Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 59, pp. 1–9.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

PROPERTIES OF MEROMORPHIC SOLUTIONS OF q-DIFFERENCE EQUATIONS

XIAOGUANG QI, LIANZHONG YANG

Abstract. In this article, we utilize Nevanlinna value distribution theory to study the solvability and the growth of meromorphic function $f(z)$ that satisfies some q-difference equations, which can be seen the q-difference analogues of Painlevé I and II equations. This article extends earlier results by Chen et al [2, 3].

1. Introduction

In this article, we use the basic notions in Nevanlinna theory of meromorphic functions, as found in [13]. In addition, we use $\delta(f)$ and $\lambda(\frac{1}{f})$ to denote the order, the exponent of convergence of poles of a meromorphic function $f(z)$, respectively.

A century ago, Painlevé [10, 11], Fuchs [4] and Gambier [5] classified a large class of second order differential equations in terms of a characteristic which is now known as the Painlevé property. Painlevé and his colleagues looked at the class

$$w''(z) = F(z, w, w'),$$

where F is rational in w and w' and (locally) analytic in z, rejecting those equations which did not have the Painlevé property. They singled out a list of 50 equations, six of which could not be integrated in terms of known functions. These equations are now known as the Painlevé equations. The first two of these equations are P_I and P_{II}:

$$w'' = 6w^2 + z, \quad w'' = 2w^2 + zw + \alpha,$$

where α is a constant. On the other hand, the equation that are now traditionally called the discrete Painlevé equations are special cases of QRT (Quispel-Robert-Thompson) difference equations. The QRT difference equations was a starting point in the discovery of the discrete Painlevé equations. As we all known, the discrete P_I and P_{II} can be expressed in the form

$$y_{n+1} + y_{n-1} = \frac{an+b}{y_n} + c, \quad y_{n+1} + y_{n-1} = \frac{(an+b)y_n + c}{1 - y_n^2}.$$

2000 *Mathematics Subject Classification.* 39A05, 30D35.
Key words and phrases. Meromorphic functions; q-difference equation; growth; zero order.

Submitted August 19, 2014. Published March 10, 2015.

Recently, Halburd and Korhonen [8] used Nevanlinna theory to single out difference equations in the form

$$w(z+1)+w(z-1)=R(z,w), \tag{1.1}$$

where $R(z,w)$ is rational in w and meromorphic in z. They obtained that if (1.1) has an admissible meromorphic solution of finite order, then either w satisfies a difference Riccati equation, or (1.1) can be transformed by a linear change in w to some difference equations, which include the difference P_I and P_{II} equations

$$\begin{gathered} f(z+1)+f(z-1)=\frac{az+b}{f}+c, \\ f(z+1)+f(z-1)=\frac{(az+b)f+c}{1-f^2}. \end{gathered} \tag{1.2}$$

Moreover, Chen et al [2, 3] investigated properties of finite-order transcendental meromorphic solutions of (1.2). Closely related to difference expressions are q-difference expressions, where the usual shift $f(z+c)$ of a meromorphic function will be replaced by the q-difference $f(qz)$, $q\in\mathbb{C}\setminus\{0\}$. The Nevanlinna theory of q-difference expressions and its applications to q-difference equations have recently been considered, see [1]. In addition, some results about solutions of zero order for complex q-difference equations, can be found in the introduction in [1].

A natural question is: what is the result if we give q-difference analogues of (1.2). Corresponding to this question, we consider the following two equations:

$$f(qz)+f(\frac{z}{q})=\frac{az+b}{f}+c, \tag{1.3}$$

$$f(qz)+f(\frac{z}{q})=\frac{(az+b)f+c}{1-f^2}. \tag{1.4}$$

Theorem 1.1. *Let $f(z)$ be a transcendental meromorphic solution with zero order of equation (1.3), and a, b, c be three constants such that a, b cannot vanish simultaneously. Then,*

- (i) *$f(z)$ has infinitely many poles.*
- (ii) *If $a\neq 0$, then $f(z)$ has infinitely many finite values.*
- (iii) *If $a=0$ and $f(z)$ takes a finite value A finitely often, then A is a solution of $2z^2-cz-b=0$.*

Theorem 1.2. *Let a, b, c and $|q|\neq 1$ be four constants,*

- (i) *if $a\neq 0$, then equation (1.3) has no rational solution;*
- (ii) *if $a=0$, then the rational solutions of the equation (1.3) must satisfy $f(z)=B+\frac{P(z)}{Q(z)}$, where $P(z)$ and $Q(z)$ are relatively prime polynomials and satisfy $\deg P<\deg Q$ and $2B^2-cB-b=0$.*

Theorem 1.3. *Let a,b,c be constants with $ac\neq 0$, and let $f(z)$ be a transcendental meromorphic solution with zero order of equation (1.4). Then $f(z)$ has infinitely many poles and infinitely many finite values.*

Using similar methods as in the proof of Theorem 1.1, we can prove Theorem 1.3. Here, we omit the details.

Theorem 1.4. *Let a, b, c be constants with $ac \neq 0$ and $|q| \neq 1$. Suppose that a rational function*

$$f(z) = \frac{P(z)}{Q(z)} = \frac{sz^m + p_{m-1}z^{m-1} + \cdots + p_0 z}{tz^n + q_{n-1}z^{m-1} + \cdots + q_0 z}$$

is a solution of (1.4)*, where $P(z)$ and $Q(z)$ are relatively prime polynomials, $s \neq 0, p_{m-1}, \dots, p_0$ and $t \neq 0, q_{n-1}, \dots q_0$ are constants. Then $n = m+1$ and $s = -\frac{c}{a}t$.*

As for the next result from this point of view, see [12]. Several papers have appeared in which the solutions of the following equation are studied

$$f(z+1) + f(z-1) = a(z)f(z)^2 + b(z)f(z) + c(z). \tag{1.5}$$

The reader is invited to see [9]. The following result can be seen as a q-difference counterpart to (1.5).

Theorem 1.5. *Let $|q| \neq 1$ and $n \geq 2$, let $f(z)$ be a meromorphic solution of*

$$f(qz) + f(\frac{z}{q}) = a(z)f(z)^n + b(z)f(z) + c(z) \tag{1.6}$$

with meromorphic coefficients satisfying $T(r,a) = S(r,f)$, $T(r,b) = S(r,f)$ and $T(r,c) = S(r,f)$. Then $f(z)$ is of positive order of growth.

We remark that Theorem 1.5 is not true, when $n = 1$. This can be seen by considering $f(z) = z + 1$, $f(\frac{1}{2}z) = \frac{1}{2}z + 1$ and $f(2z) = 2z + 1$. Then $f(z)$ is a solution of $f(2z) + f(\frac{z}{2}) = \frac{5}{2}f(z) - \frac{1}{2}$.

Also we remark that the right hand side of the above equations are essentially like the function $A(z,f)$, where $A(z,f)$ is a rational function of $f(z)$. Reversing the order of composition on the right hand side results in a functional q-difference equation. The following theorem gives an example.

Theorem 1.6. *Let $q \neq 0$ be a complex constant, and suppose $f(z)$ be a transcendental meromorphic function of the equation*

$$A(qz, f(qz)) = f(p(z)), \tag{1.7}$$

where $p(z)$ is a polynomial of degree $k \geq 2$, $A(z,y)$ is a rational with meromorphic coefficients of growth $S(r,f)$ such that $A(z,y)$ is irreducible in y. If $\deg_f A = n \geq k$, then for any $\varepsilon > 0$

$$T(r,f) = O((\log r)^{\alpha+\varepsilon}),$$

where $\alpha = \frac{\log n}{\log k}$.

Some ideas in this paper come from [2, 9].

2. Some Lemmas

Lemma 2.1 ([1, Theorem 2.1])**.** *Let $f(z)$ be a non-constant zero order meromorphic solution of*

$$f(z)^n P(z,f) = Q(z,f),$$

where $P(z,f)$ and $Q(z,f)$ are q-difference polynomials in $f(z)$. If the degree of $Q(z,f)$ as a polynomial in $f(z)$ and its q-shifts is at most n, then

$$m(r, P(z,f)) = o(T(r,f))$$

on a set of logarithmic density 1.

Lemma 2.2 ([1, Theorem 2.2]). *Let $f(z)$ be a non-constant zero order meromorphic solution of*

$$H(z,f)=0,$$

where $H(z,f)$ is a q-difference polynomial in $f(z)$. If $H(z,a)\not\equiv 0$ for slowly moving target $a(z)$, then

$$m(\frac{1}{f-a})=o(T(r,f))$$

on a set of logarithmic density 1.

Lemma 2.3 ([14, Theorems 1.1 and 1.3]). *Let $f(z)$ be a zero order meromorphic function, and $q\in\mathbb{C}\setminus\{0\}$. Then*

$$T(r,f(qz))=(1+o(1))T(r,f(z)) \tag{2.1}$$

$$N(r,f(qz))=(1+o(1))N(r,f(z)) \tag{2.2}$$

on a set of lower logarithmic density 1.

We recall some notation and a lemma from [8]. $\omega(z)$ has more than $S(r,\omega)$ poles of a certain type, which means that the integrated counting function of these poles is not of type $S(r,\omega)$. We use the notation $D(z_0,r)$ to denote an open disc of radius r centered at $z_0\in C$. Also, ∞^k denotes a pole of ω with multiplicity k. We now recall the following lemma from [8, Lemma 3.1].

Lemma 2.4. *Let $\omega(z)$ be a meromorphic function having more than $S(r,\omega)$ poles, and let a_s $(s=1,2,\dots,n)$ be small meromorphic functions with respect to ω. Denote by m_j the maximum order of zeros and poles of the functions as at z_j . Then for any $\varepsilon>0$, there are at most $S(r,\omega)$ points z_j such that*

$$\omega(z_j)=\infty^{k_j},$$

where $m_j\geq\varepsilon k_j$.

Lemma 2.5 ([6]). *Let $p(z)=a_kz^k+a_{k-1}z^{k-1}+\dots+a_1z+a_0$, $a_k\neq 0$, be a non-constant polynomial of degree k and let $f(z)$ be a transcendental meromorphic function. Given $0<\varrho<|a_k|$, denote $\zeta=|a_k|+\varrho$ and $\eta=|a_k|-\varrho$. Then, given $\varepsilon>0$, we have*

$$(1-\varepsilon)T(\eta r^k,f)\leq T(r,f\circ p)\leq(1+\varepsilon)T(\zeta r^k,f)$$

for all r large enough.

Lemma 2.6 ([7]). *Let $\psi:[r_0,\infty)\to(0,\infty)$ be positive and bounded in every finite interval, and suppose that $\psi(\mu r^m)\leq A\psi(r)+B$ holds for all r large enough, where $\mu>0$, $m>1$, $A>1$ and B are real constants. Then*

$$\psi(r)=O((\log r)^\alpha),$$

where $\alpha=\frac{\log A}{\log m}$.

3. Proof of Theorem 1.1

(i) Suppose that $f(z)$ is a zero order transcendental meromorphic solution of (1.3). From (1.3), we obtain

$$fP(z,f)=Q(z,f),$$

where

$$P(z,f)=f(qz)+f(\frac{z}{q}),\quad Q(z,f)=az+b+cf(z). \tag{3.1}$$

Applying Lemma 2.1, we obtain

$$m(r,P(z,f))=o(T(r,f)) \tag{3.2}$$

on a set of logarithmic density 1. Combining the Valiron-Mohon'ko theorem and (1.3),

$$T(r,P(z,f))=T(r,f)+S(r,f) \tag{3.3}$$

follows. From Lemma 2.3, we obtain

$$N(r,P(z,f))\le N(r,f(qz))+N(r,f(\frac{z}{q}))=2(1+o(1))N(r,f)$$

on a set of lower logarithmic density 1. This, together with (3.2) and (3.3), yields

$$2(1+o(1))N(r,f)\ge T(r,f)+S(r,f)$$

on a set of logarithmic density 1. Hence, $f(z)$ has infinitely many poles.

(ii) For any finite value A, and set

$$g(z)=f(z)-A.$$

This, combining with (1.3), it follows that

$$g(qz)+g(\frac{z}{q})+2A=\frac{az+b}{g(z)+A}+c.$$

Rewrite above equation in the form

$$H(z,g)=(g(qz)+g(\frac{z}{q})+2A)(g(z)+A)-c(g(z)+A)-az-b=0. \tag{3.4}$$

By the assumption that $a\ne 0$ and (3.4), we have

$$H(z,0)=2A^2-cA-b-az\not\equiv 0.$$

Hence, applying Lemma 2.2, we obtain that

$$m(r,\frac{1}{g})=o(T(r,g))$$

on a set of logarithmic density 1. That is,

$$N(r,\frac{1}{f-A})=N(r,\frac{1}{g})=T(r,g)(1+o(1))=T(r,f)(1+o(1)) \tag{3.5}$$

on a set of logarithmic density 1. The conclusion holds.

(iii) If $a=0$ and A is not a solution of $2z^2-cz-b=0$; that is,

$$H(z,0)=2A^2-cA-b\ne 0.$$

Then, we obtain $N(r,\frac{1}{f-A})=T(r,f)(1+o(1))$ as well, using the same way as the above argument. This contradicts the assumption, and the conclusion follows.

4. Proof of Theorem 1.2

Suppose that $f(z)$ is a rational solution of (1.3), and has poles $z_1,\dots z_k$. Thus, $f(z)$ can be represented in the form

$$f(z)=\sum_{i=1}^{k}\Big(\frac{d_{is_i}}{(z-z_i)^{s_i}}+\dots+\frac{d_{is_1}}{(z-z_i)}\Big)+B+B_1z+\dots+B_mz^m, \tag{4.1}$$

where $d_{is_i}\neq 0,\dots d_{is_1}$, B, $B_1\dots B_m$ are constants.

In the following, we prove that $B_1=\dots=B_m=0$. Suppose, contrary to the assertion, that $B_m\neq 0$, $(m\geq 1)$. For sufficiently large z, from (4.1), we know

$$\begin{gathered} f(z)=B_mz^m(1+o(1)),\\ f(qz)=B_mz^mq^m(1+o(1)),\\ f(qz)=B_mz^m\frac{1}{q^m}(1+o(1)). \end{gathered} \tag{4.2}$$

From (1.3), we have

$$f(z)(f(qz)+f(\frac{z}{q}))=az+b+cf(z). \tag{4.3}$$

This and (4.2) imply that

$$(q^m+\frac{1}{q^m})B_m^2z^{2m}(1+o(1))=az+b+cB_mz^m(1+o(1)),$$

Since $B_m\neq 0$ and $|q|\neq 1$, we see the above equation is a contradiction for sufficiently large z. Therefore, $B_1=\dots=B_m=0$.

Case 1. $a\neq 0$. If $B\neq 0$, then from (3.5), we have

$$f(qz)=f(z)=f(\frac{1}{q}z)=B+o(1),$$

for sufficiently large z. Substituting the above equation into (4.3), we conclude that

$$(B+o(1))(2B+o(1))=az+b+c(B+o(1)),$$

which is a contradiction to the assumption that $a\neq 0$. Thus, $B=B_1=\dots=B_m=0$. Hence, we obtain

$$f(z)=\frac{P(z)}{Q(z)}, \tag{4.4}$$

where $P(z)$, $Q(z)$ are polynomials such that $\deg P<\deg Q$. Combining (4.4) and (1.3),

$$\begin{aligned} &P(qz)P(z)Q(\frac{z}{q})+P(z)P(\frac{z}{q})Q(qz)\\ &=(az+b)Q(qz)Q(z)Q(\frac{z}{q})+cP(z)Q(qz)Q(\frac{z}{q}). \end{aligned} \tag{4.5}$$

Observing the above equation, we see that the degree of both sides of (4.5) are not equal. This is impossible. Hence, if $a\neq 0$, then (1.3) has no rational solution.

Case 2. $a=0$. From the above argument, we know that

$$f(z)=\frac{P_1(z)}{Q_1(z)}+B,$$

where $P_1(z)$, $Q_1(z)$ are polynomials such that $\deg P_1 < \deg Q_2$. Moreover, we have

$$f(qz) = f(z) = f(\frac{1}{q}z) = B + o(1),$$

for sufficiently large z. Combining above equation and (1.3), we obtain that $2B^2 - cB - b = 0$. The conclusion holds.

5. Proof of Theorem 1.4

Assume that $f(z)$ is a rational solution of (1.4), and has poles $z_1, \dots z_k$. Hence, $f(z)$ can be expressed as the following the form:

$$f(z) = \sum_{i=1}^{k} \Big(\frac{b_{is_i}}{(z - z_i)^{s_i}} + \dots + \frac{b_{is_1}}{(z - z_i)} \Big) + D + D_1 z + \dots + D_j z^j, \tag{5.1}$$

where $b_{is_i} \neq 0, \dots b_{is_1}$, D, $D_1 \dots D_j$ are constants. Using a similar method as in the proof of Theorem 1.2, we have $D_1 = \dots = D_j = 0$ as well. If $D \neq 0$, then from (5.1), we have

$$f(qz) = f(z) = f(\frac{1}{q}z) = D + o(1),$$

for sufficiently large z. Combining the above equation and (1.4),

$$(az + b)(D + o(1)) = -(D^2 + o(1))(2D + o(1)) + (2D + o(1)) - c$$

follows. From the assumption that $a \neq 0$ and $D \neq 0$, we obtain a contradiction. Therefore, $D = 0$. Furthermore, $f(z)$ can be expressed as

$$f(z) = \frac{P(z)}{Q(z)}, \tag{5.2}$$

where

$$\begin{gathered} P(z) = sz^m + p_{m-1}z^{m-1} + \dots + p_0 z, \\ Q(z) = tz^n + q_{n-1}z^{m-1} + \dots + q_0 z. \end{gathered}$$

Substituting (5.2) into (1.4), we obtain

$$\begin{aligned} & cQ(qz)Q(\frac{z}{q})Q(z)^2 + (az + b)Q(qz)Q(\frac{z}{q})P(z)Q(z) \\ & = P(qz)Q(\frac{z}{q})Q(z)^2 - P(qz)Q(\frac{z}{q})P(z)^2 + P(\frac{z}{q})Q(qz)Q(z)^2 \\ & \quad - P(\frac{z}{q})Q(qz)P(z)^2. \end{aligned} \tag{5.3}$$

As $ac \neq 0$, by comparing the degrees of all terms of (5.3), we obtain that $n = m+1$. From (5.2) and (1.4), it follows that

$$\frac{P(qz)}{Q(qz)} + \frac{P(\frac{z}{q})}{Q(\frac{z}{q})} = \frac{(az + b)P(z)Q(z) + cQ(z)^2}{Q(z)^2 - P(z)^2}. \tag{5.4}$$

From this and $n = m + 1$, we conclude that

$$\frac{P(qz)}{Q(qz)} + \frac{P(\frac{z}{q})}{Q(\frac{z}{q})} \to 0,$$

and

$$\frac{(az + b)P(z)Q(z) + cQ(z)^2}{Q(z)^2 - P(z)^2} = \frac{(ast + ct^2)z^{2n}(1 + o(1))}{s^2 z^{2n}(1 + o(1))}$$

for sufficiently large z. Comparing the two above equations, we know $ast + ct^2 = 0$, that is, $s = -\frac{c}{a}t$.

6. Proof of Theorem 1.5

Let $f(z)$ be a meromorphic solution of (1.6). Suppose, contrary to the assertion, that $\sigma(f) = 0$. If $n > 2$, then the conclusion holds by Lemma 2.3, clearly. Now we consider the case $n = 2$. Writing (1.6) in the form

$$af^2 = f(qz) + f(\frac{z}{q}) - bf - c.$$

By Lemma 2.1, we obtain that

$$m(r, f) = o(T(r, f))$$

on a set of logarithmic density 1. Hence, $f(z)$ has more than $S(r, f)$ poles counting multiplicities. We use z_j to denote points in the pole sequence. Lemma 2.4 implies that there are more than $S(r, f)$ points such that the multiplicity of $f(z_j) = \infty$ is k_j, where $\varepsilon k_j > m_j$. Here m_j refers to a, b, c. Denoting the sequence of such poles by $z_{1,j}$, we take this sequence as our starting point. For $\varepsilon < 1/2$, by (1.6), we have at least one of the points $qz_{1,j}$, $z_{1,j}/q$ is a pole of f of multiplicity $k_{2,j} > (2-\varepsilon)k_{1,j}$. By Lemma 2.4, we obtain that $f(z)$ has more than $S(r, f)$ such points $z_{2,j}$ such that the multiplicity of $f(z_{2,j}) = \infty$ is $k_{2,j}$, where $\varepsilon k_{2,j} > m_{2,j}$. Then we only choose one of these points and denote it by $z_{2,j}$. Continuing to the next phase. By (1.6), we have that $z_{3,j} = qz_{2,j}$ or $z_{3,j} = \frac{z_{2,j}}{q}$ is a pole of f of multiplicity $k_{3,j}$, where

$$k_{3,j} \geq (2 - \varepsilon)k_{2,j} > (2 - \varepsilon)^2 k_{1,j}.$$

By induction, we can choose a sequence z_n of poles of $f(z)$, the multiplicity of which is k_n, and $k_n \geq (2 - \varepsilon)^{n-1} k_{1,j} \geq (2 - \varepsilon)^{n-1}$.

Case a. If $|q| > 1$, then by a simple geometric observation, we obtain

$$z_n \in B(0, |q|^n |z_1|) = B(0, |q|^n r_1) = B(0, r_n).$$

Therefore,

$$n(r_n, f) \geq (\frac{3}{2})^{n-1} = (\frac{3}{2})^{\frac{\log r_n - \log r_1 - \log |q|}{\log |q|}}.$$

So, we obtain $\lambda(\frac{1}{f}) = \frac{\log \frac{3}{2}}{\log |q|} > 0$, contradicting our hypothesis $\sigma(f) = 0$. Hence $\sigma(f) > 0$.

Case b. If $|q| < 1$, then by a simple geometric observation, it follows

$$z_n \in B(0, |\frac{1}{q}|^n |z_1|) = B(0, |\frac{1}{q}|^n r_1) = B(0, r_n).$$

Using the same argument as Case a, we can get the conclusion as well.

7. Proof of Theorem 1.6

We first replace z by z/q in equation (1.7), then applying the Valiron-Mohon'ko theorem to the left hand side of (1.7), and combining Lemma 2.5, we obtain that

$$nT(r, f) + S(r, f) = T(r, A) = T(r, f(p(\frac{z}{q}))) \geq (1 - \varepsilon)T(\mu r^k, f).$$

Since we may assume r to be large enough to satisfy

$$n(1 + \varepsilon)T(r, f) \geq (1 - \varepsilon)T(\mu r^k, f)$$

outside of a possible exceptional set of finite linear measure. We conclude that, for every $\lambda > 1$, there exists an $r_0 > 0$ such that

$$n(1+\varepsilon)T(\lambda r, f) \geq (1-\varepsilon)T(\mu r^k, f) \tag{7.1}$$

holds for all $r \geq r_0$. Denoting $t = \lambda r$, then (7.1) can be written as

$$T(\frac{\mu}{\lambda^k}t^k, f) \leq \frac{n(1+\varepsilon)}{1-\varepsilon}T(t, f).$$

Thus, we apply Lemma 2.6 to conclude that

$$\alpha = \frac{\log n(1+\varepsilon)/(1-\varepsilon)}{\log k} = \frac{\log n}{\log k} + o(1).$$

Acknowledgments. This work was supported by the National Natural Science Foundation of China (No. 11301220 and No. 11371225), the NSF of Shandong Province, China (No. ZR2012AQ020) and the Fund of Doctoral Program Research of University of Jinan (XBS1211).

References

[1] D.C. Barnett, R. G. Halburd, R. J. Korhonen, W. Morgan; *Nevanlinna theory for the q-difference operator and meromorphic solutions of q-difference equations*, Proc. Roy. Soc. Edinburgh Sect. A **137** (2007), 457–474.

[2] Z. X. Chen, K. H. Shon; *Value distribution of meromorphic solutions of certain difference Painlevé equations*, J. Math. Anal. Appl. **364** (2010), 556-566.

[3] Z. X. Chen; *On properties of meromorphic solutions for some difference equations*, Kodai Math. **34** (2011), 244-256.

[4] L. Fuchs; *Sur quelques équations différentielles linéares du second ordre*, C. R. Acad. Sci. Paris **141** (1905), 555-558.

[5] B. Gambier; *Sur les équations différentielles du second ordre et du premier degré dont l'intégrale générale est àpoints critiques fixes*, Acta Math. **33** (1910), 1-55.

[6] R. Goldstein; *Some results on factorisation of meromorphic functions*, J. London Math. Soc. **4** (1971), 357-364.

[7] R. Goldstein; *On meromorphic solutions of certain functional equations*, Aequationes Math. **18** (1978), 112-157.

[8] R. G. Halburd, R. J. Korhonen; *Finite order solutions and the discrete Painlevé equations*, Proc. London Math. Soc. **94** (2007), 443-474.

[9] I. Laine, C. C. Yang; *Clunie theorem for difference and q-difference polynomials*, J. London. Math. Soc. **76** (2007), 556-566.

[10] P. Painlevé; *Mémoire sur les équations différentielles dont l'intégrale générale est uniforme*, Bull. Soc. Math. France **28** (1900), 201-261.

[11] P. Painlevé; *Sur les équations différentielles du second ordre et d'ordre supérieur dont l'intégrale générale est uniforme*, Acta Math. **25** (1902), 1-85

[12] A. Ramani, B. Grammaticos, T. Tamizhmani, K. M. Tamizhmani; *The road to the discrete analogue of the Painlevé property: Nevanlinna meets singularity confinement,* Comput. Math. Appl. **45** (2003), 1001-1012.

[13] C. C. Yang, H. X. Yi; *Uniqueness Theory of Meromorphic Functions*, Kluwer Academic Publishers, Dordrecht, 2003.

[14] J. L. Zhang, R. Korhonen; *On the Nevanlinna characteristic of $f(qz)$ and its applications*, J. Math. Anal. Appl. **369** (2010), 537-544.

Xiao-Guang Qi
Jinan University, School of Mathematics, Jinan, Shandong 250022, China
E-mail address: `xiaogqi@mail.sdu.edu.cn, sms_qixg@ujn.edu.cn`

Lian-Zhong Yang
Shandong University, School of Mathematics, Jinan, Shandong 250100, China
E-mail address: `lzyang@sdu.edu.cn`