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PROPERTIES OF LYAPUNOV EXPONENTS FOR QUASIPERODIC COCYCLES WITH SINGULARITIES

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Abstract. We consider the quasi-periodic cocycles $(\omega, A(x,E)) : (x,v) \mapsto (x+\omega, A(x,E)v)$ with ω Diophantine. Let $M_2(\mathbb{C})$ be a normed space endowed with the matrix norm, whose elements are the 2×2 matrices. Assume that $A : \mathbb{T}\times\mathscr{E} \to M_2(\mathbb{C})$ is jointly continuous, depends analytically on $x \in \mathbb{T}$ and is Hölder continuous in $E \in \mathscr{E}$, where $\mathscr{E}$ is a compact metric space and $\mathbb{T}$ is the torus. We prove that if two Lyapunov exponents are distinct at one point $E_0 \in \mathscr{E}$, then these two Lyapunov exponents are Hölder continuous at any E in a ball central at E_0. Moreover, we will give the expressions of the radius of this ball and the Hölder exponents of the two Lyapunov exponents.

1. Introduction

Denote by $\mathbb{T} := \mathbb{R}/\mathbb{Z}$ the torus equipped with its Haar measure μ, and $\mu(\mathbb{T}) = 1$. Let $M_d(\mathbb{C})$ be the set of linear operators from $\mathbb{C}^d$ to $\mathbb{C}^d$, i.e. the set of $d\times d$ complex matrices. A quasi-periodic cocycle is a pair (ω, A), where ω, the irrational number, is the frequency and $A \in C^0(\mathbb{T}, M_d(\mathbb{C}))$ is continuous, defined by a map $(\omega, A) : (x, v) \to (x+\omega, A(x)v)$. The iterates of the cocycle are given by $(\omega, A)^N = (N\omega, A_N)$, where

$$A_N(x) = \prod_{j=N-1}^{0} A(x + j\omega).$$

In this case, the dynamical system is ergodic and the Oseledets Theorem provides us with a sequence of Lyapunov exponents $L_1 \le L_2 \le \dots \le L_m$, and for almost every $x \in \mathbb{T}$ there exist an invariant measurable decomposition $\mathbb{C}^d = \oplus_{j=1}^n E_x^j$, and a non decreasing surjective map $k : \{1,2,\dots,d\} \to \{1,\dots,n\}$ such that for almost every $x \in \mathbb{T}$, every $1 \le i \le m$ and every $v \in E_x^{k_i}\backslash\{0\}$ we have $L_j = \lim_{N\to\infty} \frac{1}{N}\log\|A_N(x)v\|$. Moreover, $L_i = L_{i+1}$ if and only if $k_i = k_{i+1}$, and the subspace E_x^j has dimension equal to $\sharp k^{-1}(j)$, where $\sharp k^{-1}(j)$ is the number of the elements in the set $\{i|k_i = j\}$.

In the past several years, some researchers focused on the continuity of Lyapunov exponent for the Schrödinger equation:

$$(S_{x,\omega}\phi)(n) = \phi(n+1) + \phi(n-1) + v(x+n\omega)\phi(n) = E\phi(n), \quad n \in \mathbb{Z},$$

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with the cocycle

$$A(x,E)=\begin{pmatrix} v(x)-E & -1\\ 1 & 0\end{pmatrix}.$$

Goldstein and Schlag [2] proved that if $v(x)$ is analytic and ω is a Diophantine, which will mean that

$$\|n\omega\|\ge \frac{c(\omega)}{n(\log n)^a},\quad \forall n\ge 2 \tag{1.1}$$

with $a>1$ arbitrary but fixed, the Lyapunov exponent $L(E)$ is Hölder continuous. Then You and Zhang showed the similar result with more general ω in [11]. Observing that for the Schrödinger cocycle, $\det A\equiv 1$ and $L_1(E)+L_2(E)=0$ for any E, it makes that we can only study the Lyapunov exponent defined by

$$L(E):=\lim_{N\to\infty}\frac{1}{N}\int_{\mathbb{T}}\log\|A_N(x,E)\|dx. \tag{1.2}$$

Recall that we say a function $f(E)$ is Hölder continuous, when there are nonnegative real constants C, α, such that

$$|f(E_1)-f(E_2)|<C|E_1-E_2|^{\alpha}$$

for all E_1 and E_2 in the domain of $f(E)$. The number α is called the exponent of the Hölder condition or the Hölder exponent. Compared with the Schrödinger equation, the Jacobi operator is more complicated with the cocycle

$$A(x,E)=\begin{pmatrix} a(x)-E & -b(x)\\ b(x+\omega) & 0\end{pmatrix}.$$

The author in [8] showed that the Lyapunov exponent $L(E)$ defined in (1.2) is Hölder continuous, and the Hölder exponent does not depend on the $L(E)$. It is a better result than what in [2], as the Hölder exponent of the Lyapunov exponent in [2] depends on $L(E)$. Later, for the 2×2 analytic quasi-periodic cocycles $A(x)\in C^{\omega}_r(\mathbb{T},M_2(\mathbb{C}))$ which have a holomorphic extension to a neighborhood of the strip $\mathbb{S}_r:=\{z\in\mathbb{C}:|\mathrm{Im}z|\le r\}$ and is endowed with the norm

$$\|A\|_r:=\sup_{z\in\mathbb{S}_r}\|A(z)\|,$$

Jitomirskaya, Koslover and Schulteis [5] proved that the Lyapunov exponent $L(A)$, which is defined by

$$L(A):=\lim_{N\to\infty}\frac{1}{N}\int_{\mathbb{T}}\log\|\prod_{j=N-1}^{0}A(x+j\omega)\|dx, \tag{1.3}$$

is a continuous for the fixed determinant with Diophantine ω. The author gave a proof of this result for the High dimensional torus in [9]. Similar problems for higher dimensional quasi-periodic cocycles have been studied by Schlag [7]. In that paper, the Hölder continuity is proven if all Lyapunov exponents defined in Oseledets Theorem are unequal.

This article concerns certain quasi-periodic cocycles $(\omega, A(x,E))$ defined as follow. The ω is defined as (1.1) and it is well know that almost every $\omega\in(0,1)$ satisfies this condition. The function $x\to A(x,E)$ is a element of the Banach space $C^{\omega}_r(\mathbb{T},M_2(\mathbb{C}))$. The variable E in the matrix $A(x,E)$ is defined as a parameter, and the parameter space $(\mathscr{E},\mathrm{d})$ is a compact metric space. The Lyapunov exponents $L_1(E)$ and $L_2(E)$ which are concerned in this paper are defined by the Oseledets

Theorem. The purpose of this paper is to study some properties of these two Lyapunov exponents which can be regarded as functions of the parameter E. Well, we proof the following main theorem:

Theorem 1.1. *Let $A(x,E)$ be jointly continuous in $\mathbb{T}\times\mathscr{E}$, analytic uniformly in $E\in\mathscr{E}$ as a function $x\mapsto A(x,E)$, and Hölder continuous with Hölder exponent β as a function $E\mapsto A(x,E)$. Assume $L_1(E_0)-L_2(E_0):=2\tilde{L}(E_0)\neq 0$. Then there exists $\rho>0$ such that for any $E\in(E_0-\rho,E_0+\rho)$, $L_1(E)$ and $L_2(E)$ are Hölder continuous with Hölder exponent α. Moreover, $\alpha=c\beta$, where c is a positive and small constant depending only on $A(x,E)$ but not on $\tilde{L}(E_0)$; and the ρ has an expression as following:*

$$\rho(A,\tilde{L}(E_0))=\Big[\tilde{L}(E_0)\exp\Big(-\frac{C_{\rho,1}}{\tilde{L}^{C_{\rho,2}}(E_0)}\Big)\Big]^{1/\beta},$$

where $C_{\rho,1},C_{\rho,2}$ are the big constants depending only on $A(x,E)$.

Remark 1.2. The parameter space $(\mathscr{E},\mathrm{d})$ here is always being the spectral spaces of operators, such like the Schrödinger equation [2], the extended Happer's model [4] and the Jacobi operators [8]. It is well known that the cocycle is uniform hyperbolic when the energy E is not in the spectral. Thus, we assume that this parameter space is compact, as the discrete operators' spectrums are always bounded.

The study of the Hölder exponent of the Lyapunov exponents is a hot spot in our field. For the almost Mathieu operator with the cocycle

$$A(x,E)=\begin{pmatrix}\lambda\cos x-E & -1\\ 1 & 0\end{pmatrix},$$

Avila and Jitomirskaya proved that the Lyapunov exponent defined by (1.2) is Hölder continuous with Hölder exponent 1/2, provided ω is a Diophantine number [1]. For the general Schrödinger operator with the cocycle

$$A(x,E)=\begin{pmatrix}v(x)-E & -1\\ 1 & 0\end{pmatrix},$$

supposing $v(x)$ is a small perturbation of a trigonometric polynomial $v_0(x)$ of degree k_0 and $\tilde{L}(E)>0$, Goldstein and Schlag proved that Lyapunov exponent is Hölder continuous with Hölder exponent $\frac{1}{2k_0}-\kappa$ for any $\kappa>0$ [3]. In this article, we study the continuity of the Lyapunov exponent from a new perspective. We want to show the relationship between the continuity of the function $E\to A(x,E)$ and the continuity of the Lyapunov exponents. [10] showed that the assumption that $E\mapsto A(x,E)$ is analytic, is necessary. Here only consider the condition that $A(x,E)$ is a 2×2 matrix, but we believe that the main theorem also work if $A(x,E)\in M_d(\mathbb{C})$.

This article is organized as follows. In section 2 we get the Large Deviation Theorem and some upper bound estimate by using some propositions of the subharmonic functions. In section 3 we apply Avalanche principle twice to prove the sharp large deviation theorem. The proof of the main theorem is presented in section 4.

Some other common senses about analytic functions which will be applied in this article are presented as follows. Because $\mathscr{E}$ is compact, there exist $D(A)$, $C_2(A)$ and $C_{\max}(A)$, such that for any $E\in\mathscr{E}$,

$$\|\log|\det A(\cdot,E)|\|_1\le D(A),$$

$$\|\log\|A(\cdot,E)\|-\frac{1}{2}\log|\det A(\cdot,E)|\|_2=C_2(A,E)\le C_2(A)$$

and for any $x \in \mathbb{T}$ and any $E \in \mathscr{E}$,

$$\|A(x,E)\| \le C_{\max}(A,E) \le C_{\max}(A).$$

Also, by the suppose that $E \mapsto A(x,E)$ is Hölder continuous in $x \in \mathbb{T}$ with Hölder exponent β, it is easy to see that

$$|D(E) - D(E')| \le C_D(A)|E - E'|^\beta, \quad \forall E, E' \in \mathscr{E},$$

where $D(E) := \int_{\mathbb{T}} \log|\det A(x,E)|dx$.

2. Large deviation theorem and the upper bound estimate

It is obvious that we can define $L_1(E) = \lim_{N\to\infty} \frac{1}{N}\int_{\mathbb{T}} \log\|A_N(x,E)\|dx$. For easy notation, we replace $L_1(E)$ by $L(E)$.

It is also convenient to replace $A_N(x,E)$ by $A_N(e(x),E)$ (with $e(x) = e^{2\pi i x}$), where $A_N(z,E)$ is analytic function in the annulus $\mathscr{A}_\rho = \{z \in \mathbb{C} : 1-\rho < |z| < 1+\rho\}$ uniformly in $E \in \mathscr{E}$.

Set $u_n(z,E) = \frac{1}{n}\log\|A_N(z,E)\|$, $d_N(z,E) = \frac{1}{n}\log|\det A_N(z,E)|$. Sometimes we use $u_N(z)$ or u_N for short, and the same for $d_N(z,E)$. Let $L_{N,r}(E) =< u_N(re(\cdot)) >$, $D_r(E) =< \log|\det A(re(\cdot)) >$. For $r = 1$ we use notations $L_N(E)$ and $D(E)$.

Note that $u_N(z)$ and $d_N(z)$ are subharmonic functions in $\mathscr{A}_\rho$ and then the following Large Deviation Theorem for the subharmonic functions applies, provided ω satisfies (1.1).

Theorem 2.1 ([8, Theorem 2.15, Remark 2.16]). *There exists $\check{N}(A,\omega)$ such that for any $N \ge \check{N}$, any $1 - \frac{\rho}{2} \le r \le 1 + \frac{\rho}{2}$ and $\delta < 1$ holds*

$$\text{meas}(\{x : |u_N(re(x)) - L_r| > \delta\}) < \exp(-\check{c}\delta^2 N),$$
$$\text{meas}(\{x : |d_N(re(x)) - D_r| > \delta\}) < \exp(-\check{c}\delta^2 N),$$

where $\check{c} = \check{c}(A)$.

Remark 2.2. (1) For the avalanche principle in the next section, we need to define the unimodular matrix of $A_N(z,E)$:

$$\tilde{A}_N(z,E) := \frac{1}{|\det A_N(z,E)|^{1/2}} A_N(z,E), \tag{2.1}$$

and the unimodular function of $u_N(z,E)$

$$\tilde{u}_N(z,E) := \frac{1}{N}\log\|\tilde{A}_N(z,E)\| = u_N(z,E) - \frac{1}{2N}\log|\det(A_N(z,E))|.$$

By Theorem 2.1, for any $N \ge \check{N}$

$$\text{meas}(\{x : |\tilde{u}_N(re(x)) - \tilde{L}_{N,r}(E)| > \delta\}) < \exp(-\tilde{c}\delta^2 N), \tag{2.2}$$

where $\tilde{L}_{N,r}(E) =< \tilde{u}_N(re(\cdot)) >= L_{N,r} - \frac{D_r}{2}$.

(2) In the proof of Theorem 2.1, the following lemma will be necessary and apply in the later of this paper:

Lemma 2.3 ([8, Lemma 2.12]). *For any $1 - \frac{\rho}{2} \le r \le 1 + \frac{\rho}{2}$, δ and K,*

$$\text{meas}\Big\{x : |\sum_{k=1}^{K} u_N(re(x+k\omega)) - K\tilde{L}_{N,r}| > \delta K\Big\} < \exp(-c\delta K),$$

where $c = c(A,\omega)$.

(3) let us note that the constants $c, \tilde{c}$ here do not depend on δ. In particular, one can choose here δ depending on N.

In the next part of this section, we estimate the upper bound of the subharmonic function $u_N(z, E)$ and some other functions. Now we first review a lemma from [3].

Lemma 2.4 ([3, Lemma 4.1])**.** *For any r_1, r_2 so that $1 - \frac{\rho}{2} \le r_1, r_2 \le 1 + \frac{\rho}{2}$ one has*
$$u_N(re(\cdot), E) > -L_N(E)| \le C_\rho |r - 1|.$$

Lemma 2.5. *For any $N \ge \check{N}(A)$*
$$\frac{1}{N} \log \|A_N(e(x), E)\| \le L_N(E) + C_6\big(\frac{\log N}{N}\big)^{1/2},$$
where $C_6 = C_6(A)$ and $\check{N}$ is as in Theorem 2.1.

Proof. Let $0 < \delta < \frac{\rho}{4}$ be arbitrary. Note that $e(x + iy) = e^{-2\pi y} e(x)$, $1 - \frac{\rho}{4} \le e^{-2\pi y} \le 1 + \frac{\rho}{4}$, if $|y| \le \frac{\delta}{4\pi e C'_\rho}$, where $C'_\rho = \max(1, C_\rho)$ and $e = \exp(1)$. By Lemma 2.4, one has
$$|\langle u_N(re(\cdot))\rangle - L_N(E)| \le \delta, \quad \text{if } |y| \le \frac{\delta}{4\pi e C'_\rho}. \tag{2.3}$$
Set
$$\mathbb{B}_y := \{x : |u_N(e(x + iy)) - L_N(E)| > 2\delta\}.$$
It follows from (2.3) that for $|y| \le \frac{\delta}{4\pi e C'_\rho}$, there holds
$$\mathbb{B}_y \subseteq \{x : |u_N(e(x + iy)) - < u_N(e(\cdot + iy)) > | > \delta\}.$$
By Theorem 2.1 one obtains meas $\mathbb{B}_y \le \exp(-\check{c}\delta^2 N)$. The function $u_N(e(x+iy))$ is subharmonic, for $e(x+iy) \in \mathscr{A}_\rho$. Let x_0 be arbitrary and $y_0 = 0$. Then $e(x_0) \in \mathscr{A}_{\frac{\rho}{4}}$. Due to subharmonicity one has for any $t_0 < \frac{\rho}{4}$,
$$\begin{aligned} u_N(e(x_0)) - L_N(E) &\le \frac{1}{\pi t_0^2} \iint_{|(x,y)-(x_0,0)| \le t_0} [u_N(e(x + iy)) - L_N]\, dx\, dy \\ &= \frac{1}{\pi t_0^2} \int_{|y| \le t_0} \int_{|x - x_0| \le \sqrt{t_0^2 - |y|^2}} [u_N(e(x + iy)) - L_N]\, dx\, dy. \end{aligned}$$
Furthermore
$$\begin{aligned} &\int_{|x-x_0| \le \sqrt{t_0^2 - |y|^2}} [u_N(e(x + iy)) - L_N] dx \\ &= \Big(\int_{\{|x-x_0| \le \sqrt{t_0^2 - |y|^2}\} \bigcap \mathbb{B}_y} + \int_{\{|x-x_0| \le \sqrt{t_0^2 - |y|^2}\} \setminus \mathbb{B}_y}\Big)[u_N(e(x + iy)) - L_N] dx. \end{aligned}$$
Note that
$$|u_N(e(x + iy)) - L_N| \le 2\delta, \quad \text{if } x \notin \mathbb{B}_y \text{ and } y < \frac{\delta}{4e\pi C'_\rho}.$$
So
$$\Big|\int_{\{|x-x_0| \le \sqrt{t_0^2 - |y|^2}\} \setminus \mathbb{B}_y} [u_N(e(x + iy)) - L_N] dx\Big| \le 2\delta \times (2\sqrt{t_0^2 - |y|^2}).$$
By Cauchy-Schwartz inequality,
$$\Big|\int_{\{|x-x_0| \le \sqrt{t_0^2 - |y|^2}\} \bigcap \mathbb{B}_y} [u_N(e(x + iy)) - L_N] dx\Big|$$

$$
\begin{aligned}
&\le \Big(\int_0^1 |u_N(e(x+iy)) - L_N|^2 dx\Big)^{1/2} (\operatorname{meas} \mathbb{B}_y)^{1/2} \\
&\le C_7 \exp(-\frac{\check{c}}{2}\delta^2 N).
\end{aligned}
$$

Set $t_0 = \delta/(4e\pi C'_\rho)$; then

$$
\begin{aligned}
u_N(e(x)) - L_N &\le \frac{1}{\pi t_0^2} \int_{|y|\le t_0} [C_7 \exp(-\frac{\check{c}}{2}\delta^2 N) + 2\delta \times (2\sqrt{t_0^2 - |y|^2})] dy \\
&\le \frac{1}{\pi t_0^2} \times C_7 \exp(-\frac{\check{c}}{2}\delta^2 N) \times (2t_0) + 2\delta \\
&= \frac{8eC_7 C'_\rho}{\delta} \exp(-\frac{\check{c}}{2}\delta^2 N) + 2\delta.
\end{aligned}
$$

Set $\delta = \big(\frac{C_8 \log N}{N}\big)^{1/2}$, where $C_8 > 2/\check{c}$. Then $\exp(-\frac{\check{c}}{2} C_8 \log N) < \frac{1}{N}$, and

$$
u_N(e(x)) \le L_N + 8eC_7 C'_\rho \times (\frac{N}{C_8 \log N})^{1/2} \frac{1}{N} + 2(\frac{C_8 \log N}{N})^{1/2} \le L_N + C_6(\frac{\log N}{N})^{1/2}.
$$

□

Lemma 2.6. *Set*

$$
F_N(x, E) := \frac{1}{2N} \sum_{n=0}^{N-1} \big(\log|\det A(e(x+n\omega), E)| - D(E)\big).
$$

Then

(1) *For any N and any k holds*

$$
\begin{aligned}
&|\log \|\tilde{A}_N(e(x+k\omega), E)\| - \log \|\tilde{A}_N(e(x), E)\| | \\
&\le 2k(\log C_{\max}(A, E) - \frac{1}{2}D(E)) - kF_k(x) - kF_k(x + N\omega),
\end{aligned}
$$

(2) *For any N and any $k \ge \check{N}$ holds*

$$
\begin{aligned}
&|\log \|\tilde{A}_N(e(x+k\omega), E)\| - \log \|\tilde{A}_N(e(x), E)\| | \\
&\le 2k\tilde{L}_k(E) + 2C_6(k \log k)^{1/2} - kF_k(x) - kF_k(x + N\omega).
\end{aligned}
$$

Proof. (1) Recall that any N, $u_N(e(x), E) \le \log C_{\max}(A, E)$. Then

$$
\begin{aligned}
&\log \|\tilde{A}_N(e(x), E)\| \\
&= N u_N(e(x), E) - \frac{1}{2} \sum_{n=0}^{N-1} \log|\det A(e(x+n\omega), E)| \\
&\le N \log C_{\max}(A, E) - \frac{1}{2} \sum_{n=0}^{N-1} \log|\det A(e(x+n\omega), E)| \\
&= N(\log C_{\max}(A, E) - \frac{1}{2}D(E)) - \frac{1}{2} \sum_{n=0}^{N-1} (\log|\det A(e(x+n\omega), E)| - D(E)) \\
&= N(\log C_{\max}(A, E) - \frac{1}{2}D(E)) - N F_N(x, E)
\end{aligned}
$$

One has if $\det M = 1$, then $\|M\| = \|M^{-1}\|$ and

$$
\tilde{A}_N(e(x+k\omega), E)\tilde{A}_k(e(x), E) = \tilde{A}_k(e(x+N\omega), E)\tilde{A}_N(e(x), E).
$$

Thus,

$$\begin{aligned} &|\log\|\tilde{A}_N(e(x+k\omega),E)\| - \log\|\tilde{A}_N(e(x))\||\\ &\le \log\|\tilde{A}_k(e(x),E)\| + \log\|\tilde{A}_k(e(x+N\omega),E)\|\\ &\le 2k(\log C_{\max}(A,E) - \frac{1}{2}D(E)) - kF_k(x,E) - kF_k(x+N\omega,E) \end{aligned}$$

(2) From the previous lemma, we know that for any $N \ge \check{N}(A)$,

$$\frac{1}{N}\log\|A_N(e(x),E)\| \le L_N(E) + C_6(\frac{\log N}{N})^{1/2}.$$

Then the rest follows as in part (1). □

3. Avalanche principle and the sharp large deviation theorem

For the rest of the paper, without special statement, $N \ge \check{N}$ and $N \ge \check{K}$ from now on (δ in $\check{K}$ will be defined in Lemma 3.6). Furthermore, we do not use $e(x+iy)$ with $y \ne 0$ and write x instead of $e(x)$ in all expressions.

Proposition 3.1. *Let $A_1,\dots,A_n$ be a sequence of 2×2-matrices whose determinants satisfy $\max_{1\le j\le n}|\det A_j| \le 1$. Suppose that*

$$\min_{1\le j\le n}\|A_j\| \ge \mu > n,\ \max_{1\le j<n}[\log\|A_{j+1}\| + \log\|A_j\| - \log\|A_{j+1}A_j\|] < \frac{1}{2}\log\mu$$

Then

$$\Big|\log\|A_n\cdot\dots\cdot A_1\| + \sum_{j=2}^{n-1}\log\|A_j\| - \sum_{j=1}^{n-1}\log\|A_{j+1}A_j\|\Big| < C\frac{n}{\mu} \tag{3.1}$$

with some constant C.

Proof. This lemma is called the Avalanche Principle. For the proof, see [2]. Recently, Schlag [7] gave a general Avalanche Principle for $n\times n$ matrix. □

Lemma 3.2. *Let $\tilde{c}$ be as in (2.2). Let $\tilde{L}_N(E) > 100\delta > 0$, where $\delta < 1$ is a constant not depending on N, and $\tilde{L}_{2N}(E) > \frac{9}{10}\tilde{L}_N(E)$. Let $N' = mN$, $m \in \mathbb{N}$ and $m \le \exp(\frac{\tilde{c}}{4}\delta^2 N)$. Then*

$$|\tilde{L}_{N'}(E) + \tilde{L}_N(E) - 2\tilde{L}_{2N}(E)| \le \exp(-\tilde{c}'\delta^2 N) + \frac{2}{9m}L_N(E),$$

where $\tilde{c}' = \tilde{c}'(A)$. If $\exp(\frac{\tilde{c}}{10}\delta^2 N) \le m \le \exp(\frac{\tilde{c}}{4}\delta^2 N)$, we have

$$|\tilde{L}_{N'}(E) + \tilde{L}_N(E) - 2\tilde{L}_{2N}(E)| \le \exp(-\hat{c}\delta^2 N),$$

where $\hat{c} = \hat{c}(A)$. Furthermore, if $\tilde{L}_{N_0}(E) > 100\delta > 0$, $\tilde{L}_{2N_0}(E) > \frac{9}{10}\tilde{L}_{N_0}(E)$ and $\exp(-\hat{c}\delta^2 N_0) \le \delta/12$, then there exists $\tilde{N}_0 = \tilde{N}_0(A,\delta,N_0) \le (\exp(\frac{\tilde{c}}{8}\delta^2 N_0)+1)N_0$ such that for any $N \ge \tilde{N}_0$,

$$|\tilde{L}_N(E) + \tilde{L}_{N_0}(E) - 2\tilde{L}_{2N_0}(E)| < \exp(-\bar{c}'\delta^2 N_0),$$

where $\bar{c}' = \bar{c}'(A)$. Furthermore,

$$|\tilde{L}(E) + \tilde{L}_{N_0}(E) - 2\tilde{L}_{2N_0}(E)| < \exp(-\bar{c}\delta^2 N_0),$$

where $\bar{c} = \bar{c}(A)$.

Proof. In section three in [8], we proved this lemma for Jacobi operator, which is a special 2×2 analytic matrix. It is easy to see that the proof there is suitable in this general condition. $\square$

Lemma 3.3. *Assume $\tilde{L}(E_0) > 0$. There exists $\check{C}(A, \tilde{L}(E_0))$ such that with $\rho_0' = \big[\frac{\tilde{L}(E_0)}{200N} \exp\big(-\check{C}(A, \tilde{L}(E_0))N\big)\big]^{1/\beta}$, one has*

$$|\tilde{L}_N(E_0) - \tilde{L}_N(E)| < \frac{\tilde{L}(E_0)}{100},$$

for any $|E - E_0| < \rho_0'(E_0, N)$ and any N.

Proof. Note that

$$\begin{aligned}
&|\, \|A_N(x, E_0)\| - \|A_N(x, E)\| \,| \\
&\le \|A_N(x, E_0) - A_N(x, E)\| \\
&\le \sum_{j=0}^{N-1} \big(\|A(x + (N-1)\omega, E_0) \dots A(x + (j+1)\omega, E_0)\| \\
&\quad \times \|A(x + j\omega, E_0) - A(x + j\omega, E)\| \, \|A(x + (j-1)\omega, E) \dots A(x, E)\| \big) \\
&\le N C_{\max}(A)^{N-1} \times C(A) |E_0 - E|^{\beta}.
\end{aligned} \tag{3.2}$$

By (2.1), if $|\det A_N(x, E_0)| \le |\det A_N(x, E)|$, one has

$$\begin{aligned}
|\, \|\tilde{A}_N(x, E_0)\| - \|\tilde{A}_N(x, E)\| \,| &= \Big| \frac{\|A_N(x, E_0)\|}{|\det A_N(x, E_0)|^{1/2}} - \frac{\|A_N(x, E)\|}{|\det A_N(x, E)|^{1/2}} \Big| \\
&\le \frac{N C_{\max}(A)^{N-1} C(A) |E_0 - E|^{\beta}}{|\det A_N(x, E_0)|^{1/2}}
\end{aligned} \tag{3.3}$$

Assume for instance that $\|\tilde{A}_N(x, E_0)\| \ge \|\tilde{A}_N(x, E)\|$. Then

$$\begin{aligned}
&|\log \|\tilde{A}_N(x, E_0)\| - \log \|\tilde{A}_N(x, E)\| \,| \\
&= \log \frac{\|\tilde{A}_N(x, E_0)\|}{\|\tilde{A}_N(x, E)\|} = \log\Big(1 + \frac{\|\tilde{A}_N(x, E_0)\| - \|\tilde{A}_N(x, E)\|}{\|\tilde{A}_N(x, E)\|}\Big) \\
&\le \frac{\|\tilde{A}_N(x, E_0)\| - \|\tilde{A}_N(x, E)\|}{\|\tilde{A}_N(x, E)\|} \le \|\tilde{A}_N(x, E_0)\| - \|\tilde{A}_N(x, E)\| \\
&\le \frac{N C_{\max}(A)^{N-1} C(A) |E_0 - E|^{\beta}}{|\det A_N(x, E_0)|^{1/2}}.
\end{aligned} \tag{3.4}$$

By Theorem 2.1, for any δ and any K

$$\operatorname{meas}\Big\{x : \Big|\sum_{k=1}^{K} \log|\det A(x + k\omega)| - K\langle \log|\det A(\cdot)| \rangle\Big| > \delta K\Big\} < \exp(-c'\delta K).$$

Thus if $x \notin \mathbb{B}_1$, $\operatorname{meas} \mathbb{B}_1 < \exp(-c' \times \frac{800 C_2(A)}{\tilde{L}(E_0) c'} N) = \exp(-\frac{800 C_2(A)}{\tilde{L}(E_0)} N)$, then

$$\begin{aligned}
|\log|\det A_N(x, E_0)|\,| &< |\langle \log|\det A(\cdot, E_0)| \rangle| N + \frac{800 C_2(A)}{\tilde{L}(E_0) c'} N \\
&= |D(E_0)| N + \frac{800 C_2(A)}{\tilde{L}(E_0) c'} N
\end{aligned}$$

$$\le D(A)N + \frac{800C_2(A)}{\tilde{L}(E_0)c'}N = \hat{C}(A, \tilde{L}(E_0))N. \tag{3.5}$$

It is obvious that the same estimate holds in the other three conditions.

So if $x \notin \mathbb{B}_1$, $\operatorname{meas}\mathbb{B}_1 < \exp(-\frac{800\tilde{C}(A)^{1/2}}{\tilde{L}(E_0)}N)$, then

$$\begin{aligned} |\tilde{u}_N(x, E_0) - \tilde{u}_N(x, E)| &\le C_{\max}(A)^{N-1}C(A)|E_0 - E|^\beta \exp(\hat{C}(A, \tilde{L}(E_0))N) \\ &\le \exp\left(\check{C}(A, \tilde{L}(E_0))N\right)|E_0 - E|^\beta. \end{aligned}$$

Set $\rho_0' = \left[\frac{\tilde{L}(E_0)}{200}\exp\left(-\check{C}(A, \tilde{L}(E_0))N\right)\right]^{1/\beta}$. Then, if $|E - E_0| \le \rho_0'$, we have $|\tilde{u}_N(x, E_0) - \tilde{u}_N(x, E)| < \frac{\tilde{L}(E_0)}{200}$, and if $x \notin \mathbb{B}_1, \operatorname{meas}\mathbb{B}_1 < \exp(-\frac{800\tilde{C}(p,q)^{1/2}}{\tilde{L}(E_0)}N)$, we also have

$$\Big|\int_{\mathbb{T}\setminus\mathbb{B}_1} \tilde{u}_N(x, E_0)dx - \int_{\mathbb{T}\setminus\mathbb{B}_1} \tilde{u}_N(x, E)dx\Big| < \frac{\tilde{L}(E_0)}{200}. \tag{3.6}$$

By the Cauchy-Schwartz inequality,

$$\begin{aligned} \Big|\int_{\mathbb{B}_1} \tilde{u}_N dx\Big| &= \Big|\int_{\mathbb{T}} \tilde{u}_N 1_{\mathbb{B}_1} dx\Big| \\ &\le \|\tilde{u}_N(E)\|_2 (\operatorname{meas}\mathbb{B}_1)^{1/2} \\ &\le C_2(A)\exp(-\frac{400C_2(A)}{\tilde{L}(E_0)}N) \end{aligned}$$

for E or E_0. As $y\exp(-\xi y) \le \xi^{-1}$ for any $y, \xi > 0$. Thus

$$\Big|\int_{\mathbb{B}_1} \tilde{u}_N dx\Big| \le \frac{\tilde{L}(E_0)}{400N} \le \frac{\tilde{L}(E_0)}{400} \tag{3.7}$$

for E or E_0. Combining (3.6) with (3.7), one has

$$|\tilde{L}_N(E_0) - \tilde{L}_N(E)| < \frac{\tilde{L}(E_0)}{200} + 2\frac{\tilde{L}(E_0)}{400} = \frac{\tilde{L}(E_0)}{100}.$$

□

Lemma 3.4. *Assume $\tilde{L}(E_0) > 0$. There exists $\rho = \rho(A, \tilde{L}(E_0)) > 0$ and $\tilde{N}_0 = \tilde{N}_0(A, E_0) < +\infty$ such that for any $N \ge \tilde{N}_0$ and any $|E - E_0| < \rho$*

$$|\tilde{L}_N(E) - \tilde{L}(E)| < \frac{1}{20}\tilde{L}(E), \quad \frac{11}{10}\tilde{L}(E_0) > \tilde{L}(E) > \frac{9}{10}\tilde{L}(E_0).$$

Proof. One has $\lim_{n\to\infty} \tilde{L}(E_0) = \tilde{L}(E_0)$. Therefore, there exists $N_0 = N_0(A, E_0)$ such that the following statements hold:

(1) $|\tilde{L}_n(E_0) - \tilde{L}(E_0)| < \frac{\tilde{L}(E_0)}{100}$ for $n \ge N_0(A, E_0)$, which implies that $\tilde{L}_{N_0}(E_0) - \tilde{L}_{2N_0}(E_0) < \frac{\tilde{L}(E_0)}{100}$, as $\tilde{L}(E_0) \le \tilde{L}_{2N_0}(E_0) \le \tilde{L}_{N_0}(E_0)$;

(2) $\exp(-\hat{c}\delta^2 N_0) \le \frac{\delta}{12}$, $\exp(-\bar{c}\delta^2 N_0)) < \frac{1}{50}\tilde{L}(E_0)$, $\exp(-\bar{c}'\delta^2 N_0)) < \frac{1}{50}\tilde{L}(E_0)$ with $\delta = \min(\frac{1}{200}\tilde{L}(E_0), \frac{1}{2})$, where $\hat{c}$, $\bar{c}$ and $\bar{c}'$ are as Lemma 3.2.

Using Lemma 3.3 applied to N_0 and $2N_0$. One has for $|E - E_0| < \rho(A, \tilde{L}(E_0)) := \rho_0'(\tilde{L}(E_0), 2N_0)$,

$$\begin{aligned} \tilde{L}_{N_0}(E) &\ge \tilde{L}(E_0) - |\tilde{L}_{N_0}(E) - \tilde{L}_{N_0}(E_0)| - |\tilde{L}_{N_0}(E_0) - \tilde{L}(E_0)| \\ &> \tilde{L}(E_0) - \frac{\tilde{L}(E_0)}{100} - \frac{\tilde{L}(E_0)}{100} = \frac{49}{50}\tilde{L}(E_0), \end{aligned} \tag{3.8}$$

and

$$\begin{aligned} &|\tilde{L}_{N_0}(E) - \tilde{L}_{2N_0}(E)| \\ &\le |\tilde{L}_{N_0}(E) - \tilde{L}_{N_0}(E_0)| + |\tilde{L}_{N_0}(E_0) - \tilde{L}_{2N_0}(E_0)| + |\tilde{L}_{2N_0}(E_0) - \tilde{L}_{2N_0}(E)| \\ &< \frac{\tilde{L}(E_0)}{100} + \frac{\tilde{L}(E_0)}{100} + \frac{\tilde{L}(E_0)}{100} = \frac{3}{100}\tilde{L}(E_0) < \frac{1}{10}\tilde{L}_{N_0}(E). \end{aligned} \tag{3.9}$$

Thus Lemma 3.2 applies with $\tilde{L}_{N_0}(E)$, δ, N_0 and E. Then there exists a number $\tilde{N}_0 = \tilde{N}_0(A, \delta, N_0) \le (\exp(\frac{\bar{c}}{8}\delta^2 N_0) + 1)N_0$ such that for any $N \ge \tilde{N}_0$ there holds

$$\begin{aligned} |\tilde{L}_N(E) + \tilde{L}_{N_0}(E) - 2\tilde{L}_{2N_0}(E)| &< \exp(-\bar{c}'\delta^2 N_0), \\ |\tilde{L}(E) + \tilde{L}_{N_0}(E) - 2\tilde{L}_{2N_0}(E)| &< \exp(-\bar{c}\delta^2 N_0), \end{aligned} \tag{3.10}$$

where $\bar{c}' = \bar{c}'(A)$ and $\bar{c} = \bar{c}(A)$ are as in Lemma 3.2. These imply

$$\begin{aligned} |\tilde{L}(E) - \tilde{L}_N(E)| &\le \exp(-\bar{c}'\delta^2 N_0) + \exp(-\bar{c}\delta^2 N_0) \\ &< \frac{1}{50}\tilde{L}(E_0) + \frac{1}{50}\tilde{L}(E_0) = \frac{1}{25}\tilde{L}(E_0). \end{aligned} \tag{3.11}$$

Combining (3.8), (3.9) with (3.10), one obtains

$$\begin{aligned} &|\tilde{L}(E_0) - \tilde{L}(E)| \\ &\le |\tilde{L}(E) + \tilde{L}_{\tilde{N}_0}(E) - 2\tilde{L}_{2\tilde{N}_0}(E)| + |\tilde{L}(E_0) - \tilde{L}_{\tilde{N}_0}(E)| + 2|\tilde{L}_{\tilde{N}_0}(E) - \tilde{L}_{2\tilde{N}_0}(E)| \\ &< \frac{1}{50}\tilde{L}(E_0) + \frac{1}{50}\tilde{L}(E_0) + 2\frac{3}{100}\tilde{L}(E_0) = \frac{1}{10}\tilde{L}(E_0). \end{aligned}$$

It implies

$$\begin{gathered} \frac{11}{10}\tilde{L}(E_0) > \tilde{L}(E) > \frac{9}{10}\tilde{L}(E_0), \\ |\tilde{L}(E) - \tilde{L}_N(E)| < \frac{1}{25}\tilde{L}(E_0) < \frac{1}{25}(\frac{10}{9})\tilde{L}(E) = \frac{2}{45}\tilde{L}(E) < \frac{1}{20}\tilde{L}(E). \end{gathered}$$

□

Remark 3.5. From (3.5), $\hat{C}(A, \tilde{L}(E_0)) = D(A) + \frac{800\tilde{C}(A)^{1/2}}{\tilde{L}(E_0)c'}$. It is obvious that $\hat{C}(A, \tilde{L}(E_0))$ is a positive constant, which is smaller when $\tilde{L}(E_0)$ becomes larger. Thus $\check{C}(A, \tilde{L}(E_0)) = \hat{C}(A, \tilde{L}(E_0)) + \log C(A)$ is a positive and monotonically increasing function of $\tilde{L}(E_0)$. Recall that

$$0 < \rho(A, \tilde{L}(E_0)) = \big[\frac{\tilde{L}(E_0)}{200}\exp\big(-2\check{C}(A, \tilde{L}(E_0))N_0\big)\big]^{1/\beta} < 1. \tag{3.12}$$

N_0 depends on the rate of convergence of $\tilde{L}_n(E_0) \to \tilde{L}(E_0)$ (see the proof of Lemma 3.4). On the other hand, in Lemma 3.2, let $m = \exp(\frac{\bar{c}}{8}\delta^2 N)$, $N' = mN$, then

$$|\tilde{L}_{N'} - \tilde{L}_{2N'}| < 2\exp(-\hat{c}\delta^2 N) < \big(\frac{1}{N'}\big)^{C_1(A)}.$$

Like the induction in the proof of Lemma 3.2, one gets

$$|\tilde{L}_N - \tilde{L}| < \big(\frac{1}{N}\big)^{C_2(A)}. \tag{3.13}$$

Actually, in [7], the better estimate holds as

$$|\tilde{L}_N - \tilde{L}| < \frac{C}{N},$$

with the large constant C depending on $\tilde{L}$. Finally, combining (3.12), (3.13) with some assumptions in Lemma 3.2 and Lemma 3.4, we have

$$\rho(A, \tilde{L}(E_0)) \simeq \Big[\tilde{L}(E_0)\exp\Big(-\frac{C_{\rho,1}}{(\tilde{L}(E_0))^{C_{\rho,2}}}\Big)\Big]^{1/\beta},$$

where $C_{\rho,1}, C_{\rho,2}$ are the big constants depending only on $A(x,E)$.

The first part of this section shows that the uniform property that for any $E \in (E_0-\rho, E_0+\rho)$ and for the uniformly large N, the distance between $\tilde{L}_N(E)$ and $\tilde{L}(E)$ is controlled by a uniform constant $\tilde{L}(E)$. This property is very important and will be applied repeatedly in the following part of this paper, such like the sharp Large Deviation Theorem (Lemma 3.6) and the proof of the main theorem (Section 4).

Lemma 3.6. *Assume $\tilde{L}(E_0) > 0$. There exist $N_1(A, E_0)$ such that for any $N \ge N_1$ and any $E \in (E_0-\rho, E_0+\rho)$ holds*

$$\operatorname{meas}\{x : |\tilde{u}_N(x,E) - \tilde{L}(E)| > \frac{\tilde{L}(E)}{10}\} < \exp(-c\tilde{L}(E)N),$$

where constant c depends only on A, but does not depend on E or E_0.

Proof. Choose $\bar{N}_0$ such that

$$\begin{gathered} \bar{N}_0 > \max\Big(\frac{10^4 C_6}{\tilde{L}(E_0)}, \max_{E\in\mathscr{E}} \frac{10^6(\log C_{\max}(A,E) - \frac{1}{2}D(E))}{\tilde{L}(E_0)}, 40, \tilde{N}_0\Big), \\ \log \bar{N}_0 < \bar{N}_0^{1/3} \end{gathered} \tag{3.14}$$

where $(\log C_{\max}(A,E) - D(E))$ is as in Lemma 2.6, C_6 is as in Lemma 2.5, $\tilde{N}_0$ is as in Lemma 3.4.

Take $N \ge \bar{N}_0^3$, $K := \frac{1}{800}N \ge \bar{N}_0$. Thus for any $E \in (E_0-\rho, E_0+\rho)$,

$$\tilde{L}_K(E) < (1+\frac{1}{20})\tilde{L}(E). \tag{3.15}$$

Using Lemma 2.6 one obtains

$$\begin{aligned} &|\tilde{u}_N(x,E) - \frac{1}{K}\sum_{k=1}^{K}\tilde{u}_N(x+k\omega,E)| \\ &\le \frac{1}{KN}\Big[\sum_{k=1}^{\bar{N}_0} 2k(\log C_{\max}(A,E) - \frac{1}{2}D(E)) + \sum_{k=\bar{N}_0+1}^{K} 2k\tilde{L}_k(E) \\ &\quad + \sum_{k=\bar{N}_0+1}^{K} 2C_6(k\log k)^{1/2}\Big] - \frac{1}{KN}\sum_{k=1}^{K}(kF_k(x) + kF_k(x+N\omega)) \\ &= I + II. \end{aligned}$$

By (3.14), (3.15) and Lemma 3.4, one has

$$\begin{aligned} I &< \frac{\bar{N}_0^2(\log C_{\max}(A,E) - \frac{1}{2}D(E))}{KN} + 4\frac{21}{20}\tilde{L}(E)\frac{K}{N} + + \frac{C_6K^{1/2}(\log K)^{1/2}}{N} \\ &< \frac{\tilde{L}(E_0)}{320} + \frac{\tilde{L}(E)}{160} + \frac{\tilde{L}(E_0)}{320} = \frac{\tilde{L}(E)}{80}. \end{aligned} \tag{3.16}$$

If $\sum_{k=1}^{K} kF_k(x,E) < -\frac{KN}{160}\tilde{L}(E)$, then there is a k such that $kF_k(x,E) < -\frac{N}{160}\tilde{L}(E)$. We know that

$$\operatorname{meas}\{x : |kF_k(x) - k\langle F_k(x)\rangle| > k\delta\} < \exp(-c\delta k),$$

Since $\langle F_k\rangle = 0$,

$$\begin{aligned}\operatorname{meas}\{x : kF_k(x) < -\frac{N}{160}\tilde{L}(E)\} \\ &\le \operatorname{meas}\{x : |kF_k(x)| > \frac{N}{160}\tilde{L}(E)\} \\ &< \exp(-c\frac{N\tilde{L}(E)}{160k}k) = \exp(-c_2N\tilde{L}(E)).\end{aligned}$$

So

$$\begin{aligned}&\operatorname{meas}\Big\{x : \sum_{k=1}^{K} kF_k(x) < -\frac{KN}{160}\tilde{L}(E)\Big\} \\ &\le K\exp(-c_2N\tilde{L}(E) < \exp(-c_2'\tilde{L}(E)N),\end{aligned} \tag{3.17}$$

if N is large enough depending on $L(E_0)$ (see Lemma 3.4 and (3.14)). Combining (3.16) with (3.17) one has

$$\begin{aligned}&\operatorname{meas}\Big\{x : |\tilde{u}_N(x,E) - \frac{1}{K}\sum_{k=1}^{K}\tilde{u}_N(x+k\omega,E)| > \frac{\tilde{L}(E)}{40}\Big\} \\ &\le 2\exp(-c_2'N\tilde{L}(E)) < \exp(-c''\tilde{L}(E)N).\end{aligned} \tag{3.18}$$

On the other hand, recall that by Lemma 2.3, for any K,

$$\operatorname{meas}\Big\{x : |\sum_{k=1}^{K} u_N(x+k\omega) - K < u_N(\cdot) > | > \delta K\Big\} < \exp(-c\delta K),$$

$$\operatorname{meas}\Big\{x : |\sum_{k=1}^{K}\frac{1}{N}\log|\det A(x+k\omega)| - KD| > \delta K\Big\} < \exp(-c'\delta K).$$

By the definition of $\tilde{u}_N(x,E)$ and $\tilde{L}_N(E)$, there exists $\check{K} = \check{K}(A,E_0)$ such that for any $K > \check{K}$ holds (with $\delta := \frac{\tilde{L}(E)}{40}$)

$$\operatorname{meas}\Big\{x : |\sum_{k=1}^{K}\tilde{u}_N(x+k\omega,E) - K\langle\tilde{u}_N(\cdot,E)\rangle| > \frac{\tilde{L}(E)}{40}K\Big\} \le \exp(-\hat{c}\tilde{L}(E)K).$$

Note that if

$$\Big|\frac{1}{K}\sum_{k=1}^{K}\tilde{u}_N(x+k\omega,E) - \langle\tilde{u}_N(\cdot,E)\rangle\Big| \le \frac{\tilde{L}(E)}{40},$$

then

$$\begin{aligned}&|\frac{1}{K}\sum_{k=1}^{K}\tilde{u}_N(x+k\omega,E) - \tilde{L}(E)| \\ &\le |\frac{1}{K}\sum_{k=1}^{K}\tilde{u}_N(x+k\omega,E) - \tilde{L}_N(E)| + |\tilde{L}_N(E) - \tilde{L}(E)| \\ &< \frac{1}{40}\tilde{L}(E) + \frac{1}{20}\tilde{L}(E) = \frac{3}{40}\tilde{L}(E)\,.\end{aligned}$$

Therefore

$$\operatorname{meas}\Big\{x : |\frac{1}{K}\sum_{k=1}^{K}\tilde{u}_N(x+k\omega,E)-\tilde{L}(E)| > \frac{3}{40}\tilde{L}(E)\Big\} < \exp(-\hat{c}\tilde{L}(E)K).$$

Combining this with (3.18), there exists $N_1 = N_1(A, E_0)$ such that for any $N \geq N_1$ holds

$$\operatorname{meas}\{x : |\tilde{u}_N(x,E)-\tilde{L}(E)| > \frac{\tilde{L}(E)}{10}\} < \exp(-c_0 L(E)N),$$

where c_0 depends only on A. Here we replace c_0 by c for convenient notations. □

Lemma 3.7. *Assume $\tilde{L}(E_0) > 0$. Let $N \geq N_1$ and $E \in (E_0-\rho, E_0+\rho)$ be arbitrary. Then*

$$|\tilde{L}(E)+\tilde{L}_N(E)-2\tilde{L}_{2N}(E)| < \exp(-c\tilde{L}(E)N), \tag{3.19}$$

where $c = c(A)$.

Proof. By the sharp Large Deviation Theorem, Lemma 3.6, this lemma get better conclusion than Lemma 3.2, which comes from Theorem 2.1 and Remark 2.2, but the proof here is the same as the later one. □

4. Proof of the main theorem

Let $\tilde{L}(E_0) > 0$. By Lemma 3.4 and (3.11), one has that for any $N \geq \tilde{N}_0(A, E_0)$ and $E \in (E_0-\rho, E_0+\rho)$,

$$\tilde{L}_N(E) \leq \tilde{L}(E) + \frac{1}{25}\tilde{L}(E_0) < \frac{11}{10}\tilde{L}(E_0) + \frac{1}{25}\tilde{L}(E_0) = \frac{57}{50}\tilde{L}(E_0). \tag{4.1}$$

Let $E' \to E$ such that $|D(E)-D(E')| \leq \frac{1}{5}\tilde{L}(E_0)$. Thus,

$$\begin{aligned} L_N(E) - \frac{1}{2}D(E') &\leq L_N(E) + \frac{1}{10}\tilde{L}(E_0) - \frac{1}{2}D(E) \\ &= \tilde{L}_N(E) + \frac{1}{10}\tilde{L}(E_0) \leq \frac{62}{50}\tilde{L}(E_0). \end{aligned} \tag{4.2}$$

Assume $\|\tilde{A}_N(x,E)\| \geq \|\tilde{A}_N(x,E')\|$ and $|\det A_N(x,E)| \leq |\det A_N(x,E)|$. By (3.2), (3.3) and (3.4),

$$\begin{aligned} &|\log\|\tilde{A}_N(x,E)\| - \log\|\tilde{A}_N(x,E')\|\,| \\ &\leq \|\tilde{A}_N(x,E)\| - \|\tilde{A}_N(x,E')\| = \frac{\|A_N(x,E)\|}{|\det A_N(x,E)|^{1/2}} - \frac{\|A_N(x,E')\|}{|\det A_N(x,E')|^{1/2}} \\ &\leq \frac{\|A_N(x,E)\| - \|A_N(x,E')\|}{|\det A_N(x,E)|^{1/2}} \leq \frac{\|A_N(x,E) - A_N(x,E')\|}{|\det A_N(x,E)|^{1/2}} \\ &\leq C(A)|E-E'|^{\beta}\frac{\sum_{j=0}^{N-1}\|\prod_{m=1}^{N-j}A(x+(N-m)\omega,E)\|\,\|\prod_{m=j-1}^{0}A(x+m\omega,E')\|}{|\det A_N(x,E)|^{1/2}} \\ &= C(A)|E-E'|^{\beta}\sum_{j=0}^{N-1}\Big(\|\prod_{m=1}^{N-j}A(x+(N-m)\omega,E)\|\,\|\prod_{m=j-1}^{0}A(x+m\omega,E')\|\Big) \\ &\quad\times\exp(-\frac{N}{2}D(E))\exp(-NF_N(x,E)); \end{aligned} \tag{4.3}$$

see Lemma 2.6 for the definition of $F_N(x)$.

Let $N \geq N_2$, where $N_2(A) := \max_{E_1,E_2\in\mathscr{E}} \frac{2\log C_{\max}(A,E_1)-D(E_2)}{L(E_0)}\tilde{N}_0 \geq 2\tilde{N}_0$. Then

$$
\begin{aligned}
(4.3) \leq &\Big(\sum_{j=1}^{\tilde{N}_0} + \sum_{j=\tilde{N}_0+1}^{N-\tilde{N}_0} + \sum_{j=N-\tilde{N}_0+1}^{N}\Big)\|\prod_{m=1}^{N-j} A(x+(N-m)\omega, E)\| \\
&\times \|\prod_{m=j-1}^{0} A(x+m\omega, E')\| \exp(-\frac{N}{2}D)\exp(-NF_N(x,E))C(A)|E-E'|^{\beta} \\
&:= I + II + III.
\end{aligned}
$$

By Lemma 2.5 and Lemma 2.6, one has

$$
\begin{aligned}
I, III \leq &\sum_{j=1}^{\tilde{N}_0} \exp\Big(L_{N-j}(E)(N-j) + C_6(N-j)^{1/2}\log^{1/2}(N-j) \\
&+ \log C_{\max}(A,E')(j-1) - \frac{N}{2}D(E)\Big)\exp(-NF_N(x,E)) \times C(A)|E-E'|^{\beta} \\
\leq &\sum_{j=1}^{\tilde{N}_0} \exp\Big(\tilde{L}_{N-j}(E)(N-j) + C_6 N^{1/2}\log^{1/2} N + \big(\log C_{\max}(A,E') \\
&- \frac{1}{2}D(E)\big)(j-1) - \frac{1}{2}D(E)\Big)\exp(-NF_N(x,E)) \times C(A)|E-E'|^{\beta} \\
\leq &\sum_{j=1}^{\tilde{N}_0} \exp\Big(\frac{1}{2}\tilde{L}(E_0)N + \frac{62}{50}\tilde{L}(E_0)N + C_6N^{1/2}\log^{1/2} N - \frac{1}{2}D\Big) \\
&\times \exp(-NF_N(x,E))C(A)|E-E'|^{\beta},
\end{aligned}
$$

and

$$
\begin{aligned}
II \leq &\sum_{j=\tilde{N}_0+1}^{N-\tilde{N}_0} \exp\Big(L_{N-j}(E)(N-j) + C_6(N-j)^{1/2}\log^{1/2}(N-j) + L_{j-1}(E')(j-1) \\
&+ C_6(j-1)^{1/2}\log^{1/2}(j-1) - \frac{N}{2}D(E)\Big)\exp(-NF_N(x,E)) \times C(A)|E-E'|^{\beta} \\
\leq &\sum_{j=N-\tilde{N}_0+1}^{N} \exp\Big(\big(\log C_{\max}(A,E) - \frac{1}{2}D(E)\big)(N-j) + \Big(L_{j-1}(E') \\
&- \frac{1}{2}D(E)\Big)(j-1) + C_6N^{1/2}\log^{1/2} N - \frac{1}{2}D(E)\Big) \\
&\times \exp(-NF_N(x,E)) \times C(A)|E-E'|^{\beta} \\
\leq &\sum_{j=\tilde{N}_0+1}^{N-\tilde{N}_0} \exp\Big(\frac{62}{50}\tilde{L}(E_0)N + C_6N^{1/2}\log^{1/2} N - \frac{1}{2}D\Big) \\
&\times \exp(-NF_N(x,E))C(A)|E-E'|^{\beta}.
\end{aligned}
$$

Combining these expressions, we have

$$
\begin{aligned}
(4.3) \leq &\sum_{j=1}^{N} \exp\Big(\frac{87}{50}\tilde{L}(E_0)N + C_6N^{1/2}\log^{1/2} N - \frac{1}{2}D\Big)\exp(-NF_N(x,E)) \\
&\times C(A)|E-E'|^{\beta}.
\end{aligned}
$$

There exists $N_3 = N_3(A, E_0)$ such that for any $N \geq N_3$ there holds

$$C(A)\sum_{j=1}^{N}\exp\Big(\frac{87}{50}\tilde{L}(E_0)N + C_6N^{1/2}\log^{1/2}N - \frac{1}{2}D\Big) \leq \exp(2\tilde{L}(E_0)N).$$

It is easy to see that there are similar processes for the other three conditions. Thus

$$(4.3) \leq |E-E'|^{\beta}\exp(2\tilde{L}(E_0)N)\max\{\exp(-NF_N(x,E))\exp(-NF_N(x,E'))\}.$$

Set

$$\mathbb{B}(E) := \{x : NF_N(x,E) < -N\tilde{L}(E_0)\},$$

then

$$\begin{aligned}\operatorname{meas}(\mathbb{B}(E)) &\leq \operatorname{meas}(\{x : |NF_N(x,E) - N\langle F_N(\cdot,E)\rangle| > N\tilde{L}(E_0)\})\\ &< \exp(-c\tilde{L}(E_0)N),\end{aligned}$$

since $< F_N(\cdot,E) >= 0$. By the Cauchy-Schwartz inequality, one has

$$\begin{aligned}|\tilde{L}_N(E) - \tilde{L}_N(E')| &= \int_{\mathbb{T}\setminus(\mathbb{B}(E)\bigcup\mathbb{B}(E'))}|\tilde{u}_N(x,E) - \tilde{u}_N(x,E')|dx\\ &\quad + \int_{\mathbb{B}(E)\bigcup\mathbb{B}(E')}|\tilde{u}_N(x,E) - \tilde{u}_N(x,E')|dx\\ &< |E-E'|^{\beta}\exp(3\tilde{L}(E_0)N) + 4C_2(A)\exp(-\frac{c}{2}\tilde{L}(E_0)N).\end{aligned}$$

Let $N \geq N_4 := \max(N_1, N_2, N_3)$, where N_1 is as in Lemma 3.6. By Lemma 3.7,

$$\begin{aligned}&|\tilde{L}(E) - \tilde{L}(E')|\\ &\leq |\tilde{L}(E) + \tilde{L}_N(E) - 2\tilde{L}_{2N}(E)| + |\tilde{L}(E') + \tilde{L}_N(E') - 2\tilde{L}_{2N}(E')|\\ &\quad + |\tilde{L}_N(E) - \tilde{L}_N(E')| + 2|\tilde{L}_{2N}(E) - \tilde{L}_{2N}(E')|\\ &< 2\exp(-c\tilde{L}(E_0)N) + 3|E-E'|^{\beta}\exp(3\tilde{L}(E_0)N) + 12C_2(A)\exp(-\frac{c}{2}\tilde{L}(E_0)N)\\ &< \exp(-c_7\tilde{L}(E_0)N) + 3\exp(3\tilde{L}(E_0)N)|E-E'|^{\beta},\end{aligned}$$

where $c_7 = c_7(A)$. Then when $E' \to E$, there exists $N \geq N_4$ such that

$$\exp\big(-(6+c_7)\tilde{L}(E_0)(N+1)\big) \leq |E-E'|^{\beta} \leq \exp\big(-(6+c_7)\tilde{L}(E_0)N\big).$$

It implies

$$\begin{aligned}|\tilde{L}(E) - \tilde{L}(E')| &< 4\exp(-c_7\tilde{L}(E_0)N)\\ &= 4\exp\big(-\frac{N}{N+1}c_7\tilde{L}(E_0)(N+1)\big) < \exp\big(-\frac{2c_7}{3}\tilde{L}(E_0)(N+1)\big)\\ &= \exp\big(-\frac{2c_7}{18+3c_7}\tilde{L}(E_0)N\big) < |E-E'|^{\frac{2c_7\times\beta}{18+3c_7}}.\end{aligned}$$

By the definition, one also has

$$\begin{aligned}|L_1(E) - L_1(E')|,\ |L_2(E) - L_2(E')| &\leq |\tilde{L}(E) - \tilde{L}(E')| + \frac{1}{2}|D(E) - D(E')|\\ &< |E-E'|^{\frac{2c_7\times\beta}{18+3c_7}} + \frac{C_D(A)}{2}|E-E'|^{\beta}\\ &\leq (1+\frac{C(A)}{2})|E-E'|^{\frac{2c_7\times\beta}{18+3c_7}}.\end{aligned}$$

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