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INITIAL VALUE PROBLEMS OF FRACTIONAL ORDER HADAMARD-TYPE FUNCTIONAL DIFFERENTIAL EQUATIONS

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Abstract. The Banach fixed point theorem and a nonlinear alternative of Leray-Schauder type are used to investigate the existence and uniqueness of solutions for fractional order Hadamard-type functional and neutral functional differential equations.

1. Introduction

This article concerns the existence of solutions for initial value problems (IVP for short) of fractional order functional and neutral functional differential equations. In the first problem, we consider fractional order functional differential equations:

$$D^{\alpha}y(t) = f(t, y_t), \quad \text{for each } t \in J = [1, b],\ 0 < \alpha < 1, \tag{1.1}$$

$$y(t) = \phi(t), \quad t \in [1 - r, 1], \tag{1.2}$$

where D^{α} is the Hadamard fractional derivative, $f : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ is a given function and $\phi \in C([1-r, 1], \mathbb{R})$ with $\phi(1) = 0$. For any function y defined on $[1 - r, b]$ and any $t \in J$, we denote by y_t the element of $C([-r, 0], \mathbb{R})$ and is defined by

$$y_t(\theta) = y(t + \theta), \quad \theta \in [-r, 0].$$

Here $y_t(\cdot)$ represents the history of the state from time $t - r$ up to the present time t.

The second problem is devoted to the study of fractional neutral functional differential equation:

$$D^{\alpha}[y(t) - g(t, y_t)] = f(t, y_t), \quad t \in J, \tag{1.3}$$

$$y(t) = \phi(t), \quad t \in [1 - r, 1], \tag{1.4}$$

where f and ϕ are as in problem (1.1)–(1.2), and $g : J \times C([-r, 0], \mathbb{R}) \to \mathbb{R}$ is a given function such that $g(1, \phi) = 0$.

Functional and neutral functional differential equations arise in a variety of areas of biological, physical, and engineering applications, see, for example, the books [15, 17] and the references therein. Differential equations of fractional order have recently proved to be valuable tools in the modeling of many phenomena

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in various fields of science and engineering. Indeed, we can find numerous applications in various fields such as viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [16, 18, 20, 21]).

Fractional differential equations involving Riemann-Liouville and Caputo type fractional derivatives have extensively been studied by several researchers [1, 2, 3, 4, 5, 6, 12, 22, 23]. However, the literature on Hadamard type fractional differential equations is not enriched yet. The fractional derivative due to Hadamard, introduced in 1892 [14], differs from the aforementioned derivatives in the sense that the kernel of the integral in the definition of Hadamard derivative contains logarithmic function of arbitrary exponent. A detailed description of Hadamard fractional derivative and integral can be found in [9, 10, 11] and references cited therein.

The IVPs (1.1)–(1.2) and (1.3)–(1.4) in the case of infinite delay and Riemann-Liouville fractional derivative was studied in [8]. IVP for hybrid Hadamard fractional differential equations was studied in [7]. Here we study the problems involving Hadamard-type fractional derivatives. Our approach is based on the Banach fixed point theorem and nonlinear alternative of Leray-Schauder type [13]. The rest of this paper is organized as follows: in Section 2 we recall some useful preliminaries. In Section 3 we discuss the existence and uniqueness of solutions for the problem (1.1)–(1.2), while the existence results for the problem (1.3)–(1.4) are presented in Section 4. Finally, an example is given in Section 5 for illustration of the results.

2. Preliminaries

In this section, we introduce notation, definitions, and preliminary facts that we need in the sequel.

By $C(J,\mathbb{R})$ we denote the Banach space of all continuous functions from J into $\mathbb{R}$ with the norm

$$\|y\|_\infty := \sup\{|y(t)| : t \in J\}.$$

Also $C([-r,0],\mathbb{R})$ is endowed with the norm

$$\|\phi\|_C := \sup\{|\phi(\theta)| : -r \le \theta \le 0\}.$$

Definition 2.1 ([16])**.** The Hadamard derivative of fractional order q for a function $g : [1,\infty) \to \mathbb{R}$ is defined as

$$D^q g(t) = \frac{1}{\Gamma(n-q)}\Big(t\frac{d}{dt}\Big)^n \int_1^t \Big(\log\frac{t}{s}\Big)^{n-q-1}\frac{g(s)}{s}ds, \quad n-1 < q < n, n = [q]+1,$$

where $[q]$ denotes the integer part of the real number q and $\log(\cdot) = \log_e(\cdot)$.

Definition 2.2 ([16])**.** The Hadamard fractional integral of order q for a function g is defined as

$$I^q g(t) = \frac{1}{\Gamma(q)}\int_1^t \Big(\log\frac{t}{s}\Big)^{q-1}\frac{g(s)}{s}ds, \quad q > 0,$$

provided the integral exists.

3. Functional differential equations

Definition 3.1. A function $y \in C([1-r,b],\mathbb{R})$, is said to be a solution of (1.1)–(1.2) if y satisfies the equation $D^\alpha y(t) = f(t,y_t)$ on J, and the condition $y(t) = \phi(t)$ on $[1-r,1]$.

Our first existence result for (1.1)–(1.2) is based on the Banach contraction principle.

Theorem 3.2. *Let $f : J \times C([-r,0],\mathbb{R}) \to \mathbb{R}$. Assume that*

(H0) *there exists $\ell > 0$ such that*

$$|f(t,u) - f(t,v)| \le \ell \|u - v\|_C, \quad \text{for } t \in J \text{ and every } u, v \in C([-r,0],\mathbb{R}).$$

If $\frac{\ell(\log b)^\alpha}{\Gamma(\alpha+1)} < 1$, then there exists a unique solution for (1.1)–(1.2) on the interval $[1-r,b]$.

Proof. Transform problem (1.1)–(1.2) into a fixed point problem. Consider the operator $N : C([1-r,b],\mathbb{R}) \to C([1-r,b],\mathbb{R})$ defined by

$$N(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [1-r,1], \\ \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{f(s,y_s)}{s} ds, & \text{if } t \in [1,b]. \end{cases} \tag{3.1}$$

Let $y, z \in C([1-r,b],\mathbb{R})$. Then, for $t \in J$,

$$\begin{aligned} |N(y)(t) - N(z)(t\| &\le \frac{1}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} |f(s,y_s) - f(s,z_s)| \frac{ds}{s} \\ &\le \frac{\ell}{\Gamma(\alpha)} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \|y_s - z_s\|_C \frac{ds}{s} \\ &\le \frac{\ell}{\Gamma(\alpha)} \|y - z\|_{[1-r,b]} \int_1^t \left(\log \frac{t}{s}\right)^{\alpha-1} \frac{ds}{s} \\ &\le \frac{\ell(\log t)^\alpha}{\Gamma(\alpha+1)} \|y - z\|_{[1-r,b]}. \end{aligned}$$

Consequently,

$$\|N(y) - N(z)\|_{[1-r,b]} \le \frac{\ell(\log b)^\alpha}{\Gamma(\alpha+1)} \|y - z\|_{[1-r,b]},$$

which implies that N is a contraction, and hence N has a unique fixed point by Banach's contraction principle. □

Our second existence result for (1.1)–(1.2) is based on the nonlinear alternative of Leray-Schauder.

Lemma 3.3 (Nonlinear alternative for single valued maps [13])**.** *Let E be a Banach space, C a closed, convex subset of E, U an open subset of C and $0 \in U$. Suppose that $F : \overline{U} \to C$ is a continuous, compact (that is, $F(\overline{U})$ is a relatively compact subset of C) map. Then either*

(i) *F has a fixed point in $\overline{U}$, or*

(ii) *there is a $u \in \partial U$ (the boundary of U in C) and $\lambda \in (0,1)$ with $u = \lambda F(u)$.*

Theorem 3.4. *Assume that the following hypotheses hold:*

(H1) *$f : J \times C([-r,0],\mathbb{R}) \to \mathbb{R}$ is a continuous function;*

(H2) *there exist a continuous nondecreasing function $\psi : [0,\infty) \to (0,\infty)$ and a function $p \in C([1,b],\mathbb{R}^+)$ such that*

$$|f(t,u)| \le p(t)\psi(\|u\|_C) \quad \text{for each } (t,u) \in [1,b] \times C([-r,0],\mathbb{R});$$

(H3) *there exists a constant $M > 0$ such that*
$$\frac{M}{\psi(M)\|p\|_\infty \frac{(\log b)^\alpha}{\Gamma(\alpha+1)}} > 1.$$

Then (1.1)–(1.2) *has at least one solution on* $[1-r, b]$.

Proof. We consider the operator $N : C([1-r, b], \mathbb{R}) \to C([1-r, b], \mathbb{R})$ defined by (3.1). We shall show that the operator N is continuous and completely continuous.

Step 1: N is continuous. Let $\{y_n\}$ be a sequence such that $y_n \to y$ in $C([1-r, b], \mathbb{R})$. Let $\eta > 0$ such that $\|y_n\|_\infty \le \eta$. Then
$$\begin{aligned}
|N(y_n)(t) - N(y)(t)| &\le \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} |f(s, y_{ns}) - f(s, y_s)| \frac{ds}{s} \\
&\le \frac{1}{\Gamma(\alpha)} \int_1^b \Big(\log\frac{t}{s}\Big)^{\alpha-1} \sup_{s\in[1,b]} |f(s, y_{ns}) - f(s, y_s)| \frac{ds}{s} \\
&\le \frac{\|f(\cdot, y_{n\cdot}) - f(\cdot, y_\cdot)\|_\infty}{\Gamma(\alpha)} \int_1^b \Big(\log\frac{t}{s}\Big)^{\alpha-1} \frac{ds}{s} \\
&\le \frac{(\log b)^\alpha \|f(\cdot, y_{n\cdot}) - f(\cdot, y_\cdot)\|_\infty}{\alpha\Gamma(\alpha)}.
\end{aligned}$$
Since f is a continuous function, we have
$$\|N(y_n) - N(y)\|_\infty \le \frac{(\log b)^\alpha \|f(\cdot, y_{n\cdot}) - f(\cdot, y_\cdot)\|_\infty}{\Gamma(\alpha+1)} \to 0 \quad \text{as } n \to \infty.$$

Step 2: N maps bounded sets into bounded sets in $C([1-r, b], \mathbb{R})$. Indeed, it is sufficient to show that for any $\eta^* > 0$ there exists a positive constant $\tilde{\ell}$ such that for each $y \in B_{\eta^*} = \{y \in C([1-r, b], \mathbb{R}) : \|y\|_\infty \le \eta^*\}$, we have $\|N(y)\|_\infty \le \tilde{\ell}$. By (H2), for each $t \in [1, b]$, we have
$$\begin{aligned}
|N(y)(t)| &\le \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} |f(s, y_s)| \frac{ds}{s} \\
&\le \frac{\psi(\|y\|_{[1-r,b]})\|p\|_\infty}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} \frac{ds}{s} \\
&\le \frac{\psi(\|y\|_{[1-r,b]})\|p\|_\infty}{\Gamma(\alpha+1)} (\log b)^\alpha.
\end{aligned}$$
Thus
$$\|N(y)\|_\infty \le \frac{\psi(\eta^*)\|p\|_\infty}{\Gamma(\alpha+1)} (\log b)^\alpha := \tilde{\ell}.$$

Step 3:: N maps bounded sets into equicontinuous sets of $C([1-r, b], \mathbb{R})$. Let $t_1, t_2 \in [1, b]$, $t_1 < t_2$, B_{η^*} be a bounded set of $C([1-r, b], \mathbb{R})$ as in Step 2, and let $y \in B_{\eta^*}$. Then
$$\begin{aligned}
|N(y)(t_2) - N(y)(t_1)| \le \frac{1}{\Gamma(\alpha)} \Big| &\int_1^{t_1} \Big[\Big(\log\frac{t_2}{s}\Big)^{\alpha-1} - \Big(\log\frac{t_1}{s}\Big)^{\alpha-1}\Big] f(s, y_s) \frac{ds}{s} \\
&+ \frac{1}{\Gamma(\alpha)} \int_{t_1}^{t_2} \Big(\log\frac{t_2}{s}\Big)^{\alpha-1} f(s, y_s) \frac{ds}{s} \Big|
\end{aligned}$$

$$\begin{aligned}&\leq \frac{\psi(\eta^*)\|p\|_\infty}{\Gamma(\alpha)} \int_1^{t_1} \Big[\Big(\log\frac{t_2}{s}\Big)^{\alpha-1} - \Big(\log\frac{t_1}{s}\Big)^{\alpha-1}\Big]\frac{ds}{s}\\ &\quad + \frac{\psi(\eta^*)\|p\|_\infty}{\Gamma(\alpha)} \int_{t_1}^{t_2} \Big(\log\frac{t_2}{s}\Big)^{\alpha-1}\frac{ds}{s}.\end{aligned}$$

As $t_1 \to t_2$ the right-hand side of the above inequality tends to zero. The equicontinuity for the cases $t_1 < t_2 \leq 0$ and $t_1 \leq 0 \leq t_2$ is obvious.

In consequence of Steps 1 to 3, it follows by the Arzelá-Ascoli theorem that $N : C([1-r,b],\mathbb{R}) \to C([1-r,b],\mathbb{R})$ is continuous and completely continuous.

Step 4: We show that there exists an open set $U \subseteq C([1-r,b],\mathbb{R})$ with $y \neq \lambda N(y)$ for $\lambda \in (0,1)$ and $y \in \partial U$. Let $y \in C([1-r,b],\mathbb{R})$ and $y = \lambda N(y)$ for some $0 < \lambda < 1$. Thus, for each $t \in [1,b]$

$$y(t) = \lambda\Big(\frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} f(s,y_s)\,\frac{ds}{s}\Big).$$

By assumption (H2), for each $t \in J$, we obtain

$$\begin{aligned}|y(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} p(s)\psi(\|y_s\|_C)\frac{ds}{s}\\ &\leq \frac{\|p\|_\infty \psi(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)}(\log b)^\alpha,\end{aligned}$$

which can be expressed as

$$\frac{\|y\|_{[1-r,b]}}{\psi(\|y\|_{[1-r,b]})\|p\|_\infty \frac{(\log b)^\alpha}{\Gamma(\alpha+1)}} \leq 1.$$

In view of (H4), there exists M such that $\|y\|_{[1-r,b]} \neq M$. Let us set

$$U = \{y \in C([1-r,b],\mathbb{R}) : \|y\|_{[1-r,b]} < M\}.$$

Note that the operator $N : \overline{U} \to C([1-r,b],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $y \in \partial U$ such that $y = \lambda N y$ for some $\lambda \in (0,1)$. Consequently, by the nonlinear alternative of Leray-Schauder type (Lemma 3.3), we deduce that N has a fixed point $y \in \overline{U}$ which is a solution of (1.1)-(1.2). This completes the proof. □

4. Neutral functional differential equations

In this section, we establish the existence results for (1.3)–(1.4).

Definition 4.1. A function $y \in C([1-r,b],\mathbb{R})$, is said to be a solution of (1.3)–(1.4) if y satisfies the equation $D^\alpha[y(t) - g(t,y_t)] = f(t,y_t)$ on J, and the condition $y(t) = \phi(t)$ on $[1-r,1]$.

Theorem 4.2 (Uniqueness result)**.** *Assume that* (H0) *and the following condition hold:*

(A1) *there exists a nonnegative constant* c_1 *such that*

$$|g(t,u) - g(t,v)| \leq c_1\|u-v\|_C, \quad \text{for every } u,v \in C([-r,0],\mathbb{R}).$$

If

$$c_1 + \frac{\ell(\log b)^\alpha}{\Gamma(\alpha+1)} < 1, \tag{4.1}$$

then there exists a unique solution for (1.3)–(1.4) *on the interval* $[1-r,b]$.

Proof. Consider the operator $N_1 : C([1-r,b],\mathbb{R}) \to C([1-r,b],\mathbb{R})$ defined by:

$$N_1(y)(t) = \begin{cases} \phi(t), & \text{if } t \in [1-r,1], \\ g(t,y_t) + \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log \frac{t}{s}\Big)^{\alpha-1} f(s,y_s)ds, & \text{if } t \in [1,b]. \end{cases} \tag{4.2}$$

To show that the operator N_1 is a contraction, let $y, z \in C([1-r,b],\mathbb{R})$. Then we have

$$\begin{aligned} &|N_1(y)(t) - N_1(z)(t)| \\ &\le |g(t,y_t) - g(t,z_t)| + \frac{1}{\Gamma(\alpha)} \int_1^t |f(s,y_s) - f(s,z_s)| \Big(\log \frac{t}{s}\Big)^{\alpha-1} ds \\ &\le c_1 \|y_t - z_t\|_C + \frac{\ell}{\Gamma(\alpha)} \int_1^t \Big(\log \frac{t}{s}\Big)^{\alpha-1} \|y_s - z_s\|_C \, ds \\ &\le c_1 \|y - z\|_{[1-r,b]} + \frac{\ell}{\Gamma(\alpha)} \|y - z\|_{[1-r,b]} \int_1^t \Big(\log \frac{t}{s}\Big)^{\alpha-1} ds \\ &\le c_1 \|y - z\|_{[1-r,b]} + \frac{\ell (\log t)^\alpha}{\Gamma(\alpha+1)} \|y - z\|_{[1-r,b]}. \end{aligned}$$

Consequently we obtain

$$\|N_1(y) - N_1(z)\|_{[1-r,b]} \le \big[c_1 + \frac{\ell (\log b)^\alpha}{\Gamma(\alpha+1)}\big] \|y - z\|_{[1-r,b]},$$

which, in view of (4.1), implies that N_1 is a contraction. Hence N_1 has a unique fixed point by Banach's contraction principle. This, in turn, shows that the problem (1.3)–(1.4) has a unique solution on $[1-r,b]$. □

Theorem 4.3. *Assume that* (H1)–(H2) *hold. Further we suppose that*

(H4) *the function g is continuous and completely continuous, and for any bounded set B in $C([1-r,b],\mathbb{R})$, the set $\{t \to g(t,y_t) : y \in B\}$ is equicontinuous in $C([1,b],\mathbb{R})$, and there exist constants $0 \le d_1 < 1$, $d_2 \ge 0$ such that*

$$|g(t,u)| \le d_1 \|u\|_C + d_2, \quad t \in [1,b], \ u \in C([-r,0],\mathbb{R}).$$

(H5) *there exists a constant $M > 0$ such that*

$$\frac{(1-d_1)M}{d_2 + \frac{\|p\|_\infty \psi(M)}{\Gamma(\alpha+1)} (\log b)^\alpha} > 1 \,.$$

Then (1.3)–(1.4) *has at least one solution on* $[1-r,b]$.

Proof. We consider the operator $N_1 : C([1-r,b],\mathbb{R}) \to C([1-r,b],\mathbb{R})$ defined by (4.2) and show that the operator N_1 is continuous and completely continuous. Using (H3), it suffices to show that the operator $N_2 : C([1-r,b],\mathbb{R}) \to C([1-r,b],\mathbb{R})$ defined by

$$N_2(y)(t) = \begin{cases} \phi(t), & t \in [1-r,1], \\ \frac{1}{\Gamma(\alpha)} \int_1^t \Big(\log \frac{t}{s}\Big)^{\alpha-1} f(s,y_s)\, ds, & t \in [1,b], \end{cases}$$

is continuous and completely continuous. The proof is similar to that of Theorem 3.4. So we omit the details.

We now show that there exists an open set $U \subseteq C([1-r,b],\mathbb{R})$ with $y \neq \lambda N_1(y)$ for $\lambda \in (0,1)$ and $y \in \partial U$. Let $y \in C([1-r,b],\mathbb{R})$ and $y = \lambda N_1(y)$ for some $0<\lambda<1$. Thus, for each $t\in[1,b]$, we have

$$y(t) = \lambda\Big(g(t,y_t) + \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} f(s,y_s)\,ds\Big).$$

For each $t \in J$, it follows by (H2) and (H3) that

$$\begin{aligned}|y(t)| &\le d_1\|y_t\|_C + d_2 + \frac{1}{\Gamma(\alpha)}\int_1^t \Big(\log\frac{t}{s}\Big)^{\alpha-1} p(s)\psi(\|y_s\|_C)\frac{ds}{s}\\ &\le d_1\|y_t\|_C + d_2 + \frac{\|p\|_\infty \psi(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)}(\log b)^\alpha\,,\end{aligned}$$

which yields

$$(1-d_1)\|y\|_{[1-r,b]} \le d_2 + \frac{\|p\|_\infty \psi(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)}(\log b)^\alpha\,.$$

In consequence, we obtain

$$\frac{(1-d_1)\|y\|_{[1-r,b]}}{d_2 + \frac{\|p\|_\infty \psi(\|y\|_{[1-r,b]})}{\Gamma(\alpha+1)}(\log b)^\alpha} \le 1\,.$$

In view of (H4), there exists M such that $\|y\|_{[1-r,b]} \neq M$. Let us set

$$U = \{y \in C([1-r,b],\mathbb{R}) : \|y\|_{[1-r,b]} < M\}.$$

Note that the operator $N_1 : \overline{U} \to C([1-r,b],\mathbb{R})$ is continuous and completely continuous. From the choice of U, there is no $u \in \partial U$ such that $y = \lambda N_1 y$ for some $\lambda \in (0,1)$. Thus, by the nonlinear alternative of Leray-Schauder type (Lemma 3.3), we deduce that N_1 has a fixed point $y \in \overline{U}$ which is a solution of problem (1.3)-(1.4). This completes the proof. □

5. An example

In this section we give an example to illustrate the usefulness of our main results. Let us consider the fractional functional differential equation,

$$D^{1/2}y(t) = \frac{\|y_t\|_C}{2(1+\|y_t\|_C)}, \quad t \in J := [1,e], \tag{5.1}$$

$$y(t) = \phi(t), \quad t \in [1-r,1]. \tag{5.2}$$

Let

$$f(t,x) = \frac{x}{2(1+x)}, \quad (t,x) \in [1,e]\times[0,\infty).$$

For $x,y \in [0,\infty)$ and $t \in J$, we have

$$|f(t,x)-f(t,y)| = \frac{1}{2}\Big|\frac{x}{1+x} - \frac{y}{1+y}\Big| = \frac{|x-y|}{2(1+x)(1+y)} \le \frac{1}{2}|x-y|,$$

Hence the condition (H0) holds with $\ell = 1/2$. Since $\frac{\ell(\log b)^\alpha}{\Gamma(\alpha+1)} = \frac{1}{\sqrt{\pi}} < 1$, by Theorem 3.2, problem (5.1)-(5.2) has a unique solution on $[1-r,e]$.

References

[1] R. P. Agarwal, Y. Zhou, Y. He; Existence of fractional neutral functional differential equations, *Comput. Math. Appl.* **59** (2010), 1095-1100.

[2] R. P. Agarwal, Y. Zhou, J. R. Wang, X. Luo; Fractional functional differential equations with causal operators in Banach spaces, *Math. Comput. Modelling* **54** (2011), 1440-1452.

[3] B. Ahmad, S. K. Ntouyas; Nonlinear fractional differential equations and inclusions of arbitrary order and multi-strip boundary conditions, *Electron. J. Diff. Eqns.*, Vol. 2012 (2012), No. 98, pp. 1-22.

[4] B. Ahmad, S. K. Ntouyas; Integro-differential equations of fractional order with nonlocal fractional boundary conditions associated with financial asset model, *Electron. J. Diff. Eqns.*, Vol. 2013 (2013), No. 60, pp. 1-10.

[5] B. Ahmad, S. K. Ntouyas, A. Alsaedi; New existence results for nonlinear fractional differential equations with three-point integral boundary conditions, *Adv. Differ. Equ.* (2011) Art. ID 107384, 11 pp.

[6] B. Ahmad, S. Sivasundaram; On four-point nonlocal boundary value problems of nonlinear integro-differential equations of fractional order, *Appl. Math. Comput.* **217** (2010), 480-487.

[7] B. Ahmad, S. K. Ntouyas; Initial value problems for hybrid Hadamard fractional differential equations, *Electron. J. Diff. Equ.*, Vol. 2014 (2014), No. 161, pp. 1-8

[8] M. Benchohra, J. Henderson, S. K. Ntouyas, A. Ouahab; Existence results for fractional order functional differential equations with infinite delay, *J. Math. Anal. Appl.* **338** (2008), 1340-1350.

[9] P. L. Butzer, A. A. Kilbas, J. J. Trujillo; Compositions of Hadamard-type fractional integration operators and the semigroup property, *J. Math. Anal. Appl.* **269** (2002), 387-400.

[10] P. L. Butzer, A. A. Kilbas, J. J. Trujillo; Fractional calculus in the Mellin setting and Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **269** (2002), 1-27.

[11] P. L. Butzer, A. A. Kilbas, J. J. Trujillo; Mellin transform analysis and integration by parts for Hadamard-type fractional integrals, *J. Math. Anal. Appl.* **270** (2002), 1-15.

[12] A. M. A. El-Sayed; Nonlinear functional differential equations of arbitrary orders, *Nonlin. Anal.* **33** (1998), 181-186.

[13] A. Granas, J. Dugundji; *Fixed Point Theory*, Springer-Verlag, New York, 2003.

[14] J. Hadamard; Essai sur l'etude des fonctions donnees par leur developpment de Taylor, *J. Mat. Pure Appl. Ser.* **8** (1892) 101-186.

[15] J. Hale, S. M. Verduyn Lunel; *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, 99, Springer-Verlag, New York, 1993.

[16] A. A. Kilbas, Hari M. Srivastava, J. J. Trujillo; *Theory and Applications of Fractional Differential Equations*. North-Holland Mathematics Studies, 204. Elsevier Science B.V., Amsterdam, 2006.

[17] V. Kolmanovskii, A. Myshkis; *Introduction to the Theory and Applications of Functional-Differential Equations*. Mathematics and its Applications, 463. Kluwer Academic Publishers, Dordrecht, 1999.

[18] K. S. Miller, B. Ross; *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[19] S. M. Momani, S. B. Hadid; Some comparison results for integro-fractional differential inequalities, *J. Fract. Calc.* **24** (2003), 37-44.

[20] I. Podlubny; *Fractional Differential Equations*. Academic Press, San Diego, 1999.

[21] S. G. Samko, A. A. Kilbas, O. I. Marichev; *Fractional Integrals and Derivatives. Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[22] C. Yu, G. Gao; Some results on a class of fractional functional differential equations, *Commun. Appl. Nonlinear Anal.* **11** (2004), 67-75.

[23] C. Yu, G. Gao; Existence of fractional differential equations, *J. Math. Anal. Appl.* **310** (2005), 26-29.

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