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2D ZAKHAROV-KUZNETSOV-BURGERS EQUATIONS WITH VARIABLE DISSIPATION ON A STRIP

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ABSTRACT. An initial-boundary value problem for a 2D Zakharov-Kuznetsov-Burgers type equation with dissipation located in a neighborhood of $x = -\infty$ and posed on a channel-type strip was considered. The existence and uniqueness results for regular and weak solutions in weighted spaces as well as exponential decay of small solutions without restrictions on the width of a strip were proven both for regular solutions in an elevated norm and for weak solutions in the L^2-norm.

1. Introduction

We are concerned with an initial-boundary value problem (IBVP) for the two-dimensional Zakharov-Kuznetsov-Burgers (ZKB) equation with a variable dissipativity located in a neighborhood of $x = -\infty$

$$u_t + u_x - (a(x)u_x)_x + uu_x + u_{xxx} + u_{xyy} = 0 \tag{1.1}$$

posed on a strip modeling an infinite channel $\{(x, y) \in \mathbb{R}^2 : \ x \in \mathbb{R}, \, y \in (0, B), \, B > 0\}$. Here $a(x) : \mathbb{R} \to \mathbb{R}$ is a sufficiently smooth nonnegative function such that

$$a(x) \geq 0 \text{ in } \mathbb{R}, \ \sup_{\mathbb{R}} |\partial_x^i a(x)| \leq C(i); \ a(x) \geq a_0 > 0 \ \forall x < -r, \tag{1.2}$$

where a_0, r are arbitrary positive constants, $i = 0, 1, 2$. This equation is a two-dimensional analog of the well-known Korteweg-de Vries-Burgers equation

$$u_t + u_x - u_{xx} + uu_x + u_{xxx} = 0 \tag{1.3}$$

which includes dissipation due to viscosity of a medium and dispersion and has been studied by various researchers due to its applications in Mechanics and Physics [1, 2, 3]. One can find extensive bibliography and sharp results on decay rates of solutions to the Cauchy problem (IVP) for (1.3) in [1]. Exponential decay of solutions to the initial problem for (1.1) with additional damping has been established in [3]. Equations (1.1) and (1.3) are typical examples of so-called dispersive equations which attract considerable attention of both pure and applied mathematicians in the past decades.

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Quite recently, the interest on dispersive equations became to be extended to multi-dimensional models such as Kadomtsev-Petviashvili (KP) and Zakharov-Kuznetsov (ZK) equations [28]. As far as the ZK equation and its generalizations are concerned, the results on IVPs can be found in [5, 10, 19, 20, 21, 23, 26] and IBVPs were studied in [4, 6, 9, 16, 17, 18, 26, 27]. In [17, 18] it was shown that IBVP for the ZK equation posed on a half-strip unbounded in x direction with the Dirichlet conditions on the boundaries possesses regular solutions which decay exponentially as $t \to \infty$ provided initial data are sufficiently small and the width of a half-strip is not too large. This means that the ZK equation may create an internal dissipative mechanism for some types of IBVPs.

The goal of our note is to prove that the ZKB equation posed on a strip also may create a dissipative effect without adding any artificial damping even when a variable dissipativity $(-a(x)u_x)_x$ is acting only for $x < -r$. We must mention that IBVP for the ZK equation on a strip ($x \in (0,1)$, $y \in \mathbb{R}$) has been studied in [4, 25] and IBVPs on a strip ($y \in (0,L)$, $x \in \mathbb{R}$) for the ZK equation were considered in [8] and for the ZK equation with some internal variable damping $[-(a_1(x,y)u_x)_x - (a_2(x,y)u_y)_y]$ in [7]. In the domain ($y \in (0,B)$, $x \in \mathbb{R}$, $t > 0$), the term u_x in (1.1) can be scaled out by a simple change of variables. Nevertheless, it can not be safely ignored for problems posed both on finite and semi-infinite intervals as well as on infinite in y direction bands without changes in the original domain [4, 24].

The main results of our paper are the existence and uniqueness of regular and weak global-in-time solutions for (1.1) posed on a strip with the Dirichlet boundary conditions and the exponential decay rate of these solutions as well as continuous dependence on initial data. We must say that exploiting of an exponential weight function is crucial for obtaining necessary global estimates as well as for definition of regular and weak solutions. This fact yearlier has been observed in [12] while studying the Cauchy problem for the 1D KdV equation.

This article has the following structure. Section 1 is Introduction. Section 2 contains formulation of the problem. In Section 3, we prove global existence and uniqueness theorems for regular solutions in some weighted spaces and continuous dependence on initial data. In Section 4, we prove exponential decay of small regular solutions in an elevated norm corresponding to the $H^1(\mathcal{S})$-norm. In Section 5, we prove the existence, uniqueness and continuous dependence on initial data for weak solutions as well as the exponential decay rate of the $L^2(\mathcal{S})$-norm for small solutions without limitations on the width of the strip.

2. Problem and preliminaries

Let B, T, r be finite positive numbers. Define $\mathcal{S} = \{(x,y) \in \mathbb{R}^2 :\ x \in \mathbb{R},\ y \in (0,B)\}$; $\mathcal{S}_r = \{(x,y) \in \mathbb{R}^2 :\ x \in (-r,+\infty),\ y \in (0,B)\}$ and $\mathcal{S}_T = \mathcal{S} \times (0,T)$.

Hereafter subscripts u_x, u_{xy}, etc. denote the partial derivatives, as well as ∂_x or ∂^2_{xy} when it is convenient. Operators ∇ and Δ are the gradient and Laplacian acting over $\mathcal{S}$. By $(\cdot,\cdot)$ and $\|\cdot\|$ we denote the inner product and the norm in $L^2(\mathcal{S})$, and $\|\cdot\|_{H^k}$ stands for norms in the L^2-based Sobolev spaces. We will use also the spaces $H^s \cap L^2_b$, where $L^2_b = L^2(e^{2bx}dx)$, see [12].

Consider the IBVP

$$Lu \equiv u_t - (a(x)u_x)_x + uu_x + u_{xxx} + u_{xyy} = 0 \ \text{ in } \mathcal{S}_T; \tag{2.1}$$

$$u(x,0,t)=u(x,B,t)=0,\ x\in\mathbb{R},\ t>0; \tag{2.2}$$
$$u(x,y,0)=u_0(x,y),\quad (x,y)\in\mathcal{S}. \tag{2.3}$$

3. Existence of regular solutions

3.1. **Regularized problem.** First, for fixed $h\in(0,1)$ sufficently small and $m\in\mathbb{N}$ sufficiently large consider the regularized problem

$$L_hu_h\equiv u_{ht}-(a(x)u_{hx})_x+u_hu_{hx}+u_{hxxx}+u_{hxyy}+h\partial_x^4u_h=0,\quad \text{in } \mathcal{S}_T; \tag{3.1}$$
$$u_h(x,0,t)=u_h(x,B,t)=0,\quad x\in\mathbb{R},\ t>0; \tag{3.2}$$
$$u_h(x,y,0)=u_{0m}(x,y),\quad (x,y)\in\mathcal{S}, \tag{3.3}$$

where u_{0m} is an independent of h approximation of u_0 such that for all $m\in\mathbb{N}$

$$\begin{aligned} J_m=\int_{\mathcal{S}}\{&u_{0m}^2+e^{2bx}(u_{0m}^2+\big[u_{0m}u_{0mx}+\Delta u_{0mx}\big]^2+|\nabla u_{0m}|^2\\ &+|\nabla u_{0mx}|^2+|\partial_x^4u_{0m}|^2)\}\,dx\,dy<\infty,\end{aligned} \tag{3.4}$$

$$\begin{aligned} J_{hm}=\int_{\mathcal{S}}\{&u_{0m}^2+e^{2bx}(u_{0m}^2+\big[u_{0m}u_{0mx}+\Delta u_{0mx}\big]^2+|\nabla u_{0m}|^2\\ &+|\nabla u_{0mx}|^2+h|\partial_x^4u_{0m}|^2)\}\,dx\,dy<\infty,\end{aligned} \tag{3.5}$$

$$\begin{aligned} J_{0m}=\int_{\mathcal{S}}\{&u_{0m}^2+e^{2bx}(u_{0m}^2+\big[u_{0m}u_{0mx}+\Delta u_{0mx}\big]^2+|\nabla u_{0m}|^2\\ &+|\nabla u_{0mx}|^2)\}\,dx\,dy<\infty,\end{aligned} \tag{3.6}$$

$$J_0=\int_{\mathcal{S}}\{u_0^2+e^{2bx}(u_0^2+\big[u_0u_{0x}+\Delta u_{0x}\big]^2+|\nabla u_0|^2+|\nabla u_{0x}|^2)\}\,dx\,dy<\infty \tag{3.7}$$

and $\lim_{m\to\infty}J_{0m}=J_0$.

Obviously, for $m\geq m^*$ sufficiently large,

$$J_{0m}\leq 2J_0,\quad J_{hm}\leq J_m,\quad \|u_{0m}\|^2\leq 2\|u_0\|^2,\quad (e^{2bx},u_{0m}^2)\leq 2(e^{2bx},u_0^2).$$

Approximate solutions. We will construct solutions to (3.1)-(3.3) by the Faedo-Galerkin method: let $w_j(y)$ be orthonormal in $L^2(\mathcal{S})$ eigenfunctions of the Dirichlet problem

$$w_{jyy}+\lambda_jw_j=0,\quad y\in(0,B); \tag{3.8}$$
$$w_j(0)=w_j(B)=0. \tag{3.9}$$

Define approximate solutions of (3.1)-(3.3) as follows:

$$u_h^N(x,y,t)=\sum_{j=1}^N w_j(y)g_{hj}(x,t), \tag{3.10}$$

where $g_{hj}(x,t)$ are solutions to the following Cauchy problem for the system of N nonlinear parabolic equations:

$$\begin{aligned} &\frac{\partial}{\partial t}g_{hj}(x,t)+\frac{\partial^3}{\partial x^3}g_{hj}(x,t)-\frac{\partial}{\partial x}(a(x)g_{hjx}(x,t))-\lambda_j\frac{\partial}{\partial x}g_{hj}(x,t)\\ &+\int_0^B u_h^N(x,y,t)u_{hx}^N(x,y,t)w_j(y)\,dy+h\frac{\partial^4}{\partial x^4}g_{hj}(x,t)=0,\end{aligned} \tag{3.11}$$

$$g_{hj}(x,0)=\int_0^B w_j(y)u_{0m}(x,y)\,dy,\quad j=1,\dots,N. \tag{3.12}$$

It is known that for $g_{hj}(x,0)$ sufficiently smooth the Cauchy problem for the parabolic system (3.11)-(3.12) has a unique regular solution (at least local in t) [11, 13, 22]. To prove the existence of global regular solutions for (3.1)-(3.3), we need uniform in N global in t estimates of approximate solutions $u_h^N(x,y,t)$.

Estimate I. Multiply the j-th equation of (3.11) by g_{hj}, sum up over $j=1,\dots,N$ and integrate the result with respect to x over $\mathbb{R}$ to obtain

$$\frac{d}{dt}\|u_h^N\|^2(t)+2(a(x),|u_{hx}^N|^2)(t)+2h\|u_{hxx}^N\|^2(t)=0.$$

Since $u_{0m}^N=\sum_{j=1}^N w_j(y)g_{hj}(x,0)$ is an approximation of u_0, then for N,m sufficiently large $\|u_{0m}^N\|^2\le 2\|u_{0m}\|^2\|\le 4\|u_0\|^2$. Hence it follows for N,m sufficiently large that for all $t\in(0,T)$

$$\|u_h^N\|^2(t)+2h\int_0^t\|u_{hxx}^N\|^2(s)\,ds\le\|u_{0m}^N\|^2\le 4\|u_0\|^2, \tag{3.13}$$

$$\int_0^t\int_{\mathcal{S}-\mathcal{S}_r}|u_{hx}^N(x,y,t)|^2(s)\,dxy\,ds\le\frac{\|u_{0m}^N\|^2}{2a_0}\le 2\frac{\|u_0\|^2}{a_0}. \tag{3.14}$$

In our calculations we drop the indices h,N,m where it is not ambiguous.

Estimate II. For some positive b, multiply the j-th equation of (3.11) by $e^{2bx}g_j$, sum up over $j=1,\dots,N$ and integrate the result with respect to x over $\mathbb{R}$. Dropping the indices N,h, we obtain

$$\begin{aligned}&\frac{d}{dt}(e^{2bx},u^2)(t)+(e^{2bx},[2a(x)+6b-16hb^2]u_x^2)(t)+2b(e^{2bx},u_y^2)(t)\\&+2h(e^{2bx},u_{xx}^2)(t)-\frac{4b}{3}(e^{2bx},u^3)(t)-(e^{2bx},A(b,a,h)u^2)(t)=0,\end{aligned} \tag{3.15}$$

where $A(b,a,h)$ is a continuous function depending on $b,h,$ $a(x)$ and derivatives of $a(x)$. In our calculations, we will frequently use the following multiplicative inequalities [15].

Proposition 3.1. *(i) For all $u\in H^1(\mathbb{R}^2)$,*

$$\|u\|^2_{L^4(\mathbb{R}^2)}\le 2\|u\|_{L^2(\mathbb{R}^2)}\|\nabla u\|_{L^2(\mathbb{R}^2)}. \tag{3.16}$$

(ii) For all $u\in H^1(D)$,

$$\|u\|^2_{L^4(D)}\le C_D\|u\|_{L^2(D)}\|u\|_{H^1(D)}, \tag{3.17}$$

where the constant C_D depends on a way of continuation of $u\in H^1(D)$ as $\tilde u(\mathbb{R}^2)$ such that $\tilde u(D)=u(D)$.

Extending $u_h^N(x,y,t)$ for a fixed t into exterior of $\mathcal{S}$ by 0 and exploiting inequality (3.16), we find

$$\frac{4b}{3}(e^{2bx}u^3)(t)\le b(e^{2bx},u_y^2)(t)+2b(e^{2bx},u_x^2)(t)+2(b^3+\frac{8b}{9}\|u_{0m}^N\|^2)(e^{2bx},u^2)(t). \tag{3.18}$$

Substituting this into (3.15) and taking into account (1.2), for h sufficiently small and N,m sufficiently large we arrive to

$$\begin{aligned}&\frac{d}{dt}(e^{2bx},u^2)(t)+4b(e^{2bx},u_x^2)(t)+b(e^{2bx},u_y^2)(t)+h(e^{2bx},u_{xx}^2)(t)\\&\le C(b,a)(1+\|u_0\|^2)(e^{2bx},u^2)(t).\end{aligned} \tag{3.19}$$

By Gronwall's lemma,

$$(e^{2bx},u^2)(t)\le C(b,a,T,\|u_0\|)(e^{2bx},u_0^2),$$

where $C(b,a,T,\|u_0\|)$ depends on $a(x)$ and its derivatives. Returning to (3.19),

$$\begin{aligned}&(e^{2bx},|u_h^N|^2)(t)+\int_0^t(e^{2bx},|\nabla u_h^N|^2+h|u_{hxx}^N|^2)(\tau)d\tau\\&\le C(b,a,T,\|u_0\|)(e^{2bx},u_0^2)\quad \forall t\in(0,T),\end{aligned}\tag{3.20}$$

whence

$$e^{-2br}\int_0^t\|u_x\|_{L^2(\mathcal{S}_r)}(\tau)\,d\tau\le C(b,a,T,\|u_0\|)(e^{2bx},u_0^2).$$

Adding (3.14), we obtain

$$\int_0^t\|u_{hx}^N\|_{L^2(\mathcal{S})}(\tau)\,d\tau\le C(b,a,r,T,\|u_0\|)(e^{2bx},u_0^2)\tag{3.21}$$

and the constants in (3.20), (3.21) do not depend on N,h,m.

Estimate III. Multiplying the j-th equation of (3.11) by $-(e^{2bx}g_{jx})_x$, and dropping the index N, we arrive to

$$\begin{aligned}&\frac{d}{dt}(e^{2bx},u_x^2)(t)+(e^{2bx},[2a(x)+6b-16hb^2]u_{xx}^2)(t)+2b(e^{2bx},u_{xy}^2)(t)\\&+2h(e^{2bx},u_{xxx}^2)(t)+(e^{2bx},A(b,a,h)u_x^2)(t)+(e^{2bx},u_x^3)(t)-2b(e^{2bx}u,u_x^2)(t)=0.\end{aligned}\tag{3.22}$$

Using Proposition 3.1, we estimate

$$\begin{aligned}I_1=(e^{2bx},u_x^3)(t)&\le\|u_x\|(t)\|e^{bx}u_x\|^2(t)_{L^4(\mathcal{S})}\\&\le 2\|u_x\|(t)\|e^{bx}u_x\|(t)\|\nabla(e^{bx}u_x)\|(t)\\&\le\delta(e^{2bx},2u_{xx}^2+u_{xy}^2)(t)+2\big[\delta b^2+\frac{\|u_x\|^2(t)}{2\delta}\big](e^{2bx},u_x^2)(t).\end{aligned}$$

Similarly,

$$\begin{aligned}I_2&=2b(e^{2bx},uu_x^2)(t)\\&\le\delta(e^{2bx},2u_{xx}^2+u_{xy}^2)(t)+\big[2b^2\delta+\frac{4b^2}{\delta}\|u_{0m}^N\|^2(t)\big](e^{2bx},u_x^2)(t).\end{aligned}$$

Substituting I_1,I_2 into (2.3), taking δ and a fixed $h>0$ suffuciently small and making use of (3.13), (3.21), we obtain for $\forall t\in(0,T)$:

$$\begin{aligned}&(e^{2bx},|u_{hx}^N|^2)(t)+\int_0^t(e^{2bx},|\nabla u_{hx}^N|^2+h|u_{hxxx}^N|^2)(s)\,ds\\&\le C(b,r,a,T,\|u_0\|,\|e^{bx}u_0\|)(e^{2bx},u_{0mx}^2).\end{aligned}\tag{3.23}$$

Estimate IV. Multiplying the j-th equation of (3.11) by $-2(e^{2bx}\lambda g_j)$, and dropping the index N, we arrive to

$$\begin{aligned}&\frac{d}{dt}(e^{2bx},u_y^2)(t)+(e^{2bx},[2a(x)+6b-16hb^2]u_{xy}^2)(t)+2b(e^{2bx},u_{yy}^2)(t)\\&+2(1-b)(e^{2bx},u_xu_y^2)(t)+2h(e^{2bx},u_{xxy}^2)(t)+(e^{2bx},A(b,a,h)u_y^2)(t)=0.\end{aligned}\tag{3.24}$$

Using Proposition 3.1, we estimate

$$I=2(1-b)(e^{2bx},u_xu_y^2)(t)$$

$$\begin{aligned}&\le 2C_D(1+b)\|u_x\|(t)\|e^{bx}u_y\|(t)\|e^{bx}u_y\|_{H^1(\mathcal{S})}(t)\\&\le \delta(e^{2bx},2u_{xy}^2+u_{yy}^2)(t)+\big[2\delta(1+b^2)+\frac{C_D^2(1+b)^2\|u_x\|^2(t)}{\delta}\big](e^{2bx},u_y^2)(t).\end{aligned}$$

For δ, h sufficiently small, we transform (3.24) into the inequality

$$\begin{aligned}&\frac{d}{dt}(e^{2bx},u_y^2)(t)+4b(e^{2bx},u_{xy}^2)(t)+b(e^{2bx},u_{yy}^2)(t)+2h(e^{2bx},u_{xxy}^2)(t)\\&\le C(b,a)[1+\|u_x\|(t)^2](e^{2bx},u_y^2)(t).\end{aligned}$$

Using (3.21) and the Gronwall lemma, we obtain for all $t\in(0,T)$,

$$\begin{aligned}&(e^{2bx},|u_{hy}^N|^2)(t)+\int_0^t(e^{2bx},|u_{hyy}^N|^2+h|u_{hxxy}^N|^2)(s)\,ds\\&\le C(b,r,a,T,\|u_0\|,\|e^{bx}u_0\|)(e^{2bx},u_{0my}^2).\end{aligned}\tag{3.25}$$

This and (3.23) imply that for all finite $r>0$ and all $t\in(0,T)$,

$$\|u^N\|^2(t)\le C(b,r,a,T,\|u_0\|,\|e^{bx}u_0\|)(e^{2bx},|\nabla u_{0m}|^2).\tag{3.26}$$

Estimate V. Multiplying the j-th equation of (3.11) by $(e^{2bx}g_{jxx})_{xx}$, and dropping the indices N,h, we arrive to

$$\begin{aligned}&\frac{d}{dt}(e^{2bx},u_{xx}^2)(t)+(e^{2bx},[2a(x)+6b-16hb^2]u_{xxx}^2)(t)+2b(e^{2bx},u_{xxy}^2)(t)\\&+2h(e^{2bx},u_{xxxx}^2)(t)+\sum_{i=1}^2(e^{2bx},A_i(b,a,h)|\partial_x^iu|^2)(t)\\&-2b(e^{2bx},uu_{xx}^2)(t)+5(e^{2bx}u_x,u_{xx}^2)(t)=0,\end{aligned}\tag{3.27}$$

where $A_i(b,a,h)$ are constants. Using (3.16), we find

$$\begin{aligned}I&=-2b(e^{2bx},uu_{xx}^2)(t)+5(e^{2bx}u_x,u_{xx}^2)(t)\\&\le 2\delta(e^{2bx},2u_{xxx}^2+u_{xxy}^2)(t)+\big[4b^2\delta+\frac{25}{\delta}\|u_x\|(t)^2\\&\quad+\frac{4b^2}{\delta}\|u\|^2(t)\big](e^{2bx},u_{xx}^2)(t).\end{aligned}$$

Substituting I in (3.27) for δ,h sufficiently small, we obtain

$$\begin{aligned}&\frac{d}{dt}(e^{2bx},u_{xx}^2)(t)+4b(e^{2bx},u_{xxx}^2)(t)+b(e^{2bx},u_{xxy}^2)(t)+2h(e^{2bx},u_{xxxx}^2)(t)\\&\le C(b,a)[1+\|u_x\|^2(t)+\|u\|(t)^2](e^{2bx},u_{xx}^2)(t).\end{aligned}$$

Taking into account (3.13), (3.21), we find

$$\begin{aligned}&(e^{2bx},|u_{hxx}^N|^2)(t)+\int_0^t(e^{2bx},|\nabla u_{hxx}^N|^2+h|u_{hxxxx}^N|^2)(s)\,ds\\&\le C(b,r,a,T,\|u_0\|,\|e^{bx}u_0\|)(e^{2bx},u_{0mxx}^2)\quad\forall t\in(0,T).\end{aligned}\tag{3.28}$$

Estimate VI. Differentiate (3.11) by t and multiply the result by $e^{2bx}g_{jt}$ to obtain

$$\begin{aligned}&\frac{d}{dt}(e^{2bx},u_t^2)(t)+(e^{2bx},[2a(x)+6b-16hb^2]u_{xt}^2)(t)+2b(e^{2bx},u_{ty}^2)(t)\\&+2h(e^{2bx},u_{txx}^2)(t)-(e^{2bx},A(b,a,h)u_t^2)(t)+(2-2b)(e^{2bx}u_x,u_t^2)(t)=0.\end{aligned}\tag{3.29}$$

Using (3.16), we estimate

$$\begin{aligned}I&=(2-2b)(e^{2bx}u_x,u_t^2)(t)\le 2(2+2b)\|u_x\|(t)\|e^{bx}u_t\|(t)\|\nabla(e^{bx}u_t)\|(t)\\&\le\delta(e^{2bx},2u_{xt}^2+u_{ty}^2)(t)+\big[2b^2\delta+\frac{(2+2b)^2\|u_x\|(t)^2}{\delta}\big](e^{2bx},u_t^2)(t).\end{aligned}$$

Taking δ,h sufficiently small and substituting I into (3.29), we obtain

$$\begin{aligned}&\frac{d}{dt}(e^{2bx},u_t^2)(t)+4b(e^{2bx},u_{xt}^2)(t)+b(e^{2bx},u_{ty}^2)(t)\\&\le C(b,a)[1+\|u_x\|(t)^2](e^{2bx},u_t^2)(t).\end{aligned}$$

Using (3.1) and (3.10), we calculate

$$(e^{2bx},|u_{ht}^N|^2)(0)\le CJ_{hm}.$$

This implies that for all $t\in 0,T)$,

$$\begin{aligned}&(e^{2bx},|u_{ht}^N|^2)(t)+\int_0^t(e^{2bx},|\nabla u_{hs}^N|^2+h|u_{hsxx}^N|^2)(s)\,ds\\&\le C(b,r,a,T,\|u_0\|,\|e^{bx}u_0\|)(e^{2bx},u_{ht}^2)(0)\\&\le C(b,r,a,T,\|u_0\|,\|e^{bx}u_0\|)\|)J_{hm}\\&\le C(b,r,a,T,\|u_0\|,\|e^{bx}u_0\|)\|)J_m.\end{aligned}\tag{3.30}$$

Estimate VII. Multiplying the j-th equation of (3.11) by $-e^{2bx}g_{jx}$, dropping indices h,N, we arrive, to

$$\begin{aligned}&(e^{2bx},[u_{xy}^2+(1-3hb)u_{xx}^2])(t)\\&=(e^{2bx}[u_t,u_x)(t)+(e^{2bx},uu_x^2)(t)-(e^{2bx},[4hb^3-2b^2+ba-\frac{a_x}{2}]u_x^2)(t).\end{aligned}\tag{3.31}$$

Using (3.16), we estimate

$$I=(e^{2bx},uu_x^2)(t)\le\delta(e^{2bx},2u_{xx}^2+u_{xy}^2)(t)+\big[2b^2\delta+\frac{\|u_{0m}^N\|^2}{\delta}\big](e^{2bx},u_x^2)(t).$$

Taking δ,h sufficiently small and N,m sufficiently large, using (3.23), (3.27), (3.30) and substituting I, into (3.31), we obtain

$$\begin{aligned}(e^{2bx}|u_{hxx}^N|^2+|u_{hxy}^N|^2)(t)&\le C(b,r,a,T,\|u_0\|,\|e^{bx}u_0\|)\|)J_{hm}\\&\le C(b,r,a,T,\|u_0\|,\|e^{bx}u_0\|)J_m\quad\forall t\in(0,T).\end{aligned}\tag{3.32}$$

Estimate VIII. We shall need the following lemma.

Lemma 3.2. *Let $u(x,y):\mathcal{S}\to\mathbb{R}$ be such that*

$$\int_{\mathcal{S}}e^{2bx}[u^2(x,y)+|\nabla u(x,y)|^2+u_{xy}^2(x,y)]\,dx\,dy<\infty$$

and for all $x \in \mathbb{R}$ there is some $y_0 \in [0, B]$ such that $u(x, y_0) = 0$. Then

$$\begin{aligned}\sup_{\mathcal{S}} |e^{bx}u(x,y,t)|^2 &\le \delta(1+2b^2)(e^{2bx}, u_y^2)(t) + 2\delta(e^{2bx}, u_{xy}^2)(t) \\ &+ \frac{2\delta_1}{\delta}(e^{2bx}, u_x^2)(t) + \frac{1}{\delta}\Big[\frac{1}{\delta_1} + 2\delta_1 b^2\Big](e^{2bx}, u^2)(t),\end{aligned} \tag{3.33}$$

where δ, δ_1 are arbitrary positive numbers.

Proof. Denote $v = e^{bx}u$. Then simple calculations give

$$\sup_{\mathcal{S}} v^2(x,y,t) \le \delta[\|v_y\|^2(t) + \|v_{xy}\|^2(t)] + \frac{1}{\delta}[\|v_x\|^2(t) + \|v\|^2(t)].$$

Returning to the function $u(x, y, t)$, we complete the proof. □

Multiplying the j-th equation of (3.11) by $e^{2bx}g_{jxxx}$, dropping indices h, N, we arrive to

$$\begin{aligned}(e^{2bx}, u_{xxy}^2 + (1-hb)u_{xxx}^2)(t) &= -(e^{2bx}[u_t - (a(x)u_x)_x], u_{xxx})(t) \\ &- (e^{2bx}uu_x, u_{xxx})(t) + 2b^2(e^{2bx}, u_{xy}^2)(t).\end{aligned} \tag{3.34}$$

Using Lemma 3.2 and (3.13), we estimate

$$\begin{aligned}I &= (e^{2bx}uu_x, u_{xxx})(t) \le \|u\|(t)\|e^{bx}u_{xxx}\|(t)\sup_{\mathcal{S}}|e^{bx}u_x(x,y,t)| \\ &\le \epsilon\|u_0\|^2(e^{2bx}, u_{xxx}^2)(t) + \frac{1}{4\epsilon}\Big[\frac{1}{\delta}(1+2b^2)(e^{2bx}, u_x^2)(t) \\ &\quad + \frac{2}{\delta}(e^{2bx}, u_{xx}^2)(t) + \delta(1+2b^2)(e^{2bx}, u_{xy}^2)(t) + 2\delta(e^{2bx}, u_{xxy}^2)(t)\Big].\end{aligned}$$

Taking ϵ and δ sufficiently small, positive and substituting I into (3.34), we find

$$\begin{aligned}(e^{2bx}, |\nabla u_{hxx}^N|^2)(t) &\le C(b, r, a, T, \|u_0\|)J_{hm} \\ &\le C(b, r, a, T, \|u_0\|)J_m \quad \forall t \in (0, T).\end{aligned} \tag{3.35}$$

Consequently, from the equalities

$$-(e^{2bx}[u_{ht}^N - (a(x)u_{hx}^N)_x + u_{hxxx}^N + u_{hxyy}^N + u_h^N u_{hx}^N + hu_{hxxxx}^N], u_{hyy}^N)(t) = 0$$

and

$$(e^{2bx}[u_{ht}^N - (a(x)u_{hx}^N)_x + u_{hxxx}^N + u_{hxyy}^N + u_h^N u_{hx}^N + hu_{hxxxx}^N], u_{hxyy}^N)(t) = 0$$

it follows that

$$(e^{2bx}, |u_{hyy}^N|^2 + |u_{hxyy}^N|^2)(t) \le C(b, r, a, T, \|u_0\|)J_{hm} \quad \forall t \in (0, T). \tag{3.36}$$

Jointly, estimates (3.23),(3.25), (3.28), (3.32),(3.35), (3.36) read

$$\begin{aligned}&\|u_h^N\|^2(t) + (e^{2bx}, |u_h^N|^2 + |\nabla u_h^N|^2 + |\nabla u_{hx}^N|^2 + |\nabla u_{hy}^N|^2 + |\nabla u_{hxx}^N|^2)(t) \\ &\le C(b, r, a, T, \|u_0\|, \|e^{bx}u_0\|)J_{hm} \quad \forall t \in (0, T).\end{aligned} \tag{3.37}$$

This and (3.30) imply

$$h^2(e^{2bx}, u_{hxxxx}^2)(t) \le C(b, r, a, T, \|u_0\|, \|e^{bx}u_0\|)J_{hm}. \tag{3.38}$$

In other words,

$$e^{bx}u_h^N, e^{bx}u_{hx}^N \in L^\infty(0, T; H^2(\mathcal{S})) \tag{3.39}$$

and these inclusions are uniform in $N, h > 0$ while m is fixed and sufficiently large.

Estimate IX. Differentiating the j-th equation of (3.11) with respect to x and multiplying the result by $e^{2bx}\partial_x^4 g_j$, dropping indices h, N, we arrive to

$$\begin{aligned}
&(e^{2bx}, u_{xxxy}^2 + (1-hb)u_{xxxx}^2)(t) \\
&= 2b^2(e^{2bx}, u_{xxy}^2)(t) - (e^{2bx}u_{xt}, u_{xxxx})(t) \\
&\quad + (e^{2bx}[a_{xx}u_x + 2a_x u_{xx} + a u_{xxx}], u_{xxxx})(t) - (e^{2bx}[u_x^2 + u u_{xx}], \partial_x^4 u)(t).
\end{aligned} \tag{3.40}$$

Using Lemma 3.2 and (3.37), we estimate

$$\begin{aligned}
I_1 &= (e^{2bx}, u_x^2, \partial_x^4 u)(t) \le \|u_x\|(t)\|e^{bx}\partial_x^4 u\|(t) \sup_{\mathcal{S}} |e^{bx}u_x(x,y,t)| \\
&\le \frac{\epsilon_1}{2}(e^{2bx}, |\partial_x^4 u|^2)(t) + \frac{1}{2\epsilon_1}\|u_x\|^2(t)\big[(1+2b^2)(e^{2bx}, u_x^2)(t) \\
&\quad + 2(e^{2bx}, u_{xx}^2)(t) + (1+2b^2)(e^{2bx}, u_{xy}^2)(t) + 2(e^{2bx}, u_{xxy}^2)(t)\big] \\
&\le \frac{\epsilon_1}{2}(e^{2bx}, |\partial_x^4 u|^2)(t) + \frac{1}{2\epsilon_1}C(b, T, \|u_0\|, \|e^{bx}u_0\|)J_{hm},
\end{aligned}$$

$$\begin{aligned}
I_2 &= (e^{2bx}u, u_{xx}\partial_x^4 u)(t) \le \|e^{bx}\partial_x^4 u\|(t)\|u\|(t) \sup_{\mathcal{S}} |e^{bx}u_{xx}(x,y,t| \\
&\le \frac{\epsilon_1}{2}\|u_0\|^2(t)(e^{2bx}, |\partial_x^4 u|^2)(t) + \frac{1}{2\epsilon_1}\{2\delta(e^{2bx}, u_{xxxy}^2)(t) \\
&\quad + \delta(1+2b^2)(e^{2bx}, u_{xxy}^2)(t) + \frac{2}{\delta}(e^{2bx}, u_{xxx}^2)(t) \\
&\quad + \frac{1}{\delta}(1+2b^2)(e^{2bx}, u_{xx}^2)(t)\}.
\end{aligned}$$

Applying Young's inequality, taking h, ϵ_1, δ sufficiently small positive, substituting I_1, I_2 into (3.40) and integrating the result, we arrive to

$$\int_0^t (e^{2bx}, |u_{hxxxy}^N|^2 + |u_{hxxxx}^N|^2)(s)\,ds \le C(b, r, a, T, \|u_0\|, \|e^{bx}u_0\|)J_{hm} \tag{3.41}$$

for all $t \in (0,T)$.

Estimate X. Multiplying the j-th equation of (3.11) by $-e^{2bx}\lambda^2 g_{jx}$, dropping indices h, N, we arrive to

$$\begin{aligned}
&(e^{2bx}, (1-3hb)u_{xxyy}^2 + u_{xyyy}^2)(t) \\
&= -(e^{2bx}u_{ty}, u_{xyyy})(t) + (2b^2 - 4hb^3)(e^{2bx}, u_{xyy}^2)(t) \\
&\quad + (e^{2bx}[a_x u_{xy} + a u_{xyy}], u_{xyyy})(t) \\
&\quad - (e^{2bx}u_y u_x, u_{xyyy})(t) + (e^{2bx}u u_{xy}, u_{xyyy})(t).
\end{aligned} \tag{3.42}$$

We estimate

$$\begin{aligned}
I_1 &= -(e^{2bx}u_{ty}, u_{xyyy})(t) \le \frac{\epsilon}{2}(e^{2bx}, u_{xyyy}^2)(t) + \frac{1}{2\epsilon}(e^{2bx}, u_{yt}^2)(t), \\
I_2 &= (e^{2bx}u_y u_x, u_{xyyy})(t) \le \|u_x\|(t)\|e^{bx}u_{xyyy}\|(t) \sup_{\mathcal{S}} |e^{bx}u_y(x,y,t| \\
&\le \frac{\epsilon}{2}(e^{2bx}, u_{xyyy}^2)(t) + \frac{\|u_x\|(t)^2}{2\epsilon}\big[(1+2b^2)(e^{2bx}, u_y^2)(t) \\
&\quad + 2(e^{2bx}, u_{xy}^2)(t) + (1+2b^2)(e^{2bx}, u_{yy}^2)(t) + 2(e^{2bx}, u_{xyy}^2)(t)\big],
\end{aligned}$$

$$\begin{aligned} I_3 &= (e^{2bx}uu_{xy}, u_{xyyy})(t) \le \|u\|(t)\|e^{bx}u_{xyyy})\|(t)\sup_{\mathcal{S}}|e^{bx}u_{xy}(x,y,t| \\ &\le \frac{\|u_0\|^2\epsilon_1}{2}(e^{2bx}, u^2_{xyyy})(t) + \frac{1}{2\epsilon_1}\big[2\delta(e^{2bx}, u^2_{xxyy})(t) \\ &\quad + \frac{2}{\delta}(e^{2bx}, u^2_{xxy})(t) + \delta(1+2b^2)(e^{2bx}, u^2_{xyy})(t) + \frac{1}{\delta}(1+2b^2)(e^{2bx}, u^2_{xy})(t)\big]. \end{aligned}$$

Substituting $I_1 - I_3$ in (3.42) and choosing $h, \epsilon, \epsilon_1, \delta$ sufficiently small, positive, after integration, we transform (3.42) into the inequality

$$\int_0^T (e^{2bx}, [|u^N_{hxxyy}|^2 + |u^N_{hxyyy}|^2])(t)\,dt \le C(b, r, a, T, \|u_0\|, \|e^{bx}u_0\|)J_{hm}. \tag{3.43}$$

Similarly, we obtain from the scalar product

$$(e^{2bx}\big[u^N_{ht} - (au^N_{hx})_x + u^N_{hxxx} + u^N_{hxyy} + u^N_h u^N_{hx} + h\partial^4_x u^N_h\big], u^N_{hyyyy})(t) = 0$$

the estimate

$$\int_0^T (e^{2bx}, |u^N_{hyyy}|^2)(t)\,dt \le C(b, r, a, T, \|u_0\|, \|e^{bx}u_0\|)J_{hm}. \tag{3.44}$$

Estimates (3.37), (3.39), (3.41), (3.43), (3.44) guarantee that

$$\begin{aligned} &\|e^{bx}u^N_h\|_{L^\infty(0,T;H^2(\mathcal{S}))\cap L^2(0,T;H^3(\mathcal{S}))} + \|e^{bx}u^N_{hx}\|_{L^\infty(0,T;H^2(\mathcal{S}))\cap L^2(0,T;H^3(\mathcal{S}))} \\ &+ \|e^{bx}u^N_{ht}\|_{L^\infty(0,T;L^2(\mathcal{S}))\cap L^2(0,T;H^1(\mathcal{S}))} \\ &\le C(b, r, T, \|u_0\|, \|e^{bx}u_0\|)J_{hm} \end{aligned} \tag{3.45}$$

and since by (3.4), (3.5), $J_{hm} \le J_m$, these estimates do not depend on N, h. Independence of Estimates (3.13),(3.45) of N allow us first to pass to the limit as $N \to \infty$ in (3.11) and to prove the following result:

Theorem 3.3. *Let r, B, T be finite positive numbers, $h \in (0,1)$; a smooth nonnegative function $a(x)$ be defined by (1.2) and a given function $u_{0m}(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be such that $u_{0m}(x,0) = u_{0m}(x,B) = 0$ and for some $b > 0$*

$$\begin{aligned} J_{hm} = \int_{\mathcal{S}} \{u^2_{0m} + e^{2bx}[u^2_{0m} + |\nabla u_{0m}|^2 + |\nabla u_{0mx}|^2 + u^2_{0m}u^2_{0mx} \\ + |\Delta u_{0mx}|^2 + h|\partial^4_x u_{0m}|^2]\}\,dx\,dy < \infty. \end{aligned}$$

Then for a fixed $h > 0$ sufficiently small and for a fixed m sufficiently large there exists a regular solution $u_h(x,y,t)$ to (3.1)-(3.3):

$$\begin{gathered} u_{hm} \in L^\infty(0,T;L^2(\mathcal{S})), \quad u_{hmx} \in L^2(0,T;L^2(\mathcal{S})); \\ e^{bx}u_{hm},\ e^{bx}u_{hmx} \in L^\infty(0,T;H^2(\mathcal{S})) \cap L^2(0,T;H^3(\mathcal{S})); \\ e^{bx}u_{hmt} \in L^\infty(0,T;(L^2(\mathcal{S})) \cap L^2(0,T;H^1(\mathcal{S})) \end{gathered}$$

which (dropping the index m) for a.e. $t \in (0,T)$ satisfies the equality

$$\begin{aligned} &(e^{bx}\big[u_{ht} - (a(x))u_{hx})_x + u_{hxxx} + u_h u_{hx} + u_{hxyy}\big]\phi(x,y,t))(t) \\ &+ h(e^{bx}u_{hxxxx}, \phi(x,y,t))(t) = 0, \end{aligned} \tag{3.46}$$

where $\phi(x,y,t)$ is an arbitrary function from $L^\infty(0,T;L^2_b(\mathcal{S}))$.

Proof. Dropping the indices h, m, rewrite (3.11) in the form

$$(e^{bx}\big[u_t^N - (a(x))u_x^N)_x + u_{xxx}^N + u^N u_x^N + u_{xyy}^N\big], \Phi^N(y)\Psi(x,t))(t) \tag{3.47}$$

$$+ h(e^{bx}u_{xxxx}^N, \Phi^N(y)\Psi(x,t))(t) = 0, \tag{3.48}$$

where $\Phi^N(y)$ is an arbitrary function from the linear combinations $\sum_{i=1}^N \alpha_i w_i(y)$ and $\Psi(x,t)$ is an arbitrary function from $L^\infty(0,T;L_b^2(\mathcal{S}))$. Taking into account estimates (3.13), (3.45) and fixing Φ^N, we can easily pass to the limit as $N \to \infty$ in linear terms. To pass to the limit in the nonlinear term, we must use (3.26) and repeat arguments of [12]. Since linear combinations $[\sum_{i=1}^N \alpha_i w_i(y)]\Psi(x,t)$ are dense in $L^2(0,T;L^2(\mathcal{S}) \cap L_b^2(\mathcal{S}))$, we arrive to (3.46). This proves the existence of regular solutions to (3.1)-(3.3). □

Moreover, estimates (3.13), (3.45) do not depend on N, h, hence they are valid also for the limit solution u_h. This allow us to pass to the limit as $h \to 0$ in (3.46) and to prove the following result.

Theorem 3.4. *Let r, B, T be finite positive numbers; a smooth nonnegative function $a(x)$ be defined by (1.2) and given function $u_0(x,y) : \mathbb{R}^2 \to \mathbb{R}$ be such that $u_0(x,0) = u_0(x,B) = 0$ and for some $b > 0$,*

$$J_0 = \int_{\mathcal{S}} \{u_0^2 + e^{2bx}[u_0^2 + |\nabla u_0|^2 + |\nabla u_{0x}|^2 + u_0^2 u_{0x}^2 + |\Delta u_{0x}|^2]\}\, dx\, dy < \infty.$$

Then there exists a regular solution $u(x,y,t)$ to (2.1)–(2.3) such that

$$u \in L^\infty(0,T;L^2(\mathcal{S})), \quad u_x \in L^2(0,T;L^2(\mathcal{S}));$$
$$e^{bx}u,\ e^{bx}u_x \in L^\infty(0,T;H^2(\mathcal{S})) \cap L^2(0,T;H^3(\mathcal{S}));$$
$$e^{bx}u_t \in L^\infty(0,T;L^2(\mathcal{S})) \cap L^2(0,T;H^1(\mathcal{S}))$$

which for a.e. $t \in (0,T)$ satisfies the identity

$$(e^{bx}\big[u_t - (a(x)u_x)_x + u_{xxx} + uu_x + u_{xyy}\big], \phi(x,y,t))(t) = 0, \tag{3.49}$$

where $\phi(x,y,t)$ is an arbitrary function from $L^\infty(0,T;L_b^2(\mathcal{S}))$.

Proof. First, dropping the index m, we assume that $\phi(x,y,t) \in L^\infty(0,T;L_b^2(\mathcal{S}) \cap H^1(\mathcal{S}))$ in (3.46), and rewrite it as

$$\begin{aligned}&(e^{bx}\big[u_{ht} - (a(x)u_{hx})_x + u_{hxxx} + u_h u_{hx} + u_{hxyy}\big], \phi(x,y,t))(t)\\ &- h(e^{bx}u_{hxxx}, \phi_x(x,y,t))(t) + b\phi(x,y,t))(t) = 0.\end{aligned}$$

Due to estimate (3.35),

$$\lim_{h\to 0} h(e^{bx}u_{hxxx}, \phi_x(x,y,t))(t) + b\phi(x,y,t))(t) = 0.$$

Hence, passing to limit as $h \to 0$, we obtain the equality

$$(e^{bx}\big[u_{mt} - (a(x)u_{mx})_x + u_{mxxx} + u_m u_{mx} + u_{mxyy}\big], \phi(x,y,t))(t) = 0, \tag{3.50}$$

where $\phi(x,y,t) \in L^\infty(0,T;L_b^2(\mathcal{S}))$. Obviously, for $u_{0m}(x,y)$ sufficiently smooth and m fixed, $\lim_{h\to 0} J_{hm} = J_{0m}$. Taking into account that $\lim_{m\to\infty} J_{0m} = J_0$, we have for m sufficiently large that $J_{0m} \le 2J_0$ and consequently, estimates (3.13), (3.45) do not depend on m, we can pass to the limit as $m \to \infty$ in the last equality and come to (3.49), where $\phi(x,y,t) \in L^\infty(0,T;L_b^2(\mathcal{S}))$. This proves the existence of regular solutions to (2.1)–(2.3). □

Remark 3.5. Estimates (3.13), (3.20), (3.21) are valid also for the limit function $u(x,y,t)$ in the form

$$\|u\|(t) \le \|u_0\|^2, \tag{3.51}$$

$$\begin{aligned} &\|u\|^2(t) + (e^{2bx}, u^2)(t) + \int_0^t \{\|u_x\|^2(s) + (e^{2bx}, |\nabla u|^2)(s)\}\, ds \\ &\le C(r,b,a,T,\|u_0\|,\|e^{bx}u_0\|)\big[\|u_0\|^2 + (e^{2bx}, u_0^2)\big] \quad \forall t \in (0,T). \end{aligned} \tag{3.52}$$

subsection*Uniqueness of a regular solution

Theorem 3.6. *A regular solution from Theorem 3.4 is uniquely defined.*

Proof. Let u_1, u_2 be two distinct regular solutions of (2.1)–(2.3), then $z = u_1 - u_2$ satisfies the initial-boundary value problem

$$z_t - (a(x)z_x)_x + z_{xxx} + z_{xyy} + \frac{1}{2}(u_1^2 - u_2^2)_x = 0 \quad \text{in } \mathcal{S}_T, \tag{3.53}$$

$$z(x,0,t) = z(x,B,t) = 0, \quad x \in \mathbb{R}, \quad t > 0, \tag{3.54}$$

$$z(x,y,0) = 0 \quad (x,y) \in \mathcal{S}. \tag{3.55}$$

Multiplying (3.53) by $2e^{2bx}z$, we obtain

$$\begin{aligned} &\frac{d}{dt}(e^{2bx}, z^2)(t) + (e^{2bx}[2a(x) + 6b], z_x^2)(t) - (e^{2bx}[4b^2a + 2ba_x + 8b^3], z^2)(t) \\ &+ 2b(e^{2bx}, z_y^2)(t) + (e^{2bx}[u_{1x} + u_{2x}], z^2)(t) - b(e^{2bx}(u_1 + u_2), z^2)(t) = 0. \end{aligned} \tag{3.56}$$

We estimate

$$\begin{aligned} I_1 &= (e^{2bx}(u_{1x} + u_{2x}), z^2)(t) \le \|u_{1x} + u_{2x}\|(t)\|e^{bx}z\|(t)^2_{L^4(\mathcal{S})} \\ &\le 2\|u_{1x} + u_{2x}\|(t)\|e^{bx}z\|(t)\|\nabla(e^{bx}z)\|(t) \\ &\le \delta(e^{2bx}, [2{z_x}^2 + {z_y}^2])(t) + [2b^2\delta + \frac{2}{\delta}(\|u_{1x}\|^2(t) + \|u_{2x}\|^2(t))](e^{2bx}, z^2)(t), \\ I_2 &= b(e^{2bx}(u_1 + u_2), z^2)(t) \le b\|u_1 + u_2\|(t)\|e^{bx}z\|^2_{L^4(\mathcal{S})} \\ &\le 2b\|u_1 + u_2\|(t)\|e^{bx}z\|(t)\|\nabla(e^{bx}z)\|(t) \\ &\le \delta(e^{2bx}, 2z_x^2 + z_y^2)(t) + [2b^2\delta + \frac{2b^2}{\delta}(\|u_1\|^2(t) + \|u_2\|^2(t))](e^{2bx}, z^2)(t). \end{aligned}$$

Substituting I_1, I_2 into (3.56) and taking $\delta > 0$ sufficiently small, we find

$$\begin{aligned} &\frac{d}{dt}(e^{2bx}, z^2)(t) + 2b(e^{2bx}, z_x^2)(t) + b(e^{2bx}, z_y^2)(t) \\ &\le C(b,a)\big[1 + \|u_1\|(t)^2 + \|u_2\|(t)^2 + \|u_{1x}\|(t)^2 + \|u_{2x}\|(t)^2\big](e^{2bx}, z^2)(t). \end{aligned} \tag{3.57}$$

By (3.51) and (3.52),

$$u_i \in L^\infty(0,T;L^2(\mathcal{S})), \quad u_{ix} \in L^2(0,T;L^2(\mathcal{S})) \quad i = 1,2,$$

hence by Gronwall's lemma,

$$(e^{2bx}, z^2)(t) = 0 \quad \forall\, t \in (0,T)$$

and $u_1 = u_2$ a.e. in $\mathcal{S}_T$. □

Remark 3.7. Changing initial condition (3.55) for $z(x,y,0) = z_0(x,y) \neq 0$, and repeating the proof of Theorem 3.4, from (3.57) we obtain

$$(e^{2bx}, z^2)(t) \leq C(b, r, a, T, \|u_0\|, \|e^{bx}u_0\|)(e^{2bx}, z_0^2) \quad \forall t \in (0,T).$$

This means continuous dependence of regular solutions on the initial data.

4. Decay of regular solutions

In this section we prove exponential decay of regular solutions in an elevated weighted norm corresponding to the $H^1(\mathcal{S})$ norm. We start with Theorem 4.1 which is crucial for the main result.

Theorem 4.1. *Let $b \in (0, b_0)$, $a_x(x) \leq 0$, $\|u_0\| \leq \frac{3\pi}{8B}$ and $u(x,y,t)$ be a regular solution to (2.1)–(2.3). Then for all finite $B > 0$,*

$$\|e^{bx}u\|^2(t) \leq e^{-\chi t}\|e^{bx}u_0\|^2(0), \tag{4.1}$$

$$\int_0^t e^{\chi s}(e^{2bx}, |\nabla u|^2)(s)\, ds \leq C(b, \|u_0\|)(1+t)(e^{2bx}, u_0^2), \tag{4.2}$$

where

$$b_0 = \frac{\pi^2}{4B^2}\Big[\frac{1}{\sup_{\mathbb{R}}[a(x) + \sqrt{a^2(x) + \frac{5\pi^2}{8B^2}}]}\Big], \quad \chi = b_0\frac{\pi^2}{2B^2}, \tag{4.3}$$

Proof. Multiplying (2.1) by $2e^{2bx}u$, we obtain

$$\begin{aligned} &\frac{d}{dt}(e^{2bx}, u^2)(t) + (e^{2bx}[2a(x) + 6b], u_x^2)(t) + 2b(e^{2bx}, u_y^2)(t) \\ &- \frac{4b}{3}(e^{2bx}, u^3)(t) - (e^{2bx}\{2b[2a(x)b + a_x(x)] + 8b^3\}, u^2)(t) = 0. \end{aligned} \tag{4.4}$$

The following proposition is necessary for our proof.

Proposition 4.2.

$$\int_{\mathbb{R}}\int_0^B e^{2bx}u^2(x,y,t)\, dy\, dx \leq \frac{B^2}{\pi^2}\int_{\mathbb{R}}\int_0^B e^{2bx}u_y^2(x,y,t)\, dy\, dx. \tag{4.5}$$

Proof. Since $u(x,0,t) = u(x,B,t) = 0$, fixing (x,t), we can use with respect to y the following Steklov inequality: if $f(y) \in H_0^1(0,\pi)$, then

$$\int_0^\pi f^2(y)\, dy \leq \int_0^\pi |f_y(y)|^2\, dy.$$

After a corresponding process of scaling we complete the proof. □

Taking into account (3.51), we estimate

$$I = \frac{4b}{3}(e^{2bx}, u^3)(t) \leq b(e^{2bx}, u_y^2 + 2u_x^2)(t) + \big[2b^3 + \frac{16b}{9}\|u_0\|^2\big](e^{2bx}, u^2)(t).$$

Using (4.5) and substituting I in (4.4), we arrive to

$$\begin{aligned} &\frac{d}{dt}(e^{2bx}, u^2)(t) + 4b(e^{2bx}, u_x^2)(t) + b\Big[\frac{\pi^2}{B^2} - 2a_x(x) \\ &- \{4a(x)b + 10b^2 + \frac{16}{9}\|u_0\|^2\}\Big](e^{2bx}, u^2)(t) \leq 0 \end{aligned}$$

which can be rewritten as

$$\frac{d}{dt}(e^{2bx},u^2)(t)+\chi(e^{2bx},u^2)(t)\le 0, \tag{4.6}$$

where

$$\chi=b\big[\frac{\pi^2}{B^2}-2a_x(x)-4ba(x)-10b^2-\frac{16\|u_0\|^2}{9}\big].$$

Since we need $\chi>0$, define

$$0<4ba(x)+10b^2\le\frac{\pi^2}{4B^2},\quad \frac{16\|u_0\|^2}{9}\le\frac{\pi^2}{4B^2}. \tag{4.7}$$

Solving (4.7), we find

$$b_0=\frac{\pi^2}{4B^2}\Big[\frac{1}{\sup_{\mathbb{R}}[a(x)+\sqrt{a^2(x)+\frac{5\pi^2}{8B^2}}]}\Big],\quad \chi=b_0\frac{\pi^2}{2B^2}\,. \tag{4.8}$$

From (4.6) we obtain

$$(e^{2bx},u^2)(t)\le e^{-\chi t}(e^{2bx},|u_0|^2).$$

This inequality implies (4.1).

To prove (4.2), we return to (4.4) and multiply it by $e^{\chi t}$ to obtain

$$\begin{aligned}&\frac{d}{dt}[e^{\chi t}(e^{2bx},u^2)(t)]+e^{\chi t}\big[(e^{2bx},[2a(x)+6b]u_x^2)(t)+2b(e^{2bx},u_y^2)(t)\big]\\&=\frac{4be^{\chi t}}{3}(e^{2bx},u^3)(t)+e^{\chi t}(e^{2bx},A(\chi,b,a)u^2)(t),\end{aligned} \tag{4.9}$$

where $A(\chi,b,a)=2ba_x(x)+4b^2a(x)+8b^3+\chi$. Substituting (4) in (4.9), we obtain

$$\frac{d}{dt}[e^{\chi t}(e^{2bx},u^2)(t)]+e^{\chi t}(e^{2bx},|\nabla u|^2)(t)\le C(b,\chi,a,\|u_0\|)e^{\chi t}(e^{2bx},u^2)(t). \tag{4.10}$$

Integrating and using (4.1) imply

$$e^{\chi t}(e^{2bx},u^2)(t)+\int_0^t e^{\chi s}(e^{2bx},|\nabla u|^2)(s)\,ds\le C(b,\chi,a,\|u_0\|)(1+t)(e^{2bx},u_0^2). \tag{4.11}$$

The proof is complete. □

Observe that differently from [17, 18], we do not have any restrictions on the width of a strip B. The main result of this section is the following assertion.

Theorem 4.3. *Let l the conditions of Theorem 4.1 be fulfilled. Then regular solutions of* (2.1)–(2.3) *satisfy*

$$(e^{2bx},u^2+|\nabla u|^2)(t)\le C(b,\chi,\|u_0\|)(1+t)e^{-\chi t}(e^{2bx},\big[u_0^2+|u_0|^3+|\nabla u_0|^2\big]) \tag{4.12}$$

or

$$\|e^{bx}u\|^2_{H^1(\mathcal{S})}(t)\le C(b,\chi,\|u_0\|)(1+t)e^{-\chi t}(e^{2bx},u_0^2+|u_0|^3+|\nabla u_0|^2).$$

Proof. We start with the following lemma.

Lemma 4.4. *Regular solutions of* (2.1)–(2.3) *satisfy*

$$
\begin{aligned}
&e^{\chi t}(e^{2bx},|\nabla u|^2)(t)+\int_0^t e^{\chi s}\{(e^{2bx}[2a(x)+6b],u_{xx}^2)(s)+\frac{b}{2}(e^{2bx},u^4)(s)\\
&+(e^{2bx}[2a(x)+8b],u_{xy}^2)(s)+2b(e^{2bx},u_{yy}^2)(s)\}\,ds\\
&=\frac{e^{\chi t}}{3}(e^{2bx},u^3)(t)+\int_0^t\{e^{\chi s}(e^{2bx}[\chi+2ba_x(x)+4b^2a(x)+8b^3],u_y^2)(s)\\
&\quad+4b(e^{2bx},uu_y^2)(s)-(e^{2bx}\big[\frac{4a(x)b^2+2ba_x(x)+8b^3}{3}-\chi\big],u^3)(s)\}\,ds\\
&\quad+\int_0^t e^{\chi s}(e^{2bx}[\chi+a_{xx}(x)+4b^2a(x)+8b^3],u_x^2)(s)\,ds\\
&\quad+2\int_0^t e^{\chi s}(e^{2bx}[a(x)+4b]u,u_x^2)(s)\,ds+(e^{2bx},|\nabla u_0|^2-\frac{u_0^3}{3}).
\end{aligned}
\tag{4.13}
$$

Proof. First we transform the scalar product

$$
-(e^{bx}\big[u_t-[a(x)u_x]_x+u_{xxx}+u_{xyy}+uu_x\big],\big[2(e^{bx}u_x)_x+2e^{bx}u_{yy}+e^{bx}u^2\big])(t)=0 \tag{4.14}
$$

into the equality

$$
\begin{aligned}
&\frac{d}{dt}(e^{2bx},|\nabla u|^2-\frac{u^3}{3})(t)+(e^{2bx}[2a(x)+6b],u_{xx}^2)(t)\\
&+2b(e^{2bx},u_{yy}^2)(t)+(e^{2bx}[2a(x)+8b],u_{xy}^2)(t)+\frac{b}{2}(e^{2bx},u^4)(t)\\
&=(e^{2bx}[a_{xx}(x)+4a(x)b^2+8b^3],u_x^2)(t)-(e^{2bx}\big[\frac{4a(x)b^2+2a_x(x)b+8b^3}{3}\big],u^3)(t)\\
&\quad+(e^{2bx}[4a(x)b^2+2ba_x(x)+8b^3],u_y^2)(t)+4b(e^{2bx},uu_y^2)(t)\\
&\quad+(e^{2bx}[2a(x)+8b],uu_x^2)(t).
\end{aligned}
\tag{4.15}
$$

To prove (4.15), we estimate separate terms in (4.14) as follows:

$$
\begin{aligned}
I_1&=-2(e^{bx}\big[u_t-[a(x)u_x]_x+u_{xxx}+u_{xyy}+uu_x\big],(e^{bx}u_x)_x)(t)\\
&=2(e^{2bx}\big[u_t-[a(x)u_x]_x+u_{xxx}+u_{xyy}+uu_x\big]_x,u_x)(t)\\
&=\frac{d}{dt}(e^{2bx},u_x^2)(t)+(e^{2bx}[2a(x)+6b],u_{xx}^2)(t)+2b(e^{2bx},u_{xy}^2)(t)\\
&\quad-(e^{2bx}[a_{xx}(x)+4a(x)b^2+8b^3],u_x^2)(t)+(e^{2bx}u^2,u_{xxx})(t)\\
&\quad-8b(e^{2bx},uu_x^2)(t)+\frac{8b^3}{3}(e^{2bx},u^3)(t);
\end{aligned}
$$

$$
\begin{aligned}
I_2&=-2(e^{bx}\big[u_t-[a(x)u_x]_x+u_{xxx}+u_{xyy}+uu_x\big],e^{bx}u_{yy})(t)\\
&=2(e^{bx}\big[u_t-[a(x)u_x]_x+u_{xxx}+u_{xyy}+uu_x\big]_y,e^{bx}u_y)(t)\\
&=\frac{d}{dt}(e^{2bx},u_y^2)(t)+(e^{2bx}[2a(x)+6b],u_{xy}^2)(t)+2b(e^{2bx},u_{yy}^2)(t)\\
&\quad-(e^{2bx}[2ba_x(x)+4b^2a(x)+8b^3],u_y^2)(t)+(e^{2bx}u,u_{xyy})(t)-4b(e^{2bx},uu_y^2)(t);
\end{aligned}
$$

$$
I_3=-(e^{bx}\big[u_t-[a(x)u_x]_x+u_{xxx}+u_{xyy}+uu_x\big],e^{bx}u^2)(t)
$$

$$
\begin{aligned}
&= -\frac{d}{dt}(e^{2bx}, \frac{u^3}{3})(t) + \frac{2b}{3}(e^{2bx}[2ba(x) + a_x(x)], u^3)(t) + \frac{b}{2}(e^{2bx}, u^4)(t) \\
&\quad - 2(e^{2bx}a(x), uu_x^2)(t) - (e^{2bx}, u_{xxx} + u_{xyy})(t).
\end{aligned}
$$

Summing $I_1 + I_2 + I_3$, we obtain (4.15). In turn, multiplying it by $e^{\chi t}$ and integrating the result over $(0,t)$, we arrive to (4.13). The proof of Lemma 4.4 is complete. □

Using (3.16), we estimate

$$
\begin{aligned}
I_4 &= \frac{e^{\chi t}}{3}(e^{2bx}, u^3)(t) \le \frac{2e^{\chi t}}{3}\|u_0\|\|e^{bx}u\|(t)\|\nabla(e^{bx}u)\|(t) \\
&\le \frac{e^{\chi t}}{2}\{(e^{2bx}, |\nabla u|^2)(t) + [\frac{b^2}{2} + \frac{4\|u_0\|^2}{9}](e^{2bx}, u^2)(t)\}.
\end{aligned}
$$

Substituting I_4 in (4.13), we obtain

$$
\begin{aligned}
&e^{\chi t}(e^{2bx}, |\nabla u|^2)(t) + 2\int_0^t e^{\chi s}\{(e^{2bx}[2a(x) + 6b], u_{xx}^2)(s) + \frac{b}{2}(e^{2bx}, u^4)(s) \\
&+ (e^{2bx}[2a(x) + 8b], u_{xy}^2)(s) + 2b(e^{2bx}, u_{yy}^2)(s)\}\,ds \\
&\le 2\int_0^t \{e^{\chi s}(e^{2bx}[\chi + 2ba_x(x) + 4b^2a(x) + 8b^3], u_y^2)(s) \\
&\quad + 4b(e^{2bx}, uu_y^2)(s) - (e^{2bx}\big[\frac{4a(x)b^2 + 2ba_x(x) + 8b^3}{3} - \chi\big], u^3)(s)\}\,ds \qquad (4.16)\\
&\quad + 2\int_0^t e^{\chi s}(e^{2bx}[\chi + a_{xx}(x) + 4b^2a(x) + 8b^3], u_x^2)(s)\,ds \\
&\quad + 4\int_0^t e^{\chi s}(e^{2bx}[a(x) + 4b]u, u_x^2)(s)\,ds \\
&\quad + [b^2 + \frac{8\|u_0\|^2}{9}](e^{2bx}, u^2)(t) + 2(e^{2bx}, |\nabla u_0|^2 - \frac{u_0^3}{3}).
\end{aligned}
$$

By Proposition 3.1,

$$
\begin{aligned}
I_1 &= -(e^{2bx}\big[\frac{4a(x)b^2 + 2ba_x(x) + 8b^3}{3} - \chi\big], u^3)(s) \\
&\le (e^{2bx}, |\nabla u|^2)(s) + C(\chi, b, a, \|u_0\|)(e^{2bx}, u^2)(s).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
I_2 &= 4(e^{2bx}[a(x) + 4b]u, u_x^2)(s) \\
&\le \delta(e^{2bx}, 2u_{xx}^2 + u_{xy}^2)(s) + \frac{1}{\delta}C(\chi, a, b, \|u_0\|)(e^{2bx}, u_x^2)(s)
\end{aligned}
$$

and

$$
\begin{aligned}
I_3 &= 8b(e^{2bx}, uu_y^2)(s) \le 8bC_D\|u_0\|\|e^{bx}u_y\|(s)\|e^{bx}u_y\|_{H^1(\mathcal{S})}(s) \\
&\le \delta(e^{2bx}, 2u_{xy}^2 + u_{yy}^2)(s) + \big[(2b^2+1)\delta + \frac{16b^2\|u_0\|^2C_D^2}{\delta}\big](e^{2bx}, u_y^2)(s).
\end{aligned}
$$

Taking $\delta = 2b$ and using (4.1), (4.2), from (4.16) we obtain

$$
e^{\chi t}(e^{2bx}, |\nabla u|^2)(t) \le C(b, \chi, a, \|u_0\|)(1+t)(e^{2bx}, u_0^2 + |u_0|^3 + |\nabla u_0|^2).
$$

Adding (4.1), we complete the proof of Theorem 4.3. □

5. Weak solutions

Here we prove the existence, uniqueness and continuous dependence on initial data as well as exponential decay results for weak solutions of (2.1)–(2.3) with $u_0 \in L^2(\mathcal{S}) \cap L^2_b(\mathcal{S})$.

Theorem 5.1. *Let $u_0 \in L^2(\mathcal{S}) \cap L^2_b(\mathcal{S})$. Then for all finite positive T and B there exists at least one function $u(x,y,t)$ such that*

$$u \in L^\infty(0,T;L^2(\mathcal{S})), \quad u_x \in L^2(0,T;L^2(\mathcal{S})),$$
$$e^{bx}u \in L^\infty(0,T;L^2(\mathcal{S})) \cap L^2(0,T;H^1(\mathcal{S})).$$

Moreover,

$$u^m \rightharpoonup u \quad \text{*-weakly in } L^\infty(0,T;L^2(\mathcal{S}) \cap L^2_b(\mathcal{S})),$$
$$e^{bx}u^m \rightharpoonup e^{bx}u \quad \text{weakly in } L^2(0,T;H^1(\mathcal{S})),$$

where u^m are regular solutions to (2.1)–(2.3) provided by Theorem 3.4. For a.e. $t \in (0,T)$ the following integral identity takes a place

$$\begin{aligned} &(e^{bx}u,v)(t) + \int_0^t \{-(e^{bx}u,v_s)(s) + (e^{bx}u_x,\big[v_{xx} + (a(x)+2b)v_x \\ &+ (a(x)b+b^2)v\big])(s) - \frac{1}{2}(e^{2bx}u^2, bv+v_x)(s) + (e^{bx}u_y, bv_y + v_{xy})(s)\}\,ds \\ &= (e^{bx}u_0, v(x,y,0)), \end{aligned} \tag{5.1}$$

where

$$v \in L^\infty(0,T;L^2_b(\mathcal{S})) \cap L^2(0,T;H^2(\mathcal{S})) \quad v_t \in L^2(0,T;L^2_b(\mathcal{S}))$$

is an arbitrary function.

Proof. To justify our calculations, we must operate with sufficiently smooth solutions $u^m(x,y,t)$. With this purpose, we consider first initial functions $u_{0m}(x,y)$, which satisfy conditions of Theorem 3.4, and obtain estimates (3.51), (3.52) for functions $u^m(x,y,t)$. By Theorem 3.4, we can write for a.e. $t \in (0,T)$

$$(e^{bx}\big[u^m_t - (a(x)u^m_x)_x + u^m_{xxx} + u^m u^m_x + u^m_{xyy}\big], \phi(x,y,t))(t) = 0, \tag{5.2}$$

where $\phi(x,y,t)$ is an arbitrary function from

$$L^\infty(0,T;L^2_b(\mathcal{S})) \cap L^2(0,T;H^2(\mathcal{S})),$$

and the inner product at the left-hand side of (5.2) is an integrable function on $(0,T)$. Integrating (5.2) over (0,t), after standard calculations we arrive to the integral equality

$$\begin{aligned} &(e^{bx}u^m,v)(t) + \int_0^t \{-(e^{bx}u^m,v_s)(s) + (e^{bx}u^m_x,\big[v_{xx} + (a(x)+2b)v_x \\ &+ (a(x)b+b^2)v\big])(s) - \frac{1}{2}(e^{2bx}|u^m|^2, bv+v_x)(t) + (e^{bx}u^m_y, bv_y + v_{xy})(s)\}\,ds \\ &= (e^{bx}u_{0m}, v(x,y,0)). \end{aligned} \tag{5.3}$$

Using estimates (3.51), (3.52), we pass to the limit as $m \to \infty$ and come to (5.1). $\square$

Remark 5.2. There is an alternate manner to define a weak solution as a distribution, using estimates (3.51), (3.52), and passing to the limit directly in (5.2),

$$(e^{bx}\big[u_t^m - (a(x)u_x^m)_x + u_{xxx}^m + u^m u_x^m + u_{xyy}^m\big], \phi(x,y,t))(t) = 0.$$

Here we can estimate $e^{bx}u_t^m \in L^2(0,T;H^{-2}(\mathcal{S}))$ which we will need to pass to the limit in the nonlinear term.

Remark 5.3. It is easy to verify that regular solutions from Theorem 3.4, u^m, satisfy (2.1)–(2.3)), hence (5.2) and (5.3). And vice versa, if regular solutions u^m satisfy (5.3), then for a.e. $t \in (0,T)$ they also satisfy (5.2), and consequently, (2.1)–(2.3).

Uniqueness of a weak solution.

Theorem 5.4. *A weak solution of Theorem 5.1 is uniquely defined.*

Proof. Actually, this proof is provided by Theorem 3.6. It is sufficient to approximate an initial function $u_0 \in L^2(\mathcal{S}) \cap L_b^2(\mathcal{S})$ by regular functions u_{0m} in the form:

$$\lim_{m\to\infty} \|u_{0m} - u_0\|_{L^2(\mathcal{S})\cap L_b^2(\mathcal{S})} = 0,$$

where u_{0m} satisfies the conditions of Theorem 3.4. This guarantees the existence of the unique regular solution to (2.1)–(2.3), u^m, and allows us, due to Remark 5.3, to repeat all the calculations which have been done during the proof of Theorem 3.6 and arrive to

$$\begin{aligned}
&\frac{d}{dt}(e^{2bx}, z_m^2)(t) + 2b(e^{2bx}, z_{mx}^2)(t) + b(e^{2bx}, z_{my}^2)(t) \\
&\le C(b,a)\big[1 + \|u_{1m}\|(t)^2 + \|u_{2m}\|(t)^2 + \|u_{1xm}\|(t)^2 + \|u_{2xm}\|(t)^2\big](e^{2bx}, z_m^2)(t).
\end{aligned}$$

By the generalized Gronwall‘s lemma,

$$\begin{aligned}
(e^{2bx}, z_m^2)(t) \le \exp\{\int_0^t C(b,a)\big[1 + \|u_{1m}\|^2(s) + \|u_{2m}\|^2(s) + \|u_{1xm}\|^2(s) \\
+ \|u_{2xm}\|^2(s)\big]\, ds\}(e^{2bx}, z_{0m}^2)(t).
\end{aligned}$$

Functions u_{1m} and u_{2m} for m sufficiently large satisfy the estimate (3.52),

$$\begin{aligned}
&\|u_{im}\|(t)^2 + \int_0^t \|u_{imx}\|(s)^2\, ds \\
&\le C(r,T,\|u_{0m}\|, \|e^{bx}u_{0m}\|)[\|u_{0m}\|^2 + (e^{2bx}, u_{0m}^2)] \\
&\le C(r,T,\|u_0\|, \|e^{bx}u_0\|)[\|u_0\|^2 + (e^{2bx}, u_0^2)], \quad i = 1,2.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\exp\{\int_0^t C\big[1 + \|u_{1m}\|(s)^2 + \|u_{2m}\|(s)^2 + \|u_{1xm}\|(s)^2 + \|u_{2xm}\|(s)^2\big]\, ds\} \\
&\le C(b,a,T,r,\|u_0\|, \|e^{bx}u_0\|).
\end{aligned}$$

Since $e^{bx}z(x,y,t)$ is a weak limit of regular solutions $\{e^{bx}z_m(x,y,t)\}$, then

$$(e^{2bx}, z^2)(t) \le (e^{2bx}, z_m^2)(t) = 0.$$

This implies $u_1 \equiv u_2$ a.e. in $\mathcal{S}_T$. The proof of Theorem 5.4 is complete. □

Remark 5.5. Changing initial condition $z(x,y,0) \equiv 0$ for $z(x,y,0) = z_0(x,y) \neq 0$, and repeating the proof of Theorem 5.4, we obtain that

$$(e^{2bx}, z^2)(t) \leq C(b, a, T, r, \|u_0\|, \|e^{bx}u_0\|)(e^{2bx}, z_0^2) \quad \forall t \in (0,T).$$

This means continuous dependence of weak solutions on initial data.

Decay of weak solutions.

Theorem 5.6. *Let $b \in (0, b_0)$, $a_x(x) \leq 0$, $\|u_0\| \leq 3\pi/(16B)$ and $u(x,y,t)$ be a weak solution of* (2.1)–(2.3). *Then for all finite $B > 0$,*

$$\|e^{bx}u\|^2(t) \leq e^{-\chi t}\|e^{bx}u_0\|^2(0), \tag{5.4}$$

where

$$b_0 = \frac{\pi^2}{4B^2}\Big[\frac{1}{\sup_{\mathbb{R}}[a(x) + \sqrt{a^2(x) + \frac{5\pi^2}{8B^2}}]}\Big], \quad \chi = b_0\frac{\pi^2}{2B^2}, \tag{5.5}$$

Proof. Similarly to the proof of the uniqueness result for a weak solution, we approximate $u_0 \in L^2(\mathcal{S}) \cap L^2_b(\mathcal{S})$ by sufficiently smooth functions u_{0m}, satisfying the conditions of Theorem 3.4, in order to work with regular solutions. Acting in the same manner as by the proof of Theorem 4.1, we arrive to

$$\|e^{bx}u_m\|^2(t) \leq e^{-\chi t}\|e^{bx}u_0\|^2(0), \tag{5.6}$$

where

$$\chi = b_0\frac{\pi^2}{2B^2}.$$

Since $u(x,y,t)$ is a weak limit of regular solutions $\{u_m(x,y,t)\}$,

$$(e^{2bx}, u^2)(t) \leq (e^{2bx}, u_m^2)(t) \leq e^{-\chi t}(e^{2bx}, u_0^2).$$

The proof of Theorem 5.6 is complete. □

In this Theorem we have a more strict condition for the decay of weak solutions $\|u_0\| \leq \frac{3\pi}{8\sqrt{2}B}$ instead of $\|u_0\| \leq \frac{3\pi}{8B}$ in the case of decay for regular solution which follows from (4.7). In the case of weak solutions, we use in (4.7) instead of u_0 its approximation u_{0m}. This implies for m sufficiently large

$$64B^2\|u_{0m}\|^2 \leq 64B^2 2\|u_0\|^2 \leq 9\pi^2$$

and consequently, $\|u_0\| \leq 3\pi/(\sqrt{2}8B)$.

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