for any $x \in M$, $t \in [0, +\infty)$, we have

$$
\begin{gathered}
A_k(f_n+\psi)a(t) \le \overline{A}a(\infty)(S_{R,n}+1)\int_0^{\infty} G(s,s)(\phi(s)+\psi(s))ds < +\infty, \\
B_k(f_n+\psi)b(t) \le \overline{B}b(0)(S_{R,n}+1)\int_0^{\infty} G(s,s)(\phi(s)+\psi(s))ds < +\infty.
\end{gathered} \tag{3.21}
$$

Then, from (3.20), (3.21), for any $x \in M$, $t \in [0, +\infty)$, we have

$$
\begin{aligned}
&|(T_n x)(t)| \\
&\le (1 + \overline{A}a(\infty) + \overline{B}b(0))(S_{R,n}+1)\int_0^{\infty} G(s,s)(\phi(s)+\psi(s))ds < +\infty.
\end{aligned}
$$

Therefore, $T_{n,k}M$ is bounded in K.

Next, given $\overline{a} > 0$, for any $x \in M$ and $t, t' \in [0, \overline{a}]$, by (3.11), we obtain

$$\begin{aligned}
&\Big| \int_{1/k}^{\infty} G(t,s)\Big(f\big(s, [x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds \\
&\quad - \int_{1/k}^{\infty} G(t',s)\Big(f\big(s, [x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds\Big| \\
&\le \int_{1/k}^{\infty} |G(t,s) + G(t',s)|\Big(f\big(s, [x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds \\
&\le 2(S_{R,n}+1)\int_{1/k}^{\infty} G(s,s)(\phi(s)+\psi(s))ds < +\infty,
\end{aligned}$$

and so, for any $\varepsilon' > 0$, we can find a sufficiently large H_0 ($H_0 > \frac{1}{k}$) such that

$$(S_{R,n}+1)\int_{H_0}^{\infty} G(s,s)(\phi(s)+\psi(s))ds < \frac{\varepsilon'}{12}.$$

By the uniformly continuity of $G(t,s)$ on $[0,\overline{a}] \times [\frac{1}{k}, H_0]$, for the above $\varepsilon' > 0$, there exists $\delta' > 0$ such that for any $t, t' \in [0,\overline{a}]$, $s \in [\frac{1}{k}, H_0]$ and $|t-t'| < \delta'$, we have

$$|G(t,s) - G(t',s)| < \frac{\varepsilon'}{6}\Big((S_{R,n}+1)\int_{1/k}^{H_0} (\phi(s)+\psi(s))ds\Big)^{-1}.$$

Therefore, for any $x \in M$, $t, t' \in [0,\overline{a}]$, $|t-t'| < \delta'$, we obtain

$$\begin{aligned}
&\Big| \int_{1/k}^{\infty} G(t,s)\Big(f\big(s, [x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds \\
&\quad - \int_{1/k}^{\infty} G(t',s)\Big(f\big(s, [x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds\Big| \\
&\le \int_{1/k}^{H_0} |G(t,s) - G(t',s)|\Big(f\big(s, [x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds \\
&\quad + \int_{H_0}^{\infty} |G(t,s) - G(t',s)|\Big(f\big(s, [x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds \\
&\le \frac{\varepsilon'}{6} + 2(S_{R,n}+1)\int_{1/k}^{\infty} G(s,s)(\phi(s)+\psi(s))ds < \frac{\varepsilon'}{3}.
\end{aligned} \tag{3.22}$$

Also by the uniformly continuity of $a(t), b(t)$ on $[0,\overline{a}]$, for the above $\varepsilon' > 0$, there exists $\delta'' > 0$ such that for any $t, t' \in [0,\overline{a}]$ and $|t-t'| < \delta''$, we have

$$\begin{gathered}
|a(t) - a(t')| < \frac{\varepsilon'}{3}\Big(\overline{A}(S_{R,n}+1)\int_0^{\infty} G(s,s)(\phi(s)+\psi(s))ds\Big)^{-1}, \\
|b(t) - b(t')| < \frac{\varepsilon'}{3}\Big(\overline{B}(S_{R,n}+1)\int_0^{\infty} G(s,s)(\phi(s)+\psi(s))ds\Big)^{-1}.
\end{gathered} \tag{3.23}$$

By (3.22), (3.23), for the above $\varepsilon' > 0$, let $\delta_0 = \min\{\delta', \delta''\}$, then for any $t, t' \in [0,\overline{a}]$ with $|t-t'| < \delta_0$, and for any $x \in M$, we have

$$\begin{aligned}
&|T_{n,k}x(t) - T_{n,k}x(t')| \\
&= \Big| \int_{1/k}^{\infty} G(t,s)\Big(f\big(s, [x(s)-\omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds \\
&\quad + A_k(f_n+\psi)a(t) + B_k(f_n+\psi)b(t)
\end{aligned}$$

$$
\begin{aligned}
&- \int_{1/k}^{\infty} G(t',s)\Big(f\big(s,[x(s)-\omega(s)]^{*}+\frac{1}{n}\big)+\psi(s)\Big)ds \\
&- A_k(f_n+\psi)a(t') - B_k(f_n+\psi)b(t')\Big| \\
\le & \int_{1/k}^{\infty} |G(t,s)-G(t',s)|\Big(f\big(s,[x(s)-\omega(s)]^{*}+\frac{1}{n}\big)+\psi(s)\Big)ds \\
&+ \overline{A}(S_{R,n}+1)\int_0^{\infty} G(s,s)(\phi(s)+\psi(s))ds|a(t)-a(t')| \\
&+ \overline{B}(S_{R,n}+1)\int_0^{\infty} G(s,s)(\phi(s)+\psi(s))ds|b(t)-b(t')| < \varepsilon'.
\end{aligned}
$$

So, $\{T_{n,k}x : x \in M\}$ is equicontinuous on $[0,\overline{a}]$. Since $\overline{a} > 0$ is arbitrary, $\{T_{n,k}x : x \in M\}$ is locally equicontinuous on $[0,+\infty)$.

At last, let $T_{n,k}x(+\infty) = \lim_{t\to+\infty} T_{n,k}x(t)$, by a simple calculation, we can see that $\lim_{t\to+\infty} T_{n,k}x(t) < +\infty$, so we obtain

$$
\begin{aligned}
&|T_{n,k}x(t) - T_{n,k}x(+\infty)| \\
&\le \Big|\int_{1/k}^{\infty}(G(t,s)-\overline{G}(s))\Big(f\big(s,[x(s)-\omega(s)]^{*}+\frac{1}{n}\big)+\psi(s)\Big)ds\Big| \\
&\quad + A_k(f_n+\psi)|a(t)-a(\infty)| + B_k(f_n+\psi)|b(t)-b(\infty)|.
\end{aligned}
$$

By the similar method as (3.22), for any $\overline{\varepsilon} > 0$, there exists N' such that, when $t > N'$, it is true that

$$
\Big|\int_{1/k}^{\infty}(G(t,s)-\overline{G}(s))\Big(f\big(s,[x(s)-\omega(s)]^{*}+\frac{1}{n}\big)+\psi(s)\Big)ds\Big| < \frac{\overline{\varepsilon}}{3}.
$$

Together with the continuity of $a(t), b(t)$ on $[0,+\infty)$, we obtain that for the above $\overline{\varepsilon} > 0$, there exists N' such that, when $t > N'$, we have $|T_{n,k}x(t) - T_{n,k}x(+\infty)| < \overline{\varepsilon}$. Hence, $\{T_{n,k}x : x \in M\}$ is equiconvergent at $+\infty$, which implies that $\{T_{n,k}x : x \in M\}$ is relatively compact (by Lemma 2.5).

Thus, together with the continuity of $T_{n,k}$ which we discuss in (2), we obtain that the operator $T_{n,k} : K \to K$ is completely continuous for natural numbers n, k.

(4) we show $T_n : K \to K$ is a completely continuous operator. For any $t \in [0,+\infty)$ and $x \in S = \{x \in K : \|x\| \le 1\}$, by (2.9) and (3.11), we have

$$
\begin{aligned}
&\int_0^{1/k} G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^{*}+\frac{1}{n}\big)+\psi(s)\Big)ds \\
&\le \int_0^{1/k} G(s,s)\Big(\phi(s)\Big(g\big([x(s)-\omega(s)]^{*}+\frac{1}{n}\big)+h\big([x(s)-\omega(s)]^{*}+\frac{1}{n}\big)\Big)+\psi(s)\Big)ds \\
&\le \int_0^{\frac{1}{k}} G(s,s)(\phi(s)S_{1,n}+\psi(s))ds \to 0, \quad k\to+\infty,
\end{aligned}
\tag{3.24}
$$

where $S_{1,n} := \sup\{g(u)+h(u) : \frac{1}{n} \le u \le 2\} < +\infty$. And then, for any $t \in [0,+\infty)$, $x \in S$, we obtain

$$\begin{aligned}
&|A(f_n+\psi) - A_k(f_n+\psi)|a(t) \\
&\le \frac{a(\infty)}{\Delta} \begin{vmatrix} \int_0^\infty g(t)dt & \rho - \int_0^\infty g(t)b(t)dt \\ -\int_0^\infty h(t)dt & \int_0^\infty h(t)b(t)dt \end{vmatrix} \\
&\quad \times \int_0^{1/k} G(s,s)\Big(f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + f\big(s,[x(s)-\omega(s)]^* + \frac{1}{n}\big) + 2\psi(s)\Big)ds \\
&\le 2\overline{A}(S_{1,n}+1)a(\infty)\int_0^{\frac{1}{k}} G(s,s)(\phi(s)+\psi(s))ds \to 0, \quad k \to +\infty.
\end{aligned} \tag{3.25}$$

Using the similar method, for any $t \in [0,+\infty)$, $x \in S$, we have

$$\begin{aligned}
&|B(f_n+\psi) - B_k(f_n+\psi)|b(t) \\
&\le 2\overline{B}(S_{1,n}+1)b(0)\int_0^{\frac{1}{k}} G(s,s)(\phi(s)+\psi(s))ds \to 0, \quad k \to +\infty.
\end{aligned} \tag{3.26}$$

Inequalities (3.24)–(3.26) imply that

$$\|T_n - T_{n,k}\| = \sup_{x \in S} \|T_n x - T_{n,k}x\| \to 0, \quad k \to +\infty.$$

Therefore, by $T_{n,k} : K \to K$ is a completely continuous operator, we obtain that $T_n : K \to K$ is a completely continuous operator. □

Theorem 3.2. *Assume that* (H1)–(H3) *hold. In addition, suppose that the following condition are satisfied:*

(H4) *There exists a constant* $r_1 \ge \max\{4, L_1, L_2, 4\eta\gamma^{-1}\}$, *such that*

$$\overline{h}(r_1) = \sup_{u \in [0, r_1+1]} h(u) \le \frac{r_1}{L_2},$$

where

$$L_1 = 2\left(1 + \overline{A}a(\infty) + \overline{B}b(0)\right)\int_0^\infty G(s,s)\big(\phi(s)g(\gamma\zeta(s)) + \psi(s)\big)ds,$$

$$L_2 = 2\left(1 + \overline{A}a(\infty) + \overline{B}b(0)\right)\int_0^\infty G(s,s)\phi(s)ds,$$

η *is defined by Lemma 2.3,* $\overline{A}$ *and* $\overline{B}$ *are defined by* (3.3) *and* (3.5).

(H5) *There exists a constant* $r_2 > r_1$ *and* $[z_1, z_2] \subset (0,+\infty)$, *such that*

$$f(t,u) \ge \frac{r_2}{l}, \ (t,u) \in [z_1,z_2] \times [\delta r_2, r_2+1],$$

where

$$l = \theta a(z_1)b(z_2)\int_{z_1}^{z_2} G(s,s)ds,$$

$$0 < \delta = \frac{\gamma}{4\overline{d}}\left(a(z_1)b(z_2) + Aa(z_1) + Bb(z_2)\right) < 1,$$

θ *is defined by Lemma 2.1,* $\overline{d}$, A, B *are defined by* (2.4).

(H6) $\lim_{u\to+\infty} \frac{h(u)}{u} = 0$.

Then (1.1) *has at least two positive solutions.*

Proof. Let $B_{r_1} = \{x \in X : \|x\| < r_1\}$. For any $x \in K \cap \partial B_{r_1}$, $t \in [0, +\infty)$, by the definition of $\|\cdot\|$ and Lemma 2.3, we have

$$\begin{aligned}
&[x(t) - \omega(t)]^* \le x(t) \le \|x\| \le r_1,\\
x(t) - \omega(t) &\ge x(t) - \eta\zeta(t) \ge x(t) - \frac{2\eta x(t)}{\gamma\|x\|} \ge \frac{x(t)}{2}\\
&\ge \frac{\gamma\zeta(t)\|x\|}{4} \ge \frac{\gamma\zeta(t) r_1}{4} \ge \gamma\zeta(t).
\end{aligned}$$

So, for any $x \in K \cap \partial B_{r_1}$, $t \in [0, +\infty)$, by (H4), we have

$$\begin{aligned}
&|(T_n x)(t)|\\
&= \int_0^\infty G(t,s)\Big(f\big(s, [x(s) - \omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds + A(f_n + \psi)a(t)\\
&\quad + B(f_n + \psi)b(t)\\
&\le \int_0^\infty G(s,s)\Big(\phi(s)\Big(g\big([x(s) - \omega(s)]^* + \frac{1}{n}\big) + h\big([x(s) - \omega(s)]^* + \frac{1}{n}\big)\Big) + \psi(s)\Big)ds\\
&\quad + A(f_n + \psi)a(\infty) + B(f_n + \psi)b(0)\\
&\le \big(1 + \overline{A}a(\infty) + \overline{B}b(0)\big)\int_0^\infty G(s,s)\Big(\phi(s)\Big(g\big(\gamma\zeta(s)\big) + \overline{h}(r_1)\Big) + \psi(s)\Big)ds\\
&= \big(1 + \overline{A}a(\infty) + \overline{B}b(0)\big)\int_0^\infty G(s,s)\phi(s)\Big(g\big(\gamma\zeta(s)\big) + \psi(s)\Big)ds\\
&\quad + \big(1 + \overline{A}a(\infty) + \overline{B}b(0)\big)\int_0^\infty G(s,s)\phi(s)\overline{h}(r_1)ds \le r_1.
\end{aligned}$$

Thus,

$$\|T_n x\| \le \|x\|, \quad \text{for any } x \in K \cap \partial B_{r_1}. \tag{3.27}$$

On the other hand, let $B_{r_2} = \{x \in X : \|x\| < r_2\}$. For any $x \in K \cap \partial B_{r_2}$, $t \in [0, +\infty)$, since $r_2 > r_1 > 4\eta\gamma^{-1}$, we have

$$x(t) - \omega(t) \ge x(t) - \eta\zeta(t) \ge x(t) - \frac{2\eta x(t)}{\gamma\|x\|} \ge \frac{x(t)}{2} \ge \frac{\gamma\zeta(t)\|x\|}{4} \ge \frac{\gamma\zeta(t) r_2}{4}.$$

So, for any $x \in K \cap \partial B_{r_2}$, $t \in [z_1, z_2]$, by (2.4), we have

$$\begin{aligned}
\delta r_2 &= \frac{\gamma}{4\overline{d}}\left(a(z_1)b(z_2) + Aa(z_1) + Bb(z_2)\right) r_2\\
&\le \frac{\gamma}{4\overline{d}}\left(\tau(t) + a(z_1)b(z_2) + Aa(z_1) + Bb(z_2)\right) r_2\\
&\le \frac{\gamma\zeta(t) r_2}{4} \le x(t) - \omega(t) \le x(t) \le r_2.
\end{aligned} \tag{3.28}$$

By (H5) and (3.28), for any $x \in K \cap \partial B_{r_2}$, we have

$$\begin{aligned}
|(T_n x)(t)| &= \int_0^\infty G(t,s)\Big(f\big(s, [x(s) - \omega(s)]^* + \frac{1}{n}\big) + \psi(s)\Big)ds\\
&\quad + A(f_n + \psi)a(t) + B(f_n + \psi)b(t)\\
&\ge \int_{z_1}^{z_2} \theta\tau(t)G(s,s)f\big(s, [x(s) - \omega(s)]^* + \frac{1}{n}\big)ds\\
&\ge \int_{z_1}^{z_2} \theta a(t)b(t)G(s,s)\frac{r_2}{l}ds
\end{aligned}$$

$$\geq \frac{r_2\theta a(z_1)b(z_2)}{l}\int_{z_1}^{z_2} G(s,s)ds > r_2.$$

Thus,

$$\|T_n x\| \geq \|x\|, \quad \text{for any } x \in K \cap \partial B_{r_2}. \tag{3.29}$$

On the basis of (H6) and the continuity of $h(u)$ on $[0, +\infty)$, we have

$$\lim_{u\to+\infty} \frac{\overline{h}(u)}{u} = 0.$$

For

$$\overline{c} = \max\big\{1,\ \Big(4(1+\overline{A}a(\infty)+\overline{B}b(0))\int_0^\infty G(s,s)(\phi(s)+\psi(s))ds\Big)^{-1}\big\},$$

there exists $N > 0$, such that when $x \geq N$, for any $0 \leq y \leq x$, we have $h(y) \leq \overline{c}x$. Select

$$r_3 \geq \max\big\{r_2,\ N,\ 2(1+\overline{A}a(\infty)+\overline{B}b(0))\int_0^\infty G(s,s)\phi(s)g\left(\gamma\zeta(s)\right)ds\big\}.$$

Let $B_{r_3} = \{x \in X : \|x\| < r_3\}$, for any $x \in K \cap \partial B_{r_3}$, $t \in [0, +\infty)$, we have

$$[x(t)-\omega(t)]^* \leq x(t) \leq \|x\| \leq r_3,$$
$$x(t)-\omega(t) \geq x(t)-\eta\zeta(t) \geq x(t) - \frac{2\eta x(t)}{\gamma r_3} \geq \frac{x(t)}{2} \geq \frac{\gamma\zeta(t)r_3}{4} \geq \gamma\zeta(t).$$

Hence, for any $x \in K \cap \partial B_{r_3}$, $t \in [0, +\infty)$, we obtain

$$\begin{aligned}
&|(T_n x)(t)| \\
&= \int_0^\infty G(t,s)\Big(f\big(s,[x(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds \\
&\quad + A(f_n+\psi)a(t) + B(f_n+\psi)b(t) \\
&\leq \int_0^\infty G(s,s)\Big(\phi(s)\Big(g\big([x(s)-\omega(s)]^*+\frac{1}{n}\big)+h\big([x(s)-\omega(s)]^*+\frac{1}{n}\big)\Big)+\psi(s)\Big)ds \\
&\quad + A(f_n+\psi)a(\infty) + B(f_n+\psi)b(0) \\
&\leq \big(1+\overline{A}a(\infty)+\overline{B}b(0)\big)\int_0^\infty G(s,s)\Big(\phi(s)\Big(g\left(\gamma\zeta(s)\right)+\overline{c}(r_3+1)\Big)+\psi(s)\Big)ds \\
&\leq \big(1+\overline{A}a(\infty)+\overline{B}b(0)\big)\int_0^\infty G(s,s)\phi(s)g\left(\gamma\zeta(s)\right)ds \\
&\quad + \overline{c}\big(1+\overline{A}a(\infty)+\overline{B}b(0)\big)(r_3+2)\int_0^\infty G(s,s)(\phi(s)+\psi(s))ds \leq r_3.
\end{aligned}$$

Thus,

$$\|T_n x\| \leq \|x\|, \quad \text{for any } x \in K \cap \partial B_{r_3}. \tag{3.30}$$

It follows from the above discussion, (3.27), (3.29), (3.30), Lemmas 2.6 and 3.1, that for any $n \in \mathbb{N}$, T_n has two fixed points x_{1n}, x_{2n}, such that $r_1 \leq x_{1n} \leq r_2 \leq x_{2n} \leq r_3$.

Let $\{x_{1n}\}_{n=1}^{\infty}$ be the sequence of solutions of (2.8), we know it is uniformly bounded. From $r_1 \le x_{1n} \le r_2$, we have

$$\begin{aligned} &[x_{1n}(t)-\omega(t)]^* \le x_{1n}(t) \le \|x_{1n}\| \le r_2, \quad t\in[0,+\infty),\\ &x_{1n}(t)-\omega(t) \ge x_{1n}(t)-\eta\zeta(t) \ge x_{1n}(t)-\frac{2\eta x_{1n}(t)}{\gamma r_1}\\ &\ge \frac{x_{1n}(t)}{2} \ge \gamma\zeta(t), \quad t\in[0,+\infty). \end{aligned} \tag{3.31}$$

Next, given $\overline{a}'>0$, we will prove that $\{x_{1n}\}_{n=1}^{\infty}$ is equicontinuous on $[0,\overline{a}']$. For any $\overline{\varepsilon}'>0$, by $\int_0^\infty G(s,s)(\phi(s)(g(\gamma\zeta(s))+\overline{h}(r_2)+\psi(s))ds<+\infty$, where $\overline{h}(r_2)=\sup\{h(u):0\le u\le r_2+1\}$, we can find a sufficiently large $\overline{H}_0>0$ such that

$$\int_{\overline{H}_0}^{\infty} G(s,s)(\phi(s)(g(\gamma\zeta(s))+\overline{h}(r_2)+\psi(s))ds<\frac{\overline{\varepsilon}'}{12}.$$

By the uniformly continuity of $G(t,s)$ on $[0,\overline{a}']\times[0,\overline{H}_0]$, for the above $\overline{\varepsilon}'>0$, there exists $\overline{\delta}'>0$, such that for any $t,t'\in[0,\overline{a}], s\in[0,\overline{H}_0]$ and $|t-t'|<\overline{\delta}'$, we have

$$|G(t,s)-G(t',s)|<\frac{\overline{\varepsilon}'}{6}\Big(\int_0^{\overline{H}_0}(\phi(s)(g(\gamma\zeta(s))+\overline{h}(r_2))+\psi(s))ds\Big)^{-1}.$$

Therefore, for any $n\in\mathbb{N}$, $t,t'\in[0,\overline{a}], s\in[0,\overline{H}_0]$ and $|t-t'|<\overline{\delta}'$, we obtain

$$\begin{aligned} &\Big|\int_0^\infty G(t,s)\Big(f\big(s,[x_{1n}(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\\ &-\int_0^\infty G(t',s)\Big(f\big(s,[x_{1n}(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\Big|\\ &\le \int_0^{\overline{H}_0}|G(t,s)-G(t',s)|\Big(f\big(s,[x_{1n}(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\\ &\quad+\int_{\overline{H}_0}^\infty|G(t,s)-G(t',s)|\Big(f\big(s,[x_{1n}(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\\ &\le \int_0^{\overline{H}_0}|G(t,s)-G(t',s)|\Big(\phi(s)(g(\zeta(s))+\overline{h}(r_2))+\psi(s)\Big)ds\\ &\quad+\int_{\overline{H}_0}^\infty|G(t,s)-G(t',s)|(\phi(s)(g(\gamma\zeta(s))+\overline{h}(r_2))+\psi(s))ds\\ &\le \frac{\overline{\varepsilon}'}{6}+2\int_0^\infty G(s,s)\Big(\phi(s)(g(\zeta(s))+\overline{h}(r_2))+\psi(s)\Big)ds<\frac{\overline{\varepsilon}'}{3}. \end{aligned} \tag{3.32}$$

For

$$\begin{aligned} &A_{1n}(f_n+\psi)\\ &=\frac{1}{\Delta}\begin{vmatrix} \int_0^\infty g(t)\int_0^\infty G(t,s)\Big(f\Big(s,[x_{1n}(s)-\omega(s)]^*+\frac{1}{n}\Big)+\psi(s)\Big)ds\,dt & \rho-\int_0^\infty g(t)b(t)dt\\ -\int_0^\infty h(t)\int_0^\infty G(t,s)\Big(f\Big(s,[x_{1n}(s)-\omega(s)]^*+\frac{1}{n}\Big)+\psi(s)\Big)ds\,dt & \int_0^\infty h(t)b(t)dt \end{vmatrix},\\ &B_{1n}(f_n+\psi)\\ &=\frac{1}{\Delta}\begin{vmatrix} \int_0^\infty h(t)\int_0^\infty G(t,s)\Big(f\Big(s,[x_{1n}(s)-\omega(s)]^*+\frac{1}{n}\Big)+\psi(s)\Big)ds\,dt & \rho-\int_0^\infty h(t)a(t)dt\\ -\int_0^\infty g(t)\int_0^\infty G(t,s)\Big(f\Big(s,[x_{1n}(s)-\omega(s)]^*+\frac{1}{n}\Big)+\psi(s)\Big)ds\,dt & \int_0^\infty g(t)a(t)dt \end{vmatrix}, \end{aligned}$$

through calculation, we obtain

$$A_n(f_n+\psi) \le \overline{A}\int_0^\infty G(s,s)(\phi(s)(g(\gamma\zeta(s))+\overline{h}(r_2))+\psi(s))ds<+\infty,$$
$$B_n(f_n+\psi) \le \overline{B}\int_0^\infty G(s,s)(\phi(s)(g(\gamma\zeta(s))+\overline{h}(r_2))+\psi(s))ds<+\infty,$$

where $\overline{A}$ and $\overline{B}$ are defined as (3.3) and (3.5). So by the uniformly continuity of $a(t), b(t)$ on $[0,\overline{a}']$, for the above $\overline{\varepsilon}'>0$, there exists $\overline{\delta}''>0$ such that, for any $t,t'\in[0,\overline{a}']$ and $|t-t'|<\overline{\delta}''$, we have

$$\begin{aligned} |a(t)-a(t')| &< \frac{\overline{\varepsilon}'}{3}\Big(\overline{A}\int_0^\infty G(s,s)(\phi(s)(g(\gamma\zeta(s))+\overline{h}(r_2))+\psi(s))ds\Big)^{-1},\\ |b(t)-b(t')| &< \frac{\overline{\varepsilon}'}{3}\Big(\overline{B}\int_0^\infty G(s,s)(\phi(s)(g(\gamma\zeta(s))+\overline{h}(r_2))+\psi(s))ds\Big)^{-1}. \end{aligned} \tag{3.33}$$

Then, by (3.32), (3.33), for the above $\overline{\varepsilon}'>0$, let $\overline{\delta}_0=\min\{\overline{\delta}',\overline{\delta}''\}$, then for any $n\in\mathbb{N}$, $t,t'\in[0,\overline{a}']$ and $|t-t'|<\overline{\delta}_0$, we have

$$\begin{aligned} &|x_{1n}(t)-x_{1n}(t')|\\ &=\Big|\int_0^\infty G(t,s)\Big(f\big(s,[x_{1n}(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\\ &\quad + A_{1n}(f_n+\psi)a(t)+B_{1n}(f_n+\psi)b(t)\\ &\quad -\int_0^\infty G(t',s)\Big(f\big(s,[x_{1n}(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\\ &\quad - A_{1n}(f_n+\psi)a(t')-B_{1n}(f_n+\psi)b(t')\Big|\\ &\le \int_0^\infty |G(t,s)-G(t',s)|\,\Big(f\big(s,[x_{1n}(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\\ &\quad +\overline{A}\int_0^\infty G(s,s)(\phi(s)(g(\gamma\zeta(s))+\overline{h}(r_2))+\psi(s))ds|a(t)-a(t')|\\ &\quad +\overline{B}\int_0^\infty G(s,s)(\phi(s)(g(\gamma\zeta(s))+\overline{h}(r_2))+\psi(s))ds|b(t)-b(t')|<\overline{\varepsilon}'. \end{aligned}$$

So, $\{x_{1n}\}_{n=1}^\infty$ is equicontinuous on $[0,\overline{a}']$. Since $\overline{a}'>0$ is arbitrary, $\{x_{1n}\}_{n=1}^\infty$ is locally equicontinuous on $[0,+\infty)$.

Let $x_{1n}(+\infty)=\lim_{t\to+\infty}x_{1n}(t)$. then by a simple calculation, we can see that $\lim_{t\to+\infty}x_{1n}(t)<+\infty$, and so we obtain

$$\begin{aligned} &|x_{1n}(t)-x_{1n}(+\infty)|\\ &\le\Big|\int_0^\infty (G(t,s)-\overline{G}(s))\Big(f\big(s,[x_{1n}(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\Big|\\ &\quad + A_{1n}(f_n+\psi)|a(t)-a(\infty)|+B_{1n}(f_n+\psi)|b(t)-b(\infty)|. \end{aligned}$$

By the similar method as for (3.32), we obtain that, for any $\overline{\varepsilon}_0>0$, there exists $\overline{N}'$ such that, when $t>\overline{N}'$, it follows

$$\Big|\int_0^\infty (G(t,s)-\overline{G}(s))\Big(f\big(s,[x_{1n}(s)-\omega(s)]^*+\frac{1}{n}\big)+\psi(s)\Big)ds\Big|<\frac{\overline{\varepsilon}_0}{3}.$$

Together with the continuity of $a(t), b(t)$ on $[0,+\infty)$, we obtain that for the above $\bar\varepsilon_0 > 0$, there exists $\overline{N}'$ such that, when $t > \overline{N}'$, we have $|x_{1n}(t) - x_{1n}(+\infty)| < \bar\varepsilon_0$. Hence, the functions from $\{x_{1n}\}_{n=1}^\infty$ are equiconvergent at $+\infty$, which implies that $\{x_{1n}\}_{n=1}^\infty$ is relatively compact (by Lemma 2.5). Therefore, the sequence $\{x_{1n}\}_{n=1}^\infty$ has a subsequence being uniformly convergent on $[0,+\infty)$. Without loss of generality, we still assume that $\{x_{1n}\}_{n=1}^\infty$ itself uniformly converges to x_1 on $[0,+\infty)$. Since $\{x_{1n}\}_{n=1}^\infty \in K$, we have $x_{1n} \geq 0$. By (2.8), we have

$$\begin{aligned} x_{1n}(t) &= x_{1n}(\frac{1}{2}) + x_{1n}'(\frac{1}{2})\big(t-\frac{1}{2}\big) - \int_{1/2}^t ds \int_{1/2}^s \frac{p'(\varsigma)x_{1n}'(\varsigma)}{p(\varsigma)} d\varsigma \\ &\quad - \int_{1/2}^t ds \int_{1/2}^s \frac{\big(f\big(\varsigma, [x_{1n}(\varsigma)-\omega(\varsigma)]^* + \frac{1}{n}\big) + \psi(\varsigma)\big)}{p(\varsigma)} d\varsigma, \quad t \in (0,+\infty). \end{aligned} \tag{3.34}$$

As $\{x_{1n}'(1/2)\}_{n=1}^\infty$ is bounded, without loss of generality, we may assume $x_{1n}'(\frac{1}{2}) \to c_0$ as $n \to +\infty$. Then, by (3.34) and the Lebesgue dominated convergence theorem, we have

$$\begin{aligned} x_1(t) &= x_1(\frac{1}{2}) + c_0\big(t-\frac{1}{2}\big) - \int_{1/2}^t ds \int_{1/2}^s \frac{p'(\varsigma)x_1'(\varsigma)}{p(\varsigma)} d\varsigma \\ &\quad - \lambda \int_{1/2}^t ds \int_{1/2}^s \frac{\big(f\big(\varsigma, [x_1(\varsigma)-\omega(\varsigma)]^*\big) + \psi(\varsigma)\big)}{p(\varsigma)} d\varsigma, \quad t \in (0,+\infty). \end{aligned} \tag{3.35}$$

By (3.35), a direct computation shows that

$$(p(t)x_1'(t))' + \big(f\big(t, [x_1(t)-\omega(t)]^*\big) + \psi(t)\big) = 0, \quad t \in (0,+\infty).$$

On the other hand, let $n \to +\infty$ in the following boundary conditions:

$$\begin{gathered} \alpha_1 x_{1n}(0) - \beta_1 \lim_{t\to 0^+} p(t)x_{1n}'(t) = \int_0^\infty g(t)x_{1n}(t)dt, \\ \alpha_2 \lim_{t\to+\infty} x_{1n}(t) + \beta_2 \lim_{t\to+\infty} p(t)x_{1n}'(t) = \int_0^\infty h(t)x_{1n}(t)dt. \end{gathered}$$

Therefore, we deduce that x_1 is a solution of (2.8). Let $\bar{x}_1(t) = x_1(t) - \omega(t)$. By (3.31) and the convergence of the sequence $\{x_{1n}\}_{n=1}^\infty$, we have $\bar{x}_1(t) \geq \gamma\zeta(t) > 0$, $t \in [0,+\infty)$. It then follows from Lemma 2.4 that $\bar{x}_1$ is a positive solution of (1.1). By the same method, we obtain $\bar{x}_2(t) \geq \gamma\zeta(t) > 0$, $t \in [0,+\infty)$. The proof is completed. □

Theorem 3.3. *Assume that* (H1)–(H3), (H5) *hold. In addition, suppose that the following conditions are satisfied:*

(H7) *There exists a constant $R_2 > r_2$, such that*

$$\bar{h}(R_2) = \sup_{u\in[0,R_2+1]} h(u) \leq \frac{R_2}{L_2},$$

where r_2 is defined by (H5)*, L_2 is defined by* (H4)*.*

(H8) *There exists $[z_3, z_4] \subset (0,+\infty)$, such that*

$$\lim_{u\to+\infty} \min_{t\in[z_3,z_4]} \frac{f(t,u)}{u} = +\infty.$$

Then (1.1) *has at least two positive solutions.*

The proof of Theorem 3.3 is similar to that of Theorem 3.2, and so we omit it.

Remark 3.4. In the proof of Theorems 3.2 and 3.3, we obtain the two positive solutions of (1.1), under the condition that $f(t,u)$ has singularity on t and on u. In addition, the function $f(t,u)$ is semipositive, which increases the difficulty in the analysis.

Note that by Theorems 3.2 and 3.3, the positive solutions x_i of (1.1) satisfy $x_i(t) \geq \gamma\zeta(t) > 0$, for any $t \in [0,+\infty)$, $i = 1, 2$.

4. Example

Consider the following boundary-value problem

$$\begin{gathered} ((1+t)^2 x'(t))' + f(t,x(t)) = 0, \quad t \in (0,+\infty), \\ x(0) - \lim_{t\to 0^+}(1+t)^2 x'(t) = \int_0^\infty \frac{1}{(1+t)^3} x(t)dt, \\ \lim_{t\to+\infty} x(t) + \lim_{t\to+\infty}(1+t)^2 x'(t) = \int_0^\infty \frac{t}{(1+t)^3} x(t)dt. \end{gathered} \tag{4.1}$$

By calculations, we obtain: $\alpha_1 = \alpha_2 = \beta_1 = \beta_2 = 1$, $p(t) = (1+t)^2$, $a(t) = 2 - \frac{1}{1+t}$, $b(t) = 1 + \frac{1}{1+t}$, $\rho = 3$,

$$\begin{gathered} \int_0^\infty g(t)dt = \int_0^\infty \frac{1}{(1+t)^3} dt = \frac{1}{2} < +\infty, \\ \int_0^\infty h(t)dt = \int_0^\infty \frac{t}{(1+t)^3} dt = \frac{1}{2} < +\infty, \\ \rho - \int_0^\infty g(t)b(t)dt = \frac{13}{6} > 0, \quad \rho - \int_0^\infty h(t)a(t)dt = \frac{13}{6} > 0, \\ \Delta = \begin{vmatrix} \rho - \int_0^\infty g(t)b(t)dt & \int_0^\infty g(t)a(t)dt \\ \int_0^\infty h(t)b(t)dt & \rho - \int_0^\infty h(t)a(t)dt \end{vmatrix} = \begin{vmatrix} \frac{13}{6} & \frac{2}{3} \\ \frac{1}{3} & \frac{13}{6} \end{vmatrix} = \frac{161}{36} > 0. \end{gathered}$$

So condition (H1) holds. Take

$$f(t,u) = \frac{1}{(1+t)^2} \begin{cases} \frac{u}{2} + \frac{1}{10^3 u}, & u \leq 1, \\ \frac{3u}{16a-2} + \frac{1}{2} - \frac{3}{16a-2} + \frac{1}{10^3 u}, & 1 \leq u \leq 8a, \\ \frac{2698u}{b-8a} + 2700 - \frac{2698b}{b-8a} + \frac{1}{10^3 u}, & 8a \leq u \leq b, \\ \frac{(2700+\sqrt{151})u}{151-b} + \sqrt{151} + 5400 \\ -\frac{151(2700+\sqrt{151})}{151-b} + \sqrt{u-b} + \frac{1}{10^3 u}, & b \leq u \leq 151, \\ \sqrt{151} + 5400 + \sqrt{u-b} + \frac{1}{10^3 u}, & u \geq 151, \end{cases}$$

where $a = 2.5$, $b = 22.5$, we can suppose $g(u) = \frac{1}{10^3 u}$, $\varphi(t) = \phi(t) = \frac{1}{(1+t)^2}$,

$$h(u) = \begin{cases} \frac{u}{2}, & u \leq 1, \\ \frac{3u}{16a-2} + \frac{1}{2} - \frac{3}{16a-2}, & 1 \leq u \leq 8a, \\ \frac{2698u}{b-8a} + 2700 - \frac{2698b}{b-8a}, & 8a \leq u \leq b, \\ \frac{(2700+\sqrt{151})u}{151-b} + \sqrt{151} + 5400 \\ -\frac{151(2700+\sqrt{151})}{151-b} + \sqrt{u-b}, & b \leq u \leq 151, \\ \sqrt{151} + 5400 + \sqrt{u-b}, & u \geq 151, \end{cases}$$

Since

$$\int_0^\infty \varphi(s)ds = 1, \quad \int_0^\infty G(s,s)(\varphi(s)+\phi(s))ds = \frac{13}{9}.$$

So conditions (H2) and (H3) hold. For

$$A = \frac{1}{\Delta}\begin{vmatrix} \int_0^\infty g(t)\tau(t)dt & \rho - \int_0^\infty g(t)b(t)dt \\ -\int_0^\infty h(t)\tau(t)dt & \int_0^\infty h(t)b(t)dt \end{vmatrix} = \frac{36}{161}\begin{vmatrix} \frac{13}{12} & \frac{13}{6} \\ -\frac{13}{12} & \frac{1}{3} \end{vmatrix} = \frac{195}{322},$$
$$B = \frac{1}{\Delta}\begin{vmatrix} \int_0^\infty h(t)\tau(t)dt & \rho - \int_0^\infty h(t)a(t)dt \\ -\int_0^\infty g(t)\tau(t)dt & \int_0^\infty g(t)a(t)dt \end{vmatrix} = \frac{36}{161}\begin{vmatrix} \frac{13}{12} & \frac{13}{6} \\ -\frac{13}{12} & \frac{2}{3} \end{vmatrix} = \frac{221}{322},$$

choose $\overline{d} = 7$, then $\eta = \frac{\overline{d}}{\rho}\int_0^\infty \varphi(s)ds = 7/3$, $\gamma = 1/2$, so $4\eta\gamma^{-1} = 18.67$. For

$$\overline{A} = \frac{1}{\Delta}\begin{vmatrix} \int_0^\infty g(t)dt & \rho - \int_0^\infty g(t)b(t)dt \\ -\int_0^\infty h(t)dt & \int_0^\infty h(t)b(t)dt \end{vmatrix} = \frac{36}{161}\begin{vmatrix} \frac{1}{2} & \frac{13}{6} \\ -\frac{1}{2} & \frac{1}{3} \end{vmatrix} = \frac{45}{161},$$
$$\overline{B} = \frac{1}{\Delta}\begin{vmatrix} \int_0^\infty h(t)dt & \rho - \int_0^\infty h(t)a(t)dt \\ -\int_0^\infty g(t)dt & \int_0^\infty g(t)a(t)dt \end{vmatrix} = \frac{36}{161}\begin{vmatrix} \frac{1}{2} & \frac{13}{6} \\ -\frac{1}{2} & \frac{2}{3} \end{vmatrix} = \frac{51}{161},$$
$$\zeta(t) = \frac{\tau(t)+Aa(t)+Bb(t)}{\overline{d}} = 0.43 + \frac{0.09}{1+t} - \frac{0.11}{(1+t)^2},$$

we obtain

$$\begin{aligned} L_1 &= 2\left(1+\overline{A}a(\infty)+\overline{B}b(0)\right)\int_0^\infty G(s,s)\Big(\phi(s)g(\gamma\zeta(s))+\psi(s)\Big)ds \\ &= 4.39\int_0^\infty G(s,s)\Big(\phi(s)g(\gamma\zeta(s))+\psi(s)\Big)ds < 6.34, \\ L_2 &= 2\left(1+\overline{A}a(\infty)+\overline{B}b(0)\right)\int_0^\infty G(s,s)\phi(s)ds = 3.17\,. \end{aligned}$$

Choosing $r_1 = 19$, we have

$$\overline{h}(r_1) = \sup_{u\in[0,20]} h(u) = 2 \le \frac{r_1}{L_2} = 6.31.$$

Take $r_2 = 150 > r_1$, $[z_1, z_2] = [1,2] \subset (0,+\infty)$, then

$$l = \theta a(z_1)b(z_2)\int_{z_1}^{z_2} G(s,s)ds = 0.85,$$
$$\delta = \frac{\gamma}{4\overline{d}}\left(a(z_1)b(z_2)+Aa(z_1)+Bb(z_2)\right) = 0.15,$$
$$f(t,u) \ge 300 \ge \frac{r_2}{l} = \frac{150}{0.85} = 176.48, \quad (t,u)\in[1,2]\times[22.5,151],$$
$$\lim_{u\to+\infty}\frac{h(u)}{u} = \lim_{u\to+\infty}\frac{\sqrt{151}+5400+\sqrt{u-b}}{u} = 0.$$

So all conditions of Theorem 3.2 are satisfied; Therefore, (1.1) has at least two positive solutions.

Acknowledgements. The author was supported by the National Natural Science Foundation of China (11271175) and the Applied Mathematics Enhancement Program of Linyi University

References

[1] R. P. Agarwal, D. O'Regan; *Infinite Interval Problems for Differential, Difference and Integral Equations*, Kluwer Academic Publisher, 2001.

[2] R. P. Agarwal, D. O'Regan; *Nonlinear boundary value problems on the semi-infinite interval: An upper and lower solution approach*, Mathematika, 49 (2002), 129-140.

[3] R. P. Agarwal, D. O'Regan; *Infinite interval problems modeling phenomena which arise in the theory of plasma and elesrical potential theory*, Stud. Appl. Math., 111 (2003), 339-358.

[4] H. Amann; *Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces*, SIAM Rev., 18 (1976), 620-709.

[5] S. Chen, Y. Zhang; *Singular boundary value problems on a half-line*, J. Math. Anal. Appl., 195 (1995), 449-469.

[6] S. Djebali, O. Saifl, S. Zahar; *Singular boundary value problems with variable coefficients on the positive half-line*, Elect. J. Qual. Theor. Diff. Equat., 73 (2013), 1-18.

[7] D. Guo, V. Lakshmikantham; *Nonlinear Problems in Abstract Cones*, Academic Press New York, 1988.

[8] C. Kim; *Existence and iteration of positive solutions for multi-point boundary value problems on a half-line*, Comput. Math. Appl., 61 (2011), 1898-1905.

[9] G. Kitzhofer, O. Koch, P. Lima, E. Weinmilller; *Efficient numerical solution of the density profile equation in hydrodynamics*, J. Sci. Comput., 32 (2007), 411-424.

[10] H. Lian, J. Zhao; *Existence of unbounded solutions for a third-Order boundary value problem on infinite intervals*, Disc. Dyn. Nat. Soc., 2012 (2012), 14 (Article ID 357697).

[11] P. M. Lima, N. V. Chemetov, N. B. Konyi Jdiova, A. L. Sidcov; *Analytical-numerical investigation of bubble-type solutions of nonlinear singular problems*, J. Comp. Appl. Math., 189 (2006), 260-273.

[12] L. Liu, Z. Wang, Y. Wu; *Multiple positive solutions of the singular boundary value problems for second-order differential equations on the half-line*, Nonlinear Anal., 71 (2009), 2564-2575.

[13] R. Ma; *Multiple positive solutions for nonlinear m-point boundary value problems*, Appl. Math. Comput., 148 (2004), 249-262.

[14] D. Smail, S. Ouiza, B. Yan; *Positive solutions for singular BVPs on the positive half-line arising from epidemiology and combustion theory*, Acta Math. Sci., 32 (2) (2012), 672-694.

[15] Y. Wang, L. Liu, Y. Wu; *Positive solutions for singular semipositone boundary value problems on infinite intervals*, Appl. Math. Comput., 227 (2014), 256-273.

[16] X. Zhang, L. Liu, Y. Wu; *Existence of positive solutions for second-order semipositone differential equations on the half-line*, Appl. Math. Comput., 185 (2007), 628-635.

Ying Wang

School of Science, Linyi University, Linyi 276000, Shandong, China

E-mail address: lywy1981@163.com