Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 68, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

OSCILLATION OF ARBITRARY-ORDER DERIVATIVES OF SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS TAKING SMALL FUNCTIONS IN THE UNIT DISC

PAN GONG, LI-PENG XIAO

Abstract. In this article, we study the relationship between solutions and their derivatives of the differential equation

$$f'' + A(z)f' + B(z)f = F(z),$$

where $A(z), B(z), F(z)$ are meromorphic functions of finite iterated p-order in the unit disc. We obtain some oscillation theorems for $f^{(j)}(z) - \varphi(z)$, where f is a solution and $\varphi(z)$ is a small function.

1. Introduction and results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna's value distribution theory on the complex plane and in the unit disc $\Delta = \{z \in \mathbb{C} : |z| < 1\}$ (see [11, 12, 15, 16, 19]). In addition, we need to give some definitions and discussions. Firstly, let us give two definitions about the degree of small growth order of functions in Δ as polynomials on the complex plane $\mathbb{C}$. There are many types of definitions of small growth order of functions in Δ (see [9, 10]).

Definition 1.1 ([9, 10])**.** Let f be a meromorphic function in Δ , and

$$D(f) = \limsup_{r\to 1^-} \frac{T(r,f)}{\log \frac{1}{1-r}} = b.$$

If $b < \infty$, then we say that f is of finite b degree (or is non-admissible). If $b = \infty$, then we say that f is of infinite degree (or is admissible), both defined by characteristic function $T(r, f)$.

Definition 1.2 ([9, 10])**.** Let f be an analytic function in Δ , and

$$D_M(f) = \limsup_{r\to 1^-} \frac{\log^+ M(r,f)}{\log \frac{1}{1-r}} = a \quad (\text{or } a = \infty).$$

Then we say that f is a function of finite a degree (or of infinite degree) defined by maximum modulus function $M(r, f) = \max_{|z|=r} |f(z)|$.

2000 *Mathematics Subject Classification.* 34M10, 30D35.
Key words and phrases. Unit disc; iterated order; growth; exponent of convergence.

Submitted January 6, 2015. Published March 20, 2015.

For $F \subset [0,1)$, the upper and lower densities of F are defined by

$$\overline{\mathrm{dens}}_{\triangle} F = \limsup_{r\to 1^-} \frac{m(F\cap[0,r))}{m([0,r))}, \quad \underline{\mathrm{dens}}_{\triangle} F = \liminf_{r\to 1^-} \frac{m(F\cap[0,r))}{m([0,r))}$$

respectively, where $m(G) = \int_G \frac{dt}{1-t}$ for $G \subset [0,1)$.

Now we give the definition of iterated order and growth index to classify generally the functions of fast growth in Δ as those in $\mathbb{C}$, see [3, 14, 15]. Let us define inductively, for $r \in [0,1), \exp_1 r = e^r$ and $\exp_{p+1} r = \exp(\exp_p r), p \in \mathbb{N}$. We also define for all r sufficiently large in $(0,1)$, $\log_1 r = \log r$ and $\log_{p+1} r = \log(\log_p r), p \in \mathbb{N}$. Moreover, we denote by $\exp_0 r = r, \log_0 r = r, \exp_{-1} r = \log_1 r, \log_{-1} r = \exp_1 r$.

Definition 1.3 ([4]). The iterated p-order of a meromorphic function f in Δ is defined by

$$\rho_p(f) = \limsup_{r\to 1^-} \frac{\log_p^+ T(r,f)}{\log\frac{1}{1-r}} \quad (p \geq 1).$$

For an analytic function f in Δ , we also define

$$\rho_{M,p}(f) = \limsup_{r\to 1^-} \frac{\log_{p+1}^+ M(r,f)}{\log\frac{1}{1-r}} \quad (p \geq 1).$$

Remark 1.4. It follows by Tsuji [19] that if f is an analytic function in Δ, then

$$\rho_1(f) \leq \rho_{M,1}(f) \leq \rho_1(f) + 1.$$

However it follows by [15, Proposition 2.2.2] that

$$\rho_{M,p}(f) = \rho_p(f) \quad (p \geq 2).$$

Definition 1.5 ([4]). The growth index of the iterated order of a meromorphic function f in Δ is defined by

$$i(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible;} \\ \min\{p \in \mathbb{N}, \rho_p(f) < \infty\}, & \text{if } f \text{ is admissible;} \\ \infty, & \text{if } \rho_p(f) = \infty \ \text{ for all } p \in \mathbb{N}. \end{cases}$$

For an analytic function f in Δ, we also define

$$i_M(f) = \begin{cases} 0, & \text{if } f \text{ is non-admissible;} \\ \min\{p \in \mathbb{N}, \rho_{M,p}(f) < \infty\}, & \text{if } f \text{ is admissible;} \\ \infty, & \text{if } \rho_{M,p}(f) = \infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

Definition 1.6 ([5, 6]). Let f be a meromorphic function in Δ. Then the iterated p-exponent of convergence of the sequence of zeros of $f(z)$ is defined by

$$\lambda_p(f) = \limsup_{r\to 1^-} \frac{\log_p^+ N(r,\frac{1}{f})}{\log\frac{1}{1-r}},$$

where $N(r,\frac{1}{f})$ is the integrated counting function of zeros of $f(z)$ in $\{z \in \mathbb{C} : |z| \leq r\}$. Similarly, the iterated p-exponent of convergence of the sequence of distinct zeros of $f(z)$ is defined by

$$\overline{\lambda}_p(f) = \limsup_{r\to 1^-} \frac{\log_p^+ \overline{N}(r,\frac{1}{f})}{\log\frac{1}{1-r}},$$

where $\overline{N}(r, \frac{1}{f})$ is the integrated counting function of distinct zeros of $f(z)$ in $\{z \in \mathbb{C} : |z| \leq r\}$.

Definition 1.7 ([7]). The growth index of the iterated convergence exponent of the sequence of zeros of $f(z)$ in Δ is defined by

$$i_{\lambda}(f) = \begin{cases} 0, & \text{if } N(r, \frac{1}{f}) = O(\log \frac{1}{1-r}); \\ \min\{p \in \mathbb{N}, \lambda_p(f) < \infty\}, & \text{if some } p \in \mathbb{N} \text{ with } \lambda_p(f) < \infty; \\ \infty, & \text{if } \lambda_p(f) = \infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

Similarly, the growth index of the iterated convergence exponent of the sequence of distinct zeros of $f(z)$ in Δ is defined by

$$i_{\overline{\lambda}}(f) = \begin{cases} 0, & \text{if } \overline{N}(r, \frac{1}{f}) = O(\log \frac{1}{1-r}); \\ \min\{p \in \mathbb{N}, \overline{\lambda}_p(f) < \infty\}, & \text{if some } p \in \mathbb{N} \text{ with } \overline{\lambda}_p(f) < \infty; \\ \infty, & \text{if } \overline{\lambda}_p(f) = \infty \text{ for all } p \in \mathbb{N}. \end{cases}$$

Definition 1.8 ([11]). For $a \in \overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$, the deficiency of f is defined by

$$\delta(a, f) = 1 - \limsup_{r \to 1^-} \frac{N(r, \frac{1}{f-a})}{T(r, f)},$$

provided f has unbounded characteristic.

The complex oscillation theory of solutions of linear differential equations in the complex plane $\mathbb{C}$ was started by Bank and Laine in 1982. After their well known work, many important results have been obtained on the growth and the complex oscillation theory of solutions of linear differential equation in $\mathbb{C}$. It arises naturally an interesting subject of complex oscillation theory of differential equations in the unit disc, which is more difficult to study than that in the complex plane, and there exist some results (see [1, 2, 4, 5, 6, 7, 9, 10, 12, 13, 16, 18, 21]). Recently, Latreuch and Belaïdi studied the oscillation problem of solutions and their derivatives of second-order non-homogeneous linear differential equation

$$f'' + A(z)f' + B(z)f = F(z), \tag{1.1}$$

where $A(z), B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ are meromorphic functions of finite iterated p-order in Δ. For some related papers in the complex plane on the usual order see, [20]. Before we state their results we need to define the following:

$$A_j(z) = A_{j-1}(z) - \frac{B'_{j-1}(z)}{B_{j-1}(z)}, \quad (j = 1, 2, 3, \dots) \tag{1.2}$$

$$B_j(z) = A'_{j-1}(z) - A_{j-1}(z)\frac{B'_{j-1}(z)}{B_{j-1}(z)} + B_{j-1}(z) \quad (j = 1, 2, 3, \dots) \tag{1.3}$$

$$F_j(z) = F'_{j-1}(z) - F_{j-1}(z)\frac{B'_{j-1}(z)}{B_{j-1}(z)}, \quad (j = 1, 2, 3, \dots) \tag{1.4}$$

where $A_0(z) = A(z)$, $B_0(z) = B(z)$ and $F_0(z) = F(z)$. Latreuch and Belaïdi obtained the following results.

Theorem 1.9 ([17]). *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions of finite iterated p-order in Δ such that $B_j(z) \not\equiv 0$ and $F_j(z) \not\equiv 0$ $(j = 1, 2, 3 \dots)$.*

If f is a meromorphic solution in Δ of (1.1) with $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho$, then f satisfies

$$\overline{\lambda}_p(f^{(j)}) = \lambda_p(f^{(j)}) = \rho_p(f) = \infty \quad (j = 0, 1, 2, \dots)$$

$$\overline{\lambda}_{p+1}(f^{(j)}) = \lambda_{p+1}(f^{(j)}) = \rho_{p+1}(f) = \rho \quad (j = 0, 1, 2, \dots).$$

Theorem 1.10 ([17]). *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions in Δ with finite iterated p-order such that $B_j(z) \not\equiv 0$ and $F_j(z) \not\equiv 0$ $(j = 1, 2, 3 \dots)$. If f is a meromorphic solution in Δ of (1.1) with*

$$\rho_p(f) > \max\{\rho_p(A), \rho_p(B), \rho_p(F)\},$$

then

$$\overline{\lambda}_p(f^{(j)}) = \lambda_p(f^{(j)}) = \rho_p(f) \quad (j = 0, 1, 2, \dots).$$

Theorem 1.11 ([17]). *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be analytic functions in Δ with finite iterated p-order such that $\beta = \rho_p(B) > \max\{\rho_p(A), \rho_p(F)\}$. Then all nontrivial solutions of (1.1) satisfy*

$$\rho_p(B) \le \overline{\lambda}_{p+1}(f^{(j)}) = \lambda_{p+1}(f^{(j)}) = \rho_{p+1}(f) \le \rho_{M,p}(B) \quad (j = 0, 1, 2, \dots)$$

with at most one possible exceptional solution f_0 such that

$$\rho_{p+1}(f_0) < \rho_p(B).$$

Theorem 1.12 ([17]). *Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions in Δ with finite iterated p-order such that $\sigma_p(B) > \max\{\sigma_p(A), \sigma_p(F)\}$. If f is a meromorphic solution in Δ of (1.1) with $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho$, then f satisfies*

$$\overline{\lambda}_p(f^{(j)}) = \lambda_p(f^{(j)}) = \rho_p(f) = \infty \quad (j = 0, 1, 2, \dots)$$

$$\overline{\lambda}_{p+1}(f^{(j)}) = \lambda_{p+1}(f^{(j)}) = \rho_{p+1}(f) = \rho \quad (j = 0, 1, 2, \dots),$$

where

$$\sigma_p(f) = \limsup_{r \to 1^-} \frac{\log_p m(r, f)}{\log \frac{1}{1-r}}.$$

In this article, we continue to study the oscillation problem of solutions and their derivatives of second order non-homogeneous linear differential equation of (1.1). Let $\varphi(z)$ be a meromorphic function in Δ with finite iterated p-order $\rho_p(\varphi) < \infty$. We need to define the notation

$$D_j = F_j - (\varphi'' + A_j\varphi' + B_j\varphi), \quad (j = 0, 1, 2, \dots) \tag{1.5}$$

where $A_j(z), B_j(z), F_j(z)$ are defined in (1.2)–(1.4). We obtain the following results.

Theorem 1.13. *Let $\varphi(z)$ be a meromorphic function in Δ with $\rho_p(\varphi) < \infty$. Let $A(z)$, $B(z) \not\equiv 0$ and $F(z) \not\equiv 0$ be meromorphic functions of finite iterated p-order in Δ such that $B_j(z) \not\equiv 0$ and $D_j(z) \not\equiv 0$ $(j = 0, 1, 2, \dots)$.*

(a) If f is a meromorphic solution in Δ of (1.1) with $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho < \infty$, then f satisfies

$$\overline{\lambda}_p(f^{(j)} - \varphi) = \lambda_p(f^{(j)} - \varphi) = \rho_p(f) = \infty \quad (j = 0, 1, 2, \dots),$$

$$\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \lambda_{p+1}(f^{(j)} - \varphi) = \rho_{p+1}(f) = \rho \quad (j = 0, 1, 2, \dots).$$

(b) If f is a meromorphic solution in Δ of (1.1) with

$$\max\{\rho_p(A), \rho_p(B), \rho_p(F), \rho_p(\varphi)\} < \rho_p(f) < \infty,$$

then

$$\overline{\lambda}_p(f^{(j)}-\varphi)=\lambda_p(f^{(j)}-\varphi)=\rho_p(f)\quad (j=0,1,2,\dots).$$

Next, we give some sufficient conditions on the coefficients which guarantee $B_j(z)\not\equiv 0$ and $D_j(z)\not\equiv 0$ $(j=1,2,\dots)$, and we obtain

Theorem 1.14. *Let $\varphi(z)$ be an analytic function in Δ with $\rho_p(\varphi)<\infty$ and be not a solution of (1.1). Let $A(z)$, $B(z)\not\equiv 0$ and $F(z)\not\equiv 0$ be analytic functions in Δ with finite iterated p-order such that $\beta=\rho_p(B)>\max\{\rho_p(A),\rho_p(F),\rho_p(\varphi)\}$ and $\rho_{M,p}(A)\le\rho_{M,p}(B)$. Then all nontrivial solutions of (1.1) satisfy*

$$\rho_p(B)\le\overline{\lambda}_{p+1}(f^{(j)}-\varphi)=\lambda_{p+1}(f^{(j)}-\varphi)=\rho_{p+1}(f)\le\rho_{M,p}(B)\quad (j=0,1,2,\dots)$$

with at most one possible exceptional solution f_0 such that

$$\rho_{p+1}(f_0)<\rho_p(B).$$

Theorem 1.15. *Let $\varphi(z)$ be a meromorphic function in Δ with $\rho_p(\varphi)<\infty$ and be not a solution of (1.1). Let $A(z)$, $B(z)\not\equiv 0$ and $F(z)\not\equiv 0$ be meromorphic functions in Δ with finite iterated p-order such that $\rho_p(B)>\max\{\rho_p(A),\rho_p(F),\rho_p(\varphi)\}$ and $\delta(\infty,B)>0$. If f is a meromorphic solution in Δ of (1.1) with $\rho_p(f)=\infty$ and $\rho_{p+1}(f)=\rho$, then f satisfies*

$$\overline{\lambda}_p(f^{(j)}-\varphi)=\lambda_p(f^{(j)}-\varphi)=\rho_p(f)=\infty\quad (j=0,1,2,\dots),$$
$$\overline{\lambda}_{p+1}(f^{(j)}-\varphi)=\lambda_{p+1}(f^{(j)}-\varphi)=\rho_{p+1}(f)=\rho\quad (j=0,1,2,\dots).$$

2. Preliminary Lammas

Lemma 2.1 ([2]). *Let $f(z)$ be a meromorphic function in the unit disc for which $i(f)=p\ge 1$ and $\rho_p(f)=\beta<\infty$ and let $k\in\mathbb{N}$. Then for any $\varepsilon>0$,*

$$m\Big(r,\frac{f^{(k)}}{f}\Big)=O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta+\varepsilon}\Big)$$

for all r outside a set $E_1\subset[0,1)$ with $\int_{E_1}\frac{dr}{1-r}<\infty$.

Lemma 2.2 ([6]). *Let $A_0,A_1,\dots,A_{k-1},F\not\equiv 0$ be meromorphic functions in Δ , and let f be a meromorphic solution of the differential equation*

$$f^{(k)}+A_{k-1}(z)f^{(k-1)}+\dots+A_0(z)f=F(z) \tag{2.1}$$

such that $i(f)=p(0<p<\infty)$. If either

$$\max\{i(A_j)\quad (j=0,1,\dots,k-1),i(F)\}<p$$

or

$$\max\{\rho_p(A_j)\quad (j=0,1,\dots,k-1),\rho_p(F)\}<\rho_p(f),$$

then

$$i_{\overline{\lambda}}(f)=i_\lambda(f)=i(f)=p,$$
$$\overline{\lambda}_p(f)=\lambda_p(f)=\rho_p(f).$$

Lemma 2.3 ([17]). *Let $A_0,A_1,\dots,A_{k-1},F\not\equiv 0$ be finite iterated p-order meromorphic functions in the unit disc Δ. If f is a meromorphic solution with $\rho_p(f)=\infty$ and $\rho_{p+1}(f)=\rho<\infty$ of equation (2.1), then*

$$\overline{\lambda}_p(f)=\lambda_p(f)=\rho_p(f)=\infty,$$
$$\overline{\lambda}_{p+1}(f)=\lambda_{p+1}(f)=\rho_{p+1}(f)=\rho.$$

Lemma 2.4. *Let $\varphi, A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be finite iterated p-order meromorphic functions in the unit disc Δ such that*
$$F - \varphi^{(k)} - A_{k-1}\varphi^{(k-1)} - \dots - A_1\varphi' - A_0\varphi \not\equiv 0.$$
If f is a meromorphic solution with $\rho_p(f) = \infty$ and $\rho_{p+1}(f) = \rho < \infty$ of equation (2.1), then
$$\overline{\lambda}_p(f - \varphi) = \lambda_p(f - \varphi) = \rho_p(f) = \infty,$$
$$\overline{\lambda}_{p+1}(f - \varphi) = \lambda_{p+1}(f - \varphi) = \rho_{p+1}(f) = \rho.$$

Proof. Suppose that $g = f - \varphi$, we obtain $f = g + \varphi$, then from (2.1) we have
$$g^{(k)} + A_{k-1}g^{(k-1)} + \dots + A_1g' + A_0g = F - \varphi^{(k)} - A_{k-1}\varphi^{(k-1)} - \dots - A_1\varphi' - A_0\varphi.$$
Since $\rho_p(f - \varphi) = \infty$ and $\rho_{p+1}(f - \varphi) = \rho < \infty$, then by using Lemma 2.3 we obtain
$$\overline{\lambda}_p(f - \varphi) = \lambda_p(f - \varphi) = \rho_p(f) = \infty,$$
$$\overline{\lambda}_{p+1}(f - \varphi) = \lambda_{p+1}(f - \varphi) = \rho_{p+1}(f) = \rho.$$
□

Lemma 2.5 ([6])**.** *Let $p \in \mathbb{N}$, and assume that the coefficients $A_0, \dots, A_{k-1}$ and $F \not\equiv 0$ are analytic in Δ and $\rho_p(A_j) < \rho_p(A_0)$ for all $j = 1, \dots, k-1$. Let $\alpha_M = \max\{\rho_{M,p}(A_j) : j = 0, \dots, k-1\}$. If $\rho_{M,p+1}(F) < \rho_p(A_0)$, then all solutions f of (2.1) satisfy*
$$\rho_p(A_0) \le \overline{\lambda}_{p+1}(f) = \lambda_{p+1}(f) = \rho_{M,p+1}(f) \le \alpha_M,$$
with at most one exception f_0 satisfying $\rho_{M,p+1}(f_0) < \rho_p(A_0)$.

By a similar reasoning as Lemma 2.4 and by using Lemma 2.5, we can obtain the following lemma.

Lemma 2.6. *Let $p \in \mathbb{N}$, φ be finite iterated p-order analytic functions in the unit disc Δ and assume that the coefficients $A_0, \dots, A_{k-1}$,$F \not\equiv 0$ and $F - \varphi^{(k)} - A_{k-1}\varphi^{(k-1)} - \dots - A_1\varphi' - A_0\varphi \not\equiv 0$ are analytic in Δ and $\rho_p(A_j) < \rho_p(A_0)$ for all $j = 1, \dots, k-1$. Let $\alpha_M = \max\{\rho_{M,p}(A_j) : j = 0, \dots, k-1)$. If $\rho_{M,p+1}(F - \varphi^{(k)} - A_{k-1}\varphi^{(k-1)} - \dots - A_1\varphi' - A_0\varphi) < \rho_p(A_0)$, then all solutions f of (2.1) satisfy*
$$\rho_p(A_0) \le \overline{\lambda}_{p+1}(f - \varphi) = \lambda_{p+1}(f - \varphi) = \rho_{M,p+1}(f) \le \alpha_M,$$
with at most one exception f_0 satisfying $\rho_{M,p+1}(f_0) < \rho_p(A_0)$.

Lemma 2.7. *Let φ, $A_0, \dots, A_{k-1}$, $F \not\equiv 0$ be meromorphic functions in the unit disc Δ such that $F - \varphi^{(k)} - A_{k-1}\varphi^{(k-1)} - \dots - A_1\varphi' - A_0\varphi \not\equiv 0$, and let f be a meromorphic solution of the differential equation of (2.1), such that $i(f) = p(0 < p < \infty)$. If either*
$$\max\{i(A_j) : (j = 0, 1, \dots, k-1), i(F),\ i(\varphi)\} < p$$
or
$$\max\{\rho_p(A_j) : (j = 0, 1, \dots, k-1),\ \rho_p(F), \rho_p(\varphi)\} < \rho_p(f),$$
then
$$i_{\overline{\lambda}}(f - \varphi) = i_\lambda(f - \varphi) = i(f) = p,$$
$$\overline{\lambda}_p(f - \varphi) = \lambda_p(f - \varphi) = \rho_p(f).$$

The proof of the above lemma follows a similar reasoning as in Lemmas 2.4 and 2.2.

3. Proofs of theorems

Proof of Theorem 1.13. (a) For the proof, we use the principle of mathematical induction. Since $D_0 = F - (\varphi'' + A\varphi' + B\varphi) \not\equiv 0$, then by using Lemma 2.4, we have

$$\overline{\lambda}_p(f-\varphi) = \lambda_p(f-\varphi) = \rho_p(f) = \infty,$$
$$\overline{\lambda}_{p+1}(f-\varphi) = \lambda_{p+1}(f-\varphi) = \rho_{p+1}(f) = \rho.$$

Since $B(z) \not\equiv 0$, dividing both sides of (1.1) by B, we obtain

$$\frac{1}{B}f'' + \frac{A}{B}f' + f = \frac{F}{B}. \tag{3.1}$$

Differentiating both sides of (3.1), we have

$$\frac{1}{B}f^{(3)} + \Big(\big(\frac{1}{B}\big)' + \frac{A}{B}\Big)f'' + \Big(\big(\frac{A}{B}\big)' + 1\Big)f' = \big(\frac{F}{B}\big)'. \tag{3.2}$$

Multiplying (3.2) by B, we obtain

$$f^{(3)} + A_1f'' + B_1f' = F_1, \tag{3.3}$$

where

$$A_1 = A - \frac{B'}{B}, \quad B_1 = A' - A\frac{B'}{B} + B, \quad F_1 = F' - F\frac{B'}{B}.$$

Since A_1, B_1 and F_1 are meromorphic functions with finite iterated p-order, and $D_1 = F_1 - (\varphi'' + A_1\varphi' + B_1\varphi) \not\equiv 0$, then using Lemma 2.4, we obtain

$$\overline{\lambda}_p(f'-\varphi) = \lambda_p(f'-\varphi) = \rho_p(f) = \infty,$$
$$\overline{\lambda}_{p+1}(f'-\varphi) = \lambda_{p+1}(f'-\varphi) = \rho_{p+1}(f) = \rho.$$

Since $B_1(z) \not\equiv 0$, dividing now both sides of (3.3) by B_1, we obtain

$$\frac{1}{B_1}f^{(3)} + \frac{A_1}{B_1}f'' + f' = \frac{F_1}{B_1}. \tag{3.4}$$

Differentiating both sides of equation (3.4) and multiplying by B_1, we obtain

$$f^{(4)} + A_2f^{(3)} + B_2f'' = F_2, \tag{3.5}$$

where A_2, B_2, F_2 are meromorphic functions defined in (1.2)-(1.4). Since $D_2 = F_2 - (\varphi'' + A_2\varphi' + B_2\varphi) \not\equiv 0$, by using Lemma 2.4 again, we obtain

$$\overline{\lambda}_p(f''-\varphi) = \lambda_p(f''-\varphi) = \rho_p(f) = \infty,$$
$$\overline{\lambda}_{p+1}(f''-\varphi) = \lambda_{p+1}(f''-\varphi) = \rho_{p+1}(f) = \rho.$$

Suppose now that

$$\overline{\lambda}_p(f^{(k)}-\varphi) = \lambda_p(f^{(k)}-\varphi) = \rho_p(f) = \infty, \tag{3.6}$$
$$\overline{\lambda}_{p+1}(f^{(k)}-\varphi) = \lambda_{p+1}(f^{(k)}-\varphi) = \rho_{p+1}(f) = \rho \tag{3.7}$$

for all $k = 0, 1, 2, \dots, j-1$, and we prove that (3.6) and (3.7) are true for $k = j$. By the same procedure as before, we can obtain

$$f^{(j+2)} + A_jf^{(j+1)} + B_jf^{(j)} = F_j, \tag{3.8}$$

where A_j, B_j and F_j are meromorphic functions defined in (1.2)-(1.4). Since $D_j = F_j - (\varphi'' + A_j\varphi' + B_j\varphi) \not\equiv 0$, by using Lemma 2.4, we obtain

$$\overline{\lambda}_p(f^{(j)} - \varphi) = \lambda_p(f^{(j)} - \varphi) = \rho_p(f) = \infty,$$
$$\overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \lambda_{p+1}(f^{(j)} - \varphi) = \rho_{p+1}(f) = \rho.$$

(b) Since $D_0 = F - (\varphi'' + A\varphi' + B\varphi) \not\equiv 0$, and $\max\{\rho_p(A), \rho_p(B), \rho_p(F), \rho_p(\varphi)\} < \rho_p(f) < \infty$, then by using Lemma 2.7 we have

$$\overline{\lambda}_p(f - \varphi) = \lambda_p(f - \varphi) = \rho_p(f).$$

By (a), we have (3.3) and $\max\{\rho_p(A_1), \rho_p(B_1), \rho_p(F_1), \rho_p(\varphi)\} < \rho_p(f) < \infty$. Since $D_1 \not\equiv 0$, then by using Lemma 2.7 we obtain

$$\overline{\lambda}_p(f' - \varphi) = \lambda_p(f' - \varphi) = \rho_p(f).$$

By (a), we have (3.5) and $\max\{\rho_p(A_2), \rho_p(B_2), \rho_p(F_2), \rho_p(\varphi)\} < \rho_p(f) < \infty$. Since $D_2 \not\equiv 0$, then by using Lemma 2.7 we obtain

$$\overline{\lambda}_p(f'' - \varphi) = \lambda_p(f'' - \varphi) = \rho_p(f).$$

Suppose now that

$$\overline{\lambda}_p(f^{(k)} - \varphi) = \lambda_p(f^{(k)} - \varphi) = \rho_p(f) \tag{3.9}$$

for all $k = 0, 1, 2, \dots, j-1$, and we prove that (3.9) is true for $k = j$. By (a) we have (3.8) and $\max\{\rho_p(A_j), \rho_p(B_j), \rho_p(F_j), \rho_p(\varphi)\} < \rho_p(f) < \infty$. Since $D_j \not\equiv 0$, then by using Lemma 2.7 we obtain

$$\overline{\lambda}_p(f^{(j)} - \varphi) = \lambda_p(f^{(j)} - \varphi) = \rho_p(f),$$

The proof is complete. □

Proof of Theorem 1.14. Since $F - (\varphi'' + A\varphi' + B\varphi) \not\equiv 0$, $\rho_{M,p+1}(F - (\varphi'' + A\varphi' + B\varphi)) < \rho_p(B)$. By Lemma 2.6, all nontrivial solutions of (1.1) satisfy

$$\rho_p(B) \le \overline{\lambda}_{p+1}(f - \varphi) = \lambda_{p+1}(f - \varphi) = \rho_{p+1}(f) \le \rho_{M,p}(B)$$

with at most one possible exceptional solution f_0 such that $\rho_{p+1}(f_0) < \rho_p(B)$. By using (1.2) and Lemma 2.1 we have for any $\varepsilon > 0$,

$$m(r, A_j) \le m(r, A_{j-1}) + O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta+\varepsilon}\Big) \quad (\beta = \rho_p(B_{j-1}))$$

outside a set $E_1 \subset [0,1)$ with $\int_{E_1} \frac{dr}{1-r} < \infty$, for all $j = 1, 2, 3, \dots$, which we can write as

$$m(r, A_j) \le m(r, A) + O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta+\varepsilon}\Big). \tag{3.10}$$

On the other hand, from (1.3), we have

$$\begin{aligned} B_j &= A_{j-1}\Big(\frac{A'_{j-1}}{A_{j-1}} - \frac{B'_{j-1}}{B_{j-1}}\Big) + B_{j-1} \\ &= A_{j-1}\Big(\frac{A'_{j-1}}{A_{j-1}} - \frac{B'_{j-1}}{B_{j-1}}\Big) + A_{j-2}\Big(\frac{A'_{j-2}}{A_{j-2}} - \frac{B'_{j-2}}{B_{j-2}}\Big) + B_{j-2} \\ &= \sum_{k=0}^{j-1} A_k\Big(\frac{A'_k}{A_k} - \frac{B'_k}{B_k}\Big) + B. \end{aligned} \tag{3.11}$$

Now we prove that $B_j \not\equiv 0$ for all $j = 1, 2, 3, \dots$. For that we suppose there exists $j \in \mathbb{N}$ such that $B_j = 0$. By (3.10) and (3.11) we have

$$\begin{aligned} T(r,B) = m(r,B) &\le \sum_{k=0}^{j-1} m(r,A_k) + O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta+\varepsilon}\Big) \\ &\le jm(r,A) + O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta+\varepsilon}\Big) \\ &= jT(r,A) + O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta+\varepsilon}\Big), \end{aligned} \tag{3.12}$$

which implies the contradiction $\rho_p(B) \le \rho_p(A)$. Hence $B_j \not\equiv 0$ for all $j = 1, 2, 3, \dots$. We prove that $D_j \not\equiv 0$ for all $j = 1, 2, 3, \dots$. For that we suppose there exists $j \in \mathbb{N}$ such that $D_j = 0$. We have $F_j - (\varphi'' + A_j\varphi' + B_j\varphi) = 0$ from (1.5), which implies

$$F_j = \varphi\Big(\frac{\varphi''}{\varphi} + A_j\frac{\varphi'}{\varphi} + B_j\Big) = \varphi\Big[\frac{\varphi''}{\varphi} + A_j\frac{\varphi'}{\varphi} + \sum_{k=0}^{j-1} A_k\Big(\frac{A_k'}{A_k} - \frac{B_k'}{B_k}\Big) + B\Big].$$

Here we suppose that $\varphi(z) \not\equiv 0$, otherwise by Theorem 1.11 there is nothing to prove. Therefore,

$$B = \frac{F_j}{\varphi} - \Big[\frac{\varphi''}{\varphi} + A_j\frac{\varphi'}{\varphi} + \sum_{k=0}^{j-1} A_k\Big(\frac{A_k'}{A_k} - \frac{B_k'}{B_k}\Big)\Big]. \tag{3.13}$$

On the other hand, from (1.4),

$$m(r,F_j) \le m(r,F) + O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta+\varepsilon}\Big). \quad (j = 1, 2, 3, \dots) \tag{3.14}$$

By (3.10), (3.13), (3.14) and Lemma 2.1 we have

$$\begin{aligned} T(r,B) = m(r,B) &\le m(r,\frac{1}{\varphi}) + m(r,F) + (j+1)m(r,A) \\ &\quad + O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta_1+\varepsilon}\Big), \end{aligned} \tag{3.15}$$

where β_1 is some non-negative constant, which implies the contradiction $\rho_p(B) \le \max\{\rho_p(A), \rho_p(F), \rho_p(\varphi)\}$. Hence $D_j \not\equiv 0$ for all $j = 1, 2, 3, \dots$. Since $B_j \not\equiv 0$, $D_j \not\equiv 0$ $(j = 1, 2, 3, \dots)$, then by Theorem 1.13 and Lemma 2.6 we have

$$\rho_p(B) \le \overline{\lambda}_{p+1}(f^{(j)} - \varphi) = \lambda_{p+1}(f^{(j)} - \varphi) = \rho_{p+1}(f) \le \rho_{M,p}(B) \quad (j = 0, 1, 2, \dots)$$

with at most one possible exceptional solution f_0 such that $\rho_{p+1}(f_0) < \rho_p(B)$. □

Proof of Theorem 1.15. We need only to prove that $B_j \not\equiv 0$ and $D_j \not\equiv 0$ for all $j = 1, 2, 3, \dots$. Then by Theorem 1.13 we can obtain Theorem 1.15. Consider the assumption $\delta(\infty, B) = \delta > 0$. Then for $r \to 1^-$ we have

$$T(r,B) \le \frac{2}{\delta} m(r,B). \tag{3.16}$$

Now we prove that $B_j \not\equiv 0$ for all $j = 1, 2, 3, \dots$. For that we suppose there exists $j \in \mathbb{N}$ such that $B_j = 0$. By (3.10), (3.11) and (3.16) we obtain

$$\begin{aligned} T(r,B) \le \frac{2}{\delta} m(r,B) &\le \frac{2}{\delta}\sum_{k=0}^{j-1} m(r,A_k) + \frac{2}{\delta} O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta+\varepsilon}\Big) \\ &\le \frac{2}{\delta} j m(r,A) + \frac{2}{\delta} O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta+\varepsilon}\Big) \\ &\le \frac{2}{\delta} j T(r,A) + \frac{2}{\delta} O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta+\varepsilon}\Big), \end{aligned} \tag{3.17}$$

which implies the contradiction $\rho_p(B) \le \rho_p(A)$. Hence $B_j \not\equiv 0$ for all $j = 1, 2, 3, \dots$. We prove that $D_j \not\equiv 0$ for all $j = 1, 2, 3, \dots$. For that we suppose there exists $j \in \mathbb{N}$ such that $D_j = 0$. If $\varphi(z) \not\equiv 0$, then by (3.10), (3.13), (3.14), (3.16) and Lemma 2.1 we have

$$\begin{aligned} T(r,B) &\le \frac{2}{\delta} m(r,B) \\ &\le \frac{2}{\delta}\Big[m(r,\frac{1}{\varphi}) + m(r,F) + (j+1)m(r,A) + O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta+\varepsilon}\Big)\Big], \end{aligned} \tag{3.18}$$

which implies the contradiction $\rho_p(B) \le \max\{\rho_p(A), \rho_p(F), \rho_p(\varphi)\}$. If $\varphi(z) \equiv 0$, Then from (1.4), (1.5), we have

$$F'_{j-1} - F_{j-1}\frac{B'_{j-1}(z)}{B_{j-1}(z)} = 0, \tag{3.19}$$

which implies $F_{j-1}(z) = cB_{j-1}(z)$, where c is some constant. By (3.11) and (3.19), we have

$$\frac{1}{c} F_{j-1} = \sum_{k=0}^{j-2} A_k\Big(\frac{A'_k}{A_k} - \frac{B'_k}{B_k}\Big) + B. \tag{3.20}$$

On the other hand, from (1.4),

$$m(r, F_{j-1}) \le m(r,F) + O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta+\varepsilon}\Big). \tag{3.21}$$

By (3.16), (3.20), (3.21) and Lemma 2.1, we have

$$\begin{aligned} T(r,B) &\le \frac{2}{\delta} m(r,B) \\ &\le \frac{2}{\delta}\sum_{k=0}^{j-2} m(r,A_k) + \frac{2}{\delta} m(r,F_{j-1}) + O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta+\varepsilon}\Big) \\ &\le \frac{2}{\delta}(j-1)T(r,A) + \frac{2}{\delta} T(r,F) + O\Big(\exp_{p-2}\big(\frac{1}{1-r}\big)^{\beta+\varepsilon}\Big), \end{aligned} \tag{3.22}$$

which implies the contradiction $\rho_p(B) \le \max\{\rho_p(A), \rho_p(F)\}$. Hence $D_j \not\equiv 0$ for all $j = 1, 2, 3, \dots$. By Theorem 1.13, we obtain Theorem 1.15. □

Acknowledgments. The authors would like to thank the anonymous referee for making valuable suggestions and comments to improve this article.

This research was supported by the National Natural Science Foundation of China (11301232, 11171119), by the Natural Science Foundation of Jiangxi province (20132BAB211009), and by the Youth Science Foundation of Education Burean of Jiangxi province (GJJ12207).

References

[1] B. Belaïdi, A. EI Farissi; *Fixed points and iterated order of differential polynomial generated by solutions of linear differential equations in the unit disc*, J. Adv. Res. Pure Math. 3 (2011), no. 1, 161-172.

[2] B. Belaïdi; *Oscillation of fast growing solutions of linear differential equations in the unit disc*, Acta Univ. Sapientiae Math. 2(2010), no. 1, 25-38.

[3] L. G. Bernal; *On growth k-order of solutions of a complex homogeneous linear differential equation*, Proc Amer. Math. Soc. 101(1987), no. 2, 317-322.

[4] T. B. Cao, H. Y. Yi; *The growth of solutions of linear differential equations with coefficients of iterated order in the unit disc*, J. Math. Anal. Appl. 319 (2006), no. 1, 278-294.

[5] T. B. Cao; *The growth, oscillation and fixed points of solutions of complex linear differential equations in the unit disc*, J. Math. Anal. Appl. 352 (2009), no. 2, 739-748.

[6] T. B. Cao, Z. S. Deng; *Solutions of non-homogeneous linear differential equations in the unit disc*, Ann. Polo. Math. 97 (2010), no. 1, 51-61.

[7] T. B. Cao, C. X. Zhu, K. Liu; *On the complex oscillation of meromorphic solutions of second order linear differential equations in the unit disc* , J. Math. Anal. Appl. 374 (2011), no. 1, 272-281.

[8] T. B. Cao, C. X. Zhu, H. Y. Xu; *On the complex oscillation of differential polynomials generated by meromorphic solutions of differential equations in the unit disc*, Proc. Indian Acad. Sci. 2010, 120 (4): 481-493.

[9] Z. X. Chen, K. H. Shon; *The growth of solutions of differential equations with coefficients of small growth in the disc*, J. Math. Anal. Appl. 297 (2004), no. 1, 285-304.

[10] I. E. Chyzhykov, G. G. Gundersen, J. Heittokangas; *Linear differential equations and logarithmic derivative estimates*, Proc. London Math. Soc. (3) 86 (2003), no. 3,735-754.

[11] W. K. Hayman; *Meromorphic Functions*, Oxford Mathematical Monographs Clarendon Press, Oxford, 1964.

[12] J. Heittokangas; *On complex differential equations in the unit disc*, Ann. Acad. Sci. Fenn. Math. Diss. 122 (2000), 1-54.

[13] J. Heittokangas, R. Korhonen, J. Rättyä; *Fast growing solutions of linear differential equations in the unit disc*, Results Math. 49 (2006), no. 3-4, 265-278.

[14] L. Kinnunen; *Linear differential equations with solutions of finite iterated order*, Southeast Asian Bull. Math., 22 (1998), no. 4, 385-405.

[15] I. Laine; *Nevanlinna theory and complex differential equations*, Walter de Gruyter, Berlin, 1993.

[16] I. Laine; *Complex differential equations, Handbook of differential equations: ordinary differential equations*, Vol. IV, 269-363, Handb. Differ. Equ., Elsevier/North-Holland, Amsterdam, 2008.

[17] Z. Latreuch, B. Belaïdi; *Complex oscillation of solutions and their derivatives of non-homogeneous linear differential equations in the unit disc*, Intern. J. Anal. Appl., 2 (2013), no.2, 111-123.

[18] D. Shea, L. Sons; *Value distribution theory for meromorphic functions of slow growth in the disc*, Houston J. Math. 12 (2) (1986), 249-266.

[19] M. Tsuji; *Potential Theory in Modern Function Theory*, Chelsea, New York,(1975), reprint of the 1959 edition.

[20] J. Tu, H. Y. Xu, C. Y. Zhang; *On the zeros of solutions of any order of derivative of second order linear differential equations taking small functions*, Electron. J. Qual. Theory Differ. Equ. 2011, No. 23, 1-27.

[21] G. Zhang, A. Chen; *Fixed points of the derivative and k-th power of solutions of complex linear differential equations in the unit disc*, Electron J. Qual. Theory Differ. Equ., 2009, No 48, 1-9.

Pan Gong

Institute of Mathematics and Information Science, Jiangxi Normal University, Nanchang 330022, China

E-mail address: gongpan12@163.com

Li Peng Xiao (corresponding author)
Institute of Mathematics and Information Science, Jiangxi Normal University, Nanchang 330022, China
E-mail address: 2992507211@qq.com