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BLOW-UP CRITERION FOR THE ZERO-DIFFUSIVE BOUSSINESQ EQUATIONS VIA THE VELOCITY COMPONENTS

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Abstract. This article concerns the blow up for the smooth solutions of the three-dimensional Boussinesq equations with zero diffusivity. It is shown that if any two components of the velocity field u satisfy

$$\int_0^T \frac{\||u_1|+|u_2|\|_{L^{p,\infty}}^q}{1+\ln(e+\|\nabla u\|_{L^2}^2)}ds<\infty, \quad \frac{2}{q}+\frac{3}{p}=1, \quad 3<p<\infty,$$

then the local smooth solution (u,θ) can be continuously extended to $(0,T_1)$ for some $T_1>T$.

1. Introduction

Since the famous laboratory experiments on turbulence derived by Reynolds in 1883, the mathematical models which described the motion of the viscous incompressible fluid flow have attracted more and more attention. Those mathematical models are usually controlled by the nonlinear partial differential equations. In this study, we consider a dynamical model of the ocean and atmosphere dynamics [1, 18] which is so-called Boussinesq equations

$$\begin{gathered}\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u + \theta e_3,\\ \operatorname{div} u = 0,\\ \partial_t\theta + u\cdot\nabla\theta = \kappa\Delta\theta,\end{gathered}\tag{1.1}$$

where $u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ and $\theta(x,t)$ are the unknown velocity vector field and the unknown scalar temperature, $p(x,t)$ is the unknown scalar pressure field. $\nu>0, \kappa\geq 0$ are the constants kinematic viscosity and the thermal diffusivity, $e_3=(0,0,1)^T$.

As an important mathematical model in the atmospheric sciences [1], the Boussinesq equations have play an important role in many geophysical applications [18]. When $\theta=0$, the Boussinesq equations (1.1) become the classic Navier-Stokes equations

$$\begin{gathered}\partial_t u + u\cdot\nabla u + \nabla p = \nu\Delta u,\\ \operatorname{div} u = 0.\end{gathered}\tag{1.2}$$

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From the viewpoint of mathematics, the Boussinesq system is the generalization of the Navier-Stokes equations. There is a large body of literature on the existence, uniqueness and regularity of solutions for the Boussinesq equations. In the two-dimensional case, when $\nu, \kappa > 0$, the global existence and uniqueness of smooth solution Boussinesq equations are obtained by Cannon and DiBenedetto [2]. When $\nu = 0, \kappa > 0$ or $\nu > 0, \kappa = 0$, the global regularity of local smooth solution of the Boussinesq equations is also well studied in [3, 4, 12, 16, 21].

In the three-dimensional case, corresponding three-dimensional Navier-Stokes equations [10, 15], the global regularity or finite time singularity of weak solutions for the Boussinesq equations (1.1) with positive dissipation is a big challenging problem. Therefore, it is an important problem to consider the blow-up issue for the three-dimensional Boussinesq equations (1.1) and related fluid dynamical models such as the Navier-Stokes equations and micropolar fluid flows (refer to [7, 8, 9]). Ishimura and Morimoto [13] (see also [19])first proved the Beale-Kato-Majda blow-up criteria of local smooth solution for the Boussinesq equations (1.1). That is to say, if T is the maximal existence time of the local smooth solution for the Boussinesq equations (1.1), then

$$T < \infty \Rightarrow \int_0^T \|\nabla u(s)\|_{L^\infty} ds = +\infty \tag{1.3}$$

When $\kappa = 0$, the diffusive equation in Boussinesq equations(1.1) is reduced to a transport equation

$$\partial_t \theta + u \cdot \nabla \theta = 0,$$

and Boussinesq system (1.1) namely becomes the following parabolic-hyperbolic system (for simplicity taking $\nu = 1$)

$$\begin{gathered} \partial_t u + u \cdot \nabla u + \nabla p = \Delta u + \theta e_3, \\ \operatorname{div} u = 0, \\ \partial_t \theta + u \cdot \nabla \theta = 0 \end{gathered} \tag{1.4}$$

together with the initial data

$$u(x,0) = u_0, \quad \theta(x,0) = \theta_0. \tag{1.5}$$

It should be mentioned that the temperature function $\theta(x,t)$ in the transport equation does not gain smoothness whatsoever. The blow-up issue of the zero-diffusive Boussinesq equations (1.4)-(1.5) is more difficult compared with that of Boussinesq system (1.1) with full viscosities. Fan and Zhou [11] recently studied the blow-up criterion of the local smooth solution of the zero-diffusive Boussinesq equations (1.4)-(1.5) and derived the following Beale-Kato-Majda criterion

$$\int_0^T \|\nabla \times u\|_{\dot{B}^0_{\infty,\infty}(\mathbb{R}^3)} ds < \infty \tag{1.6}$$

Jia, Zhang and Dong [14] further refined the blow-up criterion for local smooth solutions of zero-diffusive Boussinesq equations (1.4)-(1.5) in the large critical Besov space

$$\int_0^T \|u\|^p_{B^s_{q,\infty}(\mathbb{R}^3)} ds < \infty \tag{1.7}$$

with $\frac{2}{p}+\frac{3}{q}=1+s$ and

$$\frac{3}{1+s}<p\leq\infty,\quad -1<s\leq 1,\quad (p,s)\neq(\infty,1).$$

To the author's knowledge, there are a few results on the blow-up criterion for local smooth solution of zero-diffusive Boussinesq equations (1.4)-(1.5) in terms of the components of the velocity. The main purpose of this study is to investigate the blow-up criterion for local smooth solution via horizontal velocity u_1, u_2 in the critical Lorentz spaces. More precisely, we show the following blow-up criterion for local smooth solution of zero-diffusive Boussinesq equations (1.4)-(1.5)

$$\int_0^T \frac{\||u_1|+|u_2|\|_{L^{p,\infty}}^q}{1+\ln(e+\|\nabla u\|_{L^2}^2)}ds<\infty,\quad \frac{2}{q}+\frac{3}{p}=1,\quad 3<p<\infty\,,$$

where $L^{p,\infty}$ is Lorentz space (see the definition in the next section).

2. Preliminaries and main results

In this section, we first recall some basic notation. We denote by C the positive constant which may be different from line to line. We denote by $L^q(\mathbb{R}^3)$ with $1\leq p\leq\infty$ the usual vector or scalar Lebesgue space under the norm

$$\|\varphi\|_{L^p}=\begin{cases}\Big(\int_{\mathbb{R}^3}|\varphi(x)|^p\,dx\Big)^{1/p}, & 1\leq p<\infty,\\ \operatorname{ess\,sup}_{x\in\mathbb{R}^3}|\varphi(x)|, & p=\infty.\end{cases}$$

We also denote by $H^k(\mathbb{R}^3)$ the usual Sobolev space $\{\varphi\in L^2(\mathbb{R}^3);\|\nabla^k\varphi\|_{L^2}<\infty\}$.

We denote by $L^{p,q}(\mathbb{R}^3)$ with $1\leq p,\, q\leq\infty$ the Lorenz space with the norm [20]

$$\|\varphi\|_{L^{p,q}}=\Big(\int_0^\infty t^q(m(\varphi,t))^{q/p}\,\frac{dt}{t}\Big)^{1/q}<\infty\quad\text{for}\ \ 1\leq q<\infty,$$

where $m(\varphi,t)$ is the Lebesgue measure of the set $\{x\in\mathbb{R}^3:|\varphi(x)|>t\}$, *i.e.*

$$m(\varphi,t):=m\{x\in\mathbb{R}^3:|\varphi(x)|>t\}.$$

In particular, when $q=\infty$,

$$\|\varphi\|_{L^{p,\infty}}=\sup_{t\geq 0}\{t(m(\varphi,t))^{\frac{1}{p}}\}<\infty\,.$$

The Lorents space $L^{p,\infty}$ is also called weak L^p space. The norm is equivalent to the norm

$$\|f\|_{L^{q,\infty}}=\sup_{0<|E|<\infty}|E|^{1/q-1}\int_E|f(x)|dx.$$

As stated by Triebel [20], Lorentz space $L^{p,q}(\mathbb{R}^3)$ may be defined by real interpolation methods

$$L^{p,q}(\mathbb{R}^3)=(L^{p_1}(\mathbb{R}^3),\,L^{p_2}(\mathbb{R}^3))_{\alpha,q},\tag{2.1}$$

with

$$\frac{1}{p}=\frac{1-\alpha}{p_1}+\frac{\alpha}{p_2},\quad 1\leq p_1<p<p_2\leq\infty.$$

We now recall some basic inequality which will be used in the next section.

Lemma 2.1 (O'Neil [17]). *Assume* $1 \le p_a,\ p_b \le \infty$, $1 \le q_a,\ q_b \le \infty$ *and* $u \in L^{p_a,q_a}(\mathbb{R}^3)$, $v \in L^{p_b,q_b}(\mathbb{R}^3)$. *Then* $uv \in L^{p_c,q_c}(\mathbb{R}^3)$ *with*

$$\frac{1}{p_c} = \frac{1}{p_a} + \frac{1}{p_b}, \quad \frac{1}{q_c} \le \frac{1}{q_a} + \frac{1}{q_b}$$

and the inequality

$$\|uv\|_{L^{p_c,q_c}} \le C\|u\|_{L^{p_a,q_a}}\|v\|_{L^{p_b,q_b}} \tag{2.2}$$

is valid.

Our main results are read as follows.

Theorem 2.2. *Assume* (u,θ) *is the local smooth solution of zero-diffusive Boussinesq equations* (1.4)-(1.5) *satisfying that*

$$(u,\theta) \in C([0,T); H^m(\mathbb{R}^3)), \quad m > 3.$$

If T *is the maximal existence time of the solution* (u,θ), *then for*

$$\frac{2}{q} + \frac{3}{p} = 1, \quad 3 < p < \infty,$$

the following necessary blow-up condition

$$T < \infty \Rightarrow \int_0^T \frac{\||u_1| + |u_2|\|_{L^{p,\infty}}^q}{1 + \ln(e + \|\nabla u\|_{L^2}^2)} ds = +\infty \tag{2.3}$$

holds.

The above theorem obviously implies the following corollary.

Corollary 2.3. *Assume* (u,θ) *is the local smooth solution of zero-diffusive Boussinesq equations* (1.4)-(1.5) *satisfying*

$$(u,\theta) \in C([0,T); H^m(\mathbb{R}^3)), \quad m > 3.$$

If the velocity satisfies

$$\int_0^T \frac{\||u_1| + |u_2|\|_{L^{p,\infty}}^q}{1 + \ln(e + \|\nabla u\|_{L^2}^2)} ds < \infty, \quad \frac{2}{q} + \frac{3}{p} = 1, \quad 3 < p < \infty$$

then the solution (u,θ) *can be continually extended to the interval* $(0,T_1)$ *for some* $T_1 > T$.

Remark 2.4. When $\nu = \kappa = 0$, the existence and uniqueness of local smooth solution (u,θ) for zero-dissipation Boussinesq equations (1.1) have been investigated by Chae and Nam [5], therefore, we only need to prove the blow-up criterion of Theorem 2.2. Moreover, once the proof of Theorem 2.2 is obtained, the proof of Corollary 2.1 follows directly from Theorem 2.2 and we omit it here.

3. Proof of Theorem 2.2

3.1. L^p **estimate for** θ. Multiplying both sides of the transport equation of zero-diffusive Boussinesq equations (1.4)-(1.5) by $|\theta|^{p-2}\theta$ and integrating in $\mathbb{R}^3$, we have

$$\frac{d}{dt}\int_{\mathbb{R}^3} |\theta|^p\, dx = 0, \quad p \ge 2 \tag{3.1}$$

where we have used

$$\int_{\mathbb{R}^3} u \cdot \nabla\theta\theta dx = 0.$$

Integrating in time becomes

$$\operatorname{ess\,sup}_{0<t<T} \|\theta\|_{L^p} \le \|\theta_0\|_{L^p}, \quad p \ge 2 \tag{3.2}$$

3.2. **Energy estimate for (u, θ).** Taking the inner product of the zero-diffusive Boussinesq equations (1.4)-(1.5) with u, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |u|^2\,dx + \int_{\mathbb{R}^3} |\nabla u|^2\,dx = \int_{\mathbb{R}^3} \theta e_3 u dx \tag{3.3}$$

where we have also used

$$\int_{\mathbb{R}^3} u\cdot\nabla u u\,dx = 0, \quad \int_{\mathbb{R}^3} \nabla p u\,dx = 0.$$

Thanks to

$$\int_{\mathbb{R}^3} \theta e_3 u\,dx \le \|\theta\|_{L^2}\|u\|_{L^2} \le \|\theta_0\|_{L^2}\|u\|_{L^2},$$

we have

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |u|^2\,dx + \int_{\mathbb{R}^3} |\nabla u|^2\,dx \le \|\theta_0\|_2\|u\|_2;$$

the Gronwall inequality gives

$$\sup_{0\le t<T} \|u(t)\|_{L^2}^2 + 2\int_0^T \|\nabla u(\tau)\|_{L^2}^2 d\tau \ \le\ C(u_0, \theta_0). \tag{3.4}$$

3.3. **Uniform estimate for $\|\nabla u\|_{L^2}$.** Multiplying both sides of the momentum equations zero-diffusive Boussinesq equations (1.4)-(1.5) with Δu and integrating in $\mathbb{R}^3$, it follows that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \int_{\mathbb{R}^3} |\Delta u|^2\,dx = -\int_{\mathbb{R}^3} u\cdot\nabla u\Delta u\,dx \tag{3.5}$$

where we have used

$$\int_{\mathbb{R}^3} \nabla p\Delta u\,dx = 0.$$

Integrating by parts and using the divergence free condition $\sum_{k=1}^3 \partial_k u_k = 0$, we have

$$\begin{aligned}
&-\int_{\mathbb{R}^3} u\cdot\nabla u\Delta u dx\\
&= -\sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_{kk} u_j u_i \partial_i u_j\,dx\\
&= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_k(u_i\partial_i u_j)\ \partial_k u_j\,dx\\
&= \sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} \partial_k u_i \partial_k u_j \partial_i u_j\,dx + \frac{1}{2}\sum_{i,j,k=1}^3 \int_{\mathbb{R}^3} u_i\partial_i(\partial_k u_j\partial_k u_j)\,dx\\
&= I + J.
\end{aligned} \tag{3.6}$$

We now estimate I and J. When $i=1,2$ or $j=1,2$, by integrating by parts,

$$
\begin{aligned}
I &= \sum_{i,j=1,2}\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_k u_j\ \partial_i u_j dx \\
&= \sum_{i=1}^{2}\sum_{j,k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_i\partial_k u_j\partial_i u_j dx + \sum_{j=1}^{2}\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_3\partial_k u_j\partial_3 u_j dx \\
&\le C\int_{\mathbb{R}^3}(|u_1|+|u_2|)|\nabla u||\Delta u|\,dx.
\end{aligned}
\tag{3.7}
$$

When $i=j=3$, applying the fact

$$
-\partial_3 u_3 = \partial_1 u_1 + \partial_2 u_2
$$

and integrating by parts, we have

$$
\begin{aligned}
-\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i\partial_i u_j\partial_{kk}u_j\,dx &= -\sum_{i,j=3}\sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k(u_i\partial_i u_j)\ \partial_k u_j\,dx \\
&= \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_3\partial_k u_3\partial_3 u_3\,dx \\
&= \sum_{k=1}^{3}\int_{\mathbb{R}^3}\partial_k u_3\partial_k u_3(\partial_1 u_1+\partial_2 u_2)\,dx \\
&\le C\int_{\mathbb{R}^3}(|u_1|+|u_2|)|\nabla u||\Delta u|\,dx\,dx.
\end{aligned}
\tag{3.8}
$$

Inserting the inequalities (3.7) and(3.8) in (3.10), we have

$$
I \le C\int_{\mathbb{R}^3}(|u_1|+|u_2|)|\nabla u||\Delta u|\,dx\,dx. \tag{3.9}
$$

For J,

$$
\frac{1}{2}\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3} u_i\partial_i(\partial_k u_j\partial_k u_j)\,dx = -\frac{1}{2}\sum_{i,j,k=1}^{3}\int_{\mathbb{R}^3}\partial_i u_i(\partial_k u_j\partial_k u_j)\,dx = 0. \tag{3.10}
$$

Substituting the estimates I,J in the right hand side of (3.5), we obtain

$$
\frac{d}{dt}\int_{\mathbb{R}^3}|\nabla u|^2\,dx + 2\int_{\mathbb{R}^3}|\Delta u|^2\,dx \le C\int_{\mathbb{R}^3}(|u_1|+|u_2|)|\nabla u||\Delta u|\,dx\,dx. \tag{3.11}
$$

To control the right hand side of (3.11), with the aid of the Hölder inequality,the Young inequality and Lemma 2.1, it follows that

$$
\begin{aligned}
&\int_{\mathbb{R}^3}(|u_1|+|u_2|)|\nabla u||\Delta u|dx \\
&\le C\|(|u_1|+|u_2|)|\nabla u|\|_{L_2}\|\Delta u\|_{L^2} \\
&\le C\|(|u_1|+|u_2|)|\nabla u|\|_{L^2}^2 + \frac{1}{2}\|\Delta u\|_{L^2}^2 \\
&\le C\|(|u_1|+|u_2|)|\nabla u|\|_{L^2}^2 + \frac{1}{2}\|\Delta u\|_{L^{2,2}}^2 \\
&\le C\||u_1|+|u_2|\|_{L^{p,\infty}}^2\|\nabla u\|_{L^{\frac{2p}{p-2},2}}^2 + \frac{1}{2}\|\Delta u\|_{L^2}^2,
\end{aligned}
$$

thus we rewrite the inequality (3.11) as

$$\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{3}{2}\int_{\mathbb{R}^3} |\Delta u|^2\,dx \le C\||u_1| + |u_2|\|^2_{L^{p,\infty}} \|\nabla u\|^2_{L^{\frac{2p}{p-2},2}} \tag{3.12}$$

Since

$$L^{\frac{2p}{p-2},2}(\mathbb{R}^3) = \Big(L^{\frac{2p_1}{p_1-2}}(\mathbb{R}^3), L^{\frac{2p_2}{p_2-2}}(\mathbb{R}^3)\Big)_{\frac{1}{2},2}$$

with

$$3 < p_1 < p < p_2 < \infty, \quad \frac{2}{p} = \frac{1}{p_1} + \frac{1}{p_2}$$

it follows that

$$\begin{aligned} \|g\|_{L^{\frac{2p}{p-2},2}} &\le C\|g\|^{1/2}_{L^{\frac{2p_1}{p_1-2}}} \|g\|^{1/2}_{L^{\frac{2p_2}{p_2-2}}} \\ &\le C\Big(\|g\|^{\frac{p_1-3}{p_1}}_{L^2} \|\nabla g\|^{\frac{3}{p_2}}_{L^2}\Big)^{1/2} \Big(\|g\|^{\frac{p_2-3}{p_2}}_{L^2} \|\nabla g\|^{\frac{3}{p_2}}_{L^2}\Big)^{1/2} \\ &\le C\|g\|^{\frac{p-3}{p}}_{L^2} \|\nabla g\|^{\frac{3}{p}}_{L^2} \end{aligned}$$

which implies

$$\|\nabla u\|^2_{L^{\frac{2p}{p-2},2}} \le C\|\nabla u\|^{\frac{2(p-3)}{p}}_{L^2} \|\Delta u\|^{6/p}_{L^2}$$

Hence inserting the above inequality into the right hand side of (3.12) and applying the Young inequality, one shows that

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{3}{2}\int_{\mathbb{R}^3} |\Delta u|^2\,dx \\ &\le C\||u_1| + |u_2|\|^2_{L^{p,\infty}} \|\nabla u\|^{\frac{2(p-3)}{p}}_{L^2} \|\Delta u\|^{6/p}_{L^2} \\ &\le C\||u_1| + |u_2|\|^q_{L^{p,\infty}} \|\nabla u\|^2_{L^2} + \frac{1}{2}\|\Delta u\|^2_{L^2} \end{aligned} \tag{3.13}$$

where we have used that $q = 2p/(p-3)$. Thus we derive

$$\begin{aligned} &\frac{d}{dt}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \int_{\mathbb{R}^3} |\Delta u|^2\,dx \\ &\le C\||u_1| + |u_2|\|^q_{L^{p,\infty}} \|\nabla u\|^2_{L^2} \\ &\le C\frac{\||u_1| + |u_2|\|^q_{L^{p,\infty}}}{1 + \ln(e + \|\nabla u\|^2_{L^2})}(1 + \ln(e + \|\nabla u\|^2_{L^2}))\|\nabla u\|^2_{L^2}. \end{aligned} \tag{3.14}$$

Taking the Gronwall inequality into consideration, it follows that

$$\|\nabla u\|^2_{L^2} \le \|\nabla u_0\|^2_{L^2} \exp\Big\{\int_0^T \Big(\frac{\||u_1| + |u_2|\|^q_{L^{p,\infty}}}{1 + \ln(e + \|\nabla u\|^2_{L^2})}\{1 + \ln(e + \|\nabla u\|^2_{L^2})\}\Big)dt\Big\}. \tag{3.15}$$

Hence we have

$$\begin{aligned} \ln(e + \|\nabla u\|^2_{L^2}) &\le \ln(e + \|\nabla u_0\|^2_{L^2}) \\ &\quad + \int_0^T \Big(\frac{\||u_1| + |u_2|\|^q_{L^{p,\infty}}}{1 + \ln(e + \|\nabla u\|^2_{L^2})}\{1 + \ln(e + \|\nabla u\|^2_{L^2})\}\Big)dt. \end{aligned} \tag{3.16}$$

Taking the Gronwall inequality into account again, we have

$$\ln\{e + \|\nabla u\|^2_{L^2}\} \le C(u_0) \exp\Big\{\int_0^T \frac{\||u_1| + |u_2|\|^q_{L^{p,\infty}}}{1 + \ln(e + \|\nabla u\|^2_{L^2})} ds\Big\} < \infty \tag{3.17}$$

which implies the uniform estimates of ∇u,

$$\operatorname{ess\,sup}_{0<t<T}\|\nabla u\|_{L^2}^2<\infty. \tag{3.18}$$

3.4. **Uniform H^mestimate for (u,θ).** Since

$$\Delta u=\partial_t u+\nabla p+u\cdot\nabla u-\theta e_3,$$

by the standard elliptic regularity theory, we can derive

$$\operatorname{ess\,sup}_{0<t<T}\|u\|_{H^2(\mathbb{R}^3)}\le C, \tag{3.19}$$

from which and together with the standard bootstrap technique, we can obtain uniform H^m estimates

$$\sup_{0\le t<T_1}(\|u\|_{H^m}^2+\|\theta\|_{H^m}^2)\le C. \tag{3.20}$$

The detail argument can be found in [14], we omit it here. The proof of Theorem 2.2 is complete.

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