Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 58, pp. 1–15.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND STABILITY OF ALMOST PERIODIC SOLUTIONS TO DIFFERENTIAL EQUATIONS WITH PIECEWISE CONSTANT ARGUMENTS

SAMUEL CASTILLO, MANUEL PINTO

Abstract. This work concerns the existence of almost periodic solutions for certain differential equations with piecewise constant arguments. The coefficients of these equations are almost periodic and the equation can be seen as perturbations of a linear equation satisfying an exponential dichotomy on a difference equation. The stability of that solution on a semi-axis is also studied.

1. Introduction

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$ be the sets of natural, integer, real and complex numbers, respectively. Denote by $|\cdot|$ the Euclidean norm for every finite dimensional space on $\mathbb{R}$. Fix a real valued sequence $(t_n)_{n=-\infty}^{+\infty}$, such that $t_n < t_{n+1}$ and $t_n \to \pm\infty$ as $n \to \pm\infty$. For $p \in \mathbb{Z}$, let $\gamma^p : \mathbb{R} \to \mathbb{R}$ be functions such that $\gamma^p/J_n = t_{n-p}$ for all $n \in \mathbb{Z}$, where $J_n = [t_n, t_{n+1}[$, for all $n \in \mathbb{Z}$.

We are interested in the existence of almost periodic solution of the following linear differential equations with piecewise constant arguments (DEPCA)

$$y'(t) = A(t)y(t) + B(t)y(\gamma^0(t)) + f(t), \quad t \in \mathbb{R} \tag{1.1}$$

and

$$y'(t) = A(t)y(t) + B(t)y(\gamma^0(t)) + F(t, y_\gamma(t)), \quad t \in \mathbb{R}, \tag{1.2}$$

where

$$y_\gamma(t) = (y(\gamma^{p_1}(t)), y(\gamma^{p_2}(t)), \dots, y(\gamma^{p_\ell}(t))), \tag{1.3}$$

where $p_1, p_2, \dots, p_\ell \in \mathbb{N} \cup \{0\}$. Equations (1.1) and (1.2) are seen as perturbation of the linear equation

$$z'(t) = A(t)z(t) + B(t)z(\gamma^0(t)), \tag{1.4}$$

where the matrices $A, B : \mathbb{R} \to \mathcal{M}_q(\mathbb{C})$ and $f : \mathbb{R} \to \mathbb{C}^q$ are locally integrable functions, and $F : \mathbb{R} \times W \subseteq \mathbb{R} \times (\mathbb{C}^q)^\ell \to \mathbb{C}^q$ is a continuous function.

For our study, the following additional assumptions are made.

(H1) A and B are almost periodic functions.

2000 *Mathematics Subject Classification.* 34A38, 34C27, 34D09, 34D20.
Key words and phrases. Piecewise constant argument;almost periodic solutions; exponential dichotomy; stability of solutions.

Submitted August 25, 2014. Published March 10, 2015.

(H2) $\big(t_n^{(k)}\big)_{n=-\infty}^{+\infty}$, where $t_n^{(k)} = t_{n+k} - t_n$ for all $k \in \mathbb{Z}$, is equipotentially almost periodic for all $k \in \mathbb{Z}$.

(H3) (H2) holds and for all $\varepsilon > 0$,
$$T(f,\varepsilon) = \big\{\tau \in \mathbb{R} : |f(t+\tau) - f(t)| \le \varepsilon,\ \forall t \in \mathbb{R} - \big(\cup_{n\in\mathbb{Z}}]t_n - \varepsilon, t_n + \varepsilon[\big)\big\}$$
is relatively dense and there is $\delta_\varepsilon > 0$ such that $|f(t'+\tau') - f(t')| \le \varepsilon$ if $\tau' \in \mathbb{R} : |\tau'| \le \delta_\varepsilon$ and $t', t'+\tau'$ is in some of the intervals $[t_n, t_{n+1}]$.

(H4) F is uniformly almost periodic on W and there is $L > 0$ such that
$$|F(t, x_1, \dots, x_\ell) - F(t, y_1, \dots, y_\ell)| \le L \sum_{j=1}^{\ell} |x_j - y_j|, \tag{1.5}$$
for all $t \in \mathbb{R}$ and $(x_1, \dots, x_\ell), (y_1, \dots, y_\ell) \in W$.

A kind of exponential dichotomy is imposed on a part of the linear equation 1.4, which will be made explicit in the following section.

This work is motivated by the results in Fink [20, Theorems 7.7, 8.1 and 11.31]. Some extensions for piecewise constant argument can be found in [3, 22, 36]. Existence of almost periodic solutions for the impulsive case can be found in [30, 24]. Our focus is to see the almost periodic solutions for (1.1) and (1.2) in terms of the solutions of the difference equation from the Cauchy operator of the linear part (1.4), on the points t_n for all $n \in \mathbb{N}$, in the style of [22]. Other recent results are found in [6, 16, 31].

This work is different from Akhmet works [3, 6] since our emphasis is on the behavior of solutions on the points t_n. This work is different from the works by Hong-Yuan [22] and Yuan [36] since a more general y_γ is considered.

Let X be a fundamental matrix of the linear homogeneous system
$$x' = A(t)x \tag{1.6}$$
and $X(t,s) = X(t)X(s)^{-1}$. Now we follows [4] to say what is the Cauchy matrix for (1.4).

For $n \in \mathbb{Z}$ and $t \in J_n$ such that $t \ge s$, let $Z_n(t) = X(t, t_n)\mathcal{J}_n(t)$, where $\mathcal{J}_n(t) = I + \int_{t_n}^{t} X(t_n, u)B(u)du$ and assume that
$$\mathcal{J}_n(t) \text{ is invertible, for all } n \in \mathbb{Z} \text{ and } t \in [t_n, t_{n+1}]. \tag{1.7}$$
Let
$$H(n) = Z_n(t_{n+1}), \tag{1.8}$$
for all $n \in \mathbb{Z}$. For $\tau \in \mathbb{R}$, let $k(\tau) \in \mathbb{Z}$ such that $\tau \in J_{k(\tau)}$. Consider $t > s$ such that $k(t) > k(s)$. Then, we define
$$Z(t,s) = Z_{k(t)}(t)[H(k(t)-1)H(k(t)-2)\cdots H(k(s)+1)]H(k(s))^{-1}Z_{k(s)}(s)^{-1}. \tag{1.9}$$
If $t \le s$, by condition (1.7), $Z(t,s) = Z(s,t)^{-1}$ is well defined. So, $Z(t,s)$ is the Cauchy matrix for (1.4). (see [2, 3, 27, 32, 34, 35]).

Consider the difference equation
$$\phi(n+1) = H(n)\phi(n). \tag{1.10}$$
Notice that if $z : \mathbb{R} \to \mathbb{C}$, then $\phi(n) = z(t_n)$ is a solution of (1.10) if z is a solution of (1.4).

It will be prove that $H = (H(n))_{n=-\infty}^{+\infty}$ in (1.8) is almost periodic and that the sequence $h = (h(n))_{n=-\infty}^{+\infty}$, defined by

$$h(n) = \int_{t_n}^{t_{n+1}} X(t_{n+1}, u) f(u) du, \tag{1.11}$$

for all $n \in \mathbb{Z}$, is almost periodic. Based on the exponential dichotomy of (1.10) and the almost periodicity of H and h, it will be proved that the bounded solution c of the discrete system

$$c(n+1) = H(n)c(n) + h(n), \tag{1.12}$$

is almost periodic and the correspondence $h \mapsto c$ is Lipschitz continuous. Then it will be proved that the inhomogeneous linear DEPCAG (1.1) has an analogous almost periodic solution. The dependence of the almost periodic solution can be seen in terms of the almost periodic solution of the discrete part for (1.1) and (1.2), the linear continuous dependence of the almost periodic solution y of (1.1) in terms of f and the same kind of dependence of c of the almost periodic solution of (1.12) in terms of h.

By assuming that L in (1.5) is small enough, an almost periodic solution for (1.2) is obtained in terms of the solution of a difference equation. Finally, it will be proved that the almost periodic solution of (1.2) is exponentially stable as $t \to +\infty$ with respect the solutions of (1.2) for $t \geq 0$. The exponential stability is proved by using a Gronwall inequality on the mentioned difference equation.

This work is organized as follows: Section 2 provides the main definitions, assumptions and facts that will be used. In the Section 3, the existence of almost periodic solutions for (1.1) is studied. In Section 4, that study is extended for (1.2) and deals with asymptotic stability for (1.2) as $t \to +\infty$. An example is given in the last section.

2. Preliminaries

(H6) Assume that (1.10) has an *exponential dichotomy.*

This assumption is equivalent to assume that there is a projection $\Pi : \mathbb{C}^q \to \mathbb{C}^q$ and positive constants ρ, K with $\rho < 1$ such that

$$|\mathcal{G}(n,k)| \leq K\rho^{\pm(n-k)}, \tag{2.1}$$

for all $n, k \in \mathbb{Z} : \pm(n-k) \leq 0$, where

$$\mathcal{G}(n,k) = \begin{cases} \Phi(n)\Pi\Phi(k+1)^{-1}, & \text{if } n > k \\ -\Phi(n)(I-\Pi)\Phi(k+1)^{-1}, & \text{if } n \leq k \end{cases} \tag{2.2}$$

and Φ is a fundamental matrix for the system (1.10). In particular it will be said that system (1.10) is *exponentially stable* as $n \to +\infty$ if it has an exponential dichotomy with $\Pi = I$.

This definition of dichotomy definition has been adapted from that given by Papashinopulos [23] for (1.4) when $\gamma = [\cdot]$. It is an exponential dichotomy for (1.10) which is not obvious to be extended for (1.4) in terms of $Z(t,s)$ except for cases where the projection for exponential dichotomy commutes with $A(t)$ and $B(t)$. Authors did not find any reference containing a definition of exponential dichotomy for (1.4).

We start with a classical notion. A function x is a solution of

$$x'(t) = \tilde{f}(t, x_\gamma(t)), \tag{2.3}$$

where x_γ is defined in (1.3), if

(a) x is continuous on $\mathbb{R}$;
(b) the derivative x' of x exists except possibly at the points $t = t_n$ with $n \in \mathbb{Z}$, where every one-sided derivative exist;
(c) x is a solution of (2.3) except possibly at the points $t = t_n$ with $n \in \mathbb{Z}$.

If $\mathbb{E}$ is a finite dimensional space on $\mathbb{R}$, $D \subseteq \mathbb{R}$ and $g : D \to \mathbb{E}$, then $|g|_\infty = \sup_{t\in D} |g(t)|$. A set $E \subseteq \mathbb{R}$ is called *relatively dense* if there exists a positive real number l such that $E \cap [m, m + l] \neq \phi$ for all $m \in \mathbb{R}$. For $\mathbb{A} \subseteq \mathbb{R}$ an additive group and $(\mathbb{E}, |\cdot|)$ a finite dimensional linear space $g : \mathbb{A} \to \mathbb{E}$ is called *almost periodic* if it is continuous the set of translations $T(g, \varepsilon)$, defined by the set of all $\tau \in \mathbb{A}$ such that $|g(t+\tau) - g(t)| \le \varepsilon$ for all $t \in \mathbb{A}$, is relatively dense for all $\varepsilon > 0$ (see [20, Definition 1.10]). There will be considered the cases $\mathbb{A} = \mathbb{R}$ (almost periodic functions) and $\mathbb{A} = \mathbb{Z}$ (almost periodic sequences). We can notice by following [24, page 201] that (H3), is a definition of almost periodicity for *piecewise* continuous functions. An alternative definition of almost periodicity *for continuous functions* was given by Salomon Bochner [8] (see Fink [20, page 14] for more detailed reference): A function f is almost periodic if every sequence $(f(t_n + t))_{n=1}^{+\infty}$ of translations of f has a subsequence that converges uniformly for $t \in \mathbb{R}$. A function $F : \mathbb{R}\times W \subseteq \mathbb{R}\times\mathbb{E} \to \mathbb{E}^q$ is uniformly almost periodic on W, if the set $T(F, \varepsilon, W)$ which denotes the set of all $\tau \in \mathbb{R}$ such that $|F(t + \tau, w) - F(t, w)| \le \varepsilon$ for all $(t, w) \in \mathbb{R} \times W$, is relatively dense for every $\varepsilon > 0$.

Next, some notation is given. Let $\mathcal{AP}(\mathbb{A}, \mathbb{E})$ be the set of the almost periodic functions from $\mathbb{A}$ to $\mathbb{E}$. The set $(\mathcal{AP}(\mathbb{A}, \mathbb{C}^q), |\cdot|_\infty)$ is a Banach space.

We say that $\big(t_n^{(k)}\big)_{n=-\infty}^{+\infty}$ is *equipotentially almost periodic*, for all $k \in \mathbb{Z}$ if the set

$$\cap_{k\in\mathbb{N}}\big\{T \in \mathbb{Z} : |t_{T+n}^{(k)} - t_n^{(k)}| \le \varepsilon,\ \text{for all } n \in \mathbb{Z}\big\}$$

is relatively dense for all $\varepsilon > 0$.

Since A, B are almost periodic, A, B are bounded. Since $\big(t_n^{(k)}\big)_{n=-\infty}^{+\infty}$ is equipotentially almost periodic for all $k \in \mathbb{Z}$, every sequence $\big(t_n^{(k)}\big)_{n=-\infty}^{+\infty}$ is almost periodic for all $k \in \mathbb{Z}$. So, the sequences $\big(t_n^{(k)}\big)_{n=-\infty}^{+\infty}$ are bounded for all $k \in \mathbb{Z}$ (see [24, Theorem 67]) and there exists the positive real number

$$\theta = \sup_{n\in\mathbb{Z}}(t_{n+1} - t_n). \tag{2.4}$$

Since

$$|Z(t, s)| \le e^{|A|_\infty(t_{n+1}-t_n)} \Big(1 + e^{|A|_\infty(t_{n+1}-t_n)}|B|_\infty(t_{n+1} - t_n)\Big),$$

for all $t, s \in J_n$, it follows that $Z(t, s)$ is bounded. By following [4, 27], we have that $y : \mathbb{R} \to \mathbb{C}^q$ given by

$$\begin{aligned} y(t) = Z_{k(t)}(t) \times \Big(\sum_{k=-\infty}^{+\infty} \mathcal{G}(k(t), k) \int_{t_k}^{t_{k+1}} X(t_{k+1}, u)f(u)du\Big) \\ + \int_{\gamma^0(t)}^{t} X(t, u)f(u)du, \end{aligned} \tag{2.5}$$

where $t \in \mathbb{R}$, will be the unique bounded solution of (1.1) which satisfies (H3) (see Theorem 2 below). Moreover, by taking limits $t \to \gamma^0(t)^+$ and $t \to \gamma^0(t)^-$, we obtain that y is continuous on every t_n and therefore y is almost periodic.

For $\varepsilon > 0$, let Γ_ε be the set of $r \in \mathbb{R}$ such that there is $k \in \mathbb{Z}$ with

$$\sup_{n\in\mathbb{Z}} |t_n^{(k)} - r| \le \varepsilon. \tag{2.6}$$

Denote by $P_r(\varepsilon)$ the set of all $k \in \mathbb{Z}$ satisfying (2.6). Let

$$P_\varepsilon = \cup_{r\in\Gamma_\varepsilon} P_r(\varepsilon).$$

We need the following lemmas.

Lemma 2.1 ([24, Lemma 23]). *Assume that* (H2) *holds. Let $\varepsilon > 0$, $\Gamma \subseteq \Gamma_\varepsilon$, $\Gamma \neq \phi$ and $P \subseteq \cup_{r\in\Gamma} P_r(\varepsilon)$ be such that $P \cap P_r(\varepsilon) \neq \phi$ for all $r \in \Gamma$. Then the set Γ is relatively dense if and only if P is relatively dense.*

Lemma 2.2 ([24, Lemma 25]). *The following statements are equivalent.*

(a) *(H2) holds;*
(b) *The set P_ε is relatively dense for any $\varepsilon > 0$;*
(c) *The set Γ_ε is relatively dense for any $\varepsilon > 0$.*

Lemma 2.3 ([24, Lemma 29]). *Assume that f satisfies* (H3). *Then $\Gamma_\varepsilon \cap T(f, \varepsilon)$ is relatively dense.*

By mean standard arguments, we can prove the following result.

Lemma 2.4. (a) *If f_1, f_2 are functions satisfying (H3), then given $\varepsilon > 0$, $\Gamma_\varepsilon \cap T(f_1, \varepsilon) \cap T(f_2, \varepsilon)$ is relatively dense.*
(b) *If $(g_1(n))_{n=-\infty}^{+\infty}$ and $(g_2(n))_{n=-\infty}^{+\infty}$ are almost periodic solutions, then given $\varepsilon > 0$, $P_\varepsilon \cap T(g_1, \varepsilon) \cap T(g_2, \varepsilon)$ is relatively dense.*

For the following results, we recall that q is the dimension of equation (1.4). Notice that they depends only on the assumptions (H1) and (H3).

Lemma 2.5. *Let θ be as in* (2.4), *$K_0 = \exp(|A|_\infty \theta)$, $K_1 = \sup_{n\in\mathbb{Z}} \exp\big(|A|_\infty |t_{n+1}^{(p)} - \tau|\big)$ and $K_2 = K_0 K_1$. Then*

(a) *$|X(t, s)| \le \sqrt{q} K_0$, for all $t, s \in \mathbb{R}$ such that $|s - t| \le \theta$;*
(b) *If $\tau > 0$, $p \in \mathbb{N}$ and $u \in [t_n, t_{n+1}]$ then*
$$\begin{aligned} &|X(t_{n+p+1}, u + \tau) - X(t_{n+1}, u)| \\ &\le \sqrt{q} \cdot [K_1 |A|_\infty |t_n^{(p)} - \tau| + K_2 |A(\cdot + \tau) - A(\cdot)|_\infty |t_{n+1} - t_n|] \\ &\quad \times \exp\left(|A|_\infty (t_{n+1} - t_n)\right); \end{aligned}$$
(c) *If $\tau > 0$, $p \in \mathbb{N}$ and $t \in [t_n, t_{n+1}]$ then*
$$\begin{aligned} &|X(t + \tau, t_{n+p}) - X(t, t_n)| \\ &\le \sqrt{q} \cdot [K_1 |t_n^{(p)} - \tau| + K_2 |A(\cdot + \tau) - A(\cdot)|_\infty] |t_{n+1}^{(p)} - \tau - (t_{n+1} - t_n)| \\ &\quad \times \exp\Big(|A|_\infty \Big(|t_n^{(p)} - \tau| + \theta\Big)\Big); \end{aligned}$$
(d) *If $\tau > 0$ and $t, s \in \mathbb{R} : |t - s| \le \theta$ then*
$$|X(t + \tau, s + \tau) - X(t, s)| \le \sqrt{q} K_0 |A(\cdot + \tau) - A(\cdot)|_\infty;$$
(e) *If $\tau > 0$, $p \in \mathbb{N}$ and $u \in [t_n, t_{n+1}]$ then*
$$\begin{aligned} &|X(t_{n+p+1}, t_{n+p}) - X(t_{n+1}, t_n)| \\ &\le K_2 |X(u + \tau, t_{n+p}) - X(u, t_n)| + \sqrt{q} K_0 |X(t_{n+p+1}, u + \tau) - X(t_{n+p}, u)|. \end{aligned}$$

Proof. Part (a) follows immediately. To prove (b), assume without loss of generality that $t_{n+p+1} - \tau \geq t_{n+1}$. Note that for $u \in [t_n, t_{n+1}]$,

$$\begin{aligned}\Delta_n(u) &= \int_{t_{n+1}}^{t_{n+p+1}-\tau} X(t_{n+p+1}, \xi+\tau)A(\xi+\tau)d\xi \\ &\quad + \int_u^{t_{n+1}} X(t_{n+p+1}, \xi+\tau)[A(\xi+\tau) - A(\xi)]d\xi + \int_u^{t_{n+1}} \Delta_n(\xi)A(\xi)d\xi,\end{aligned}$$

where $\Delta_n(u) = X(t_{n+p+1}, u+\tau) - X(t_{n+1}, u)$. Then

$$\begin{aligned}|\Delta_n(u)| &\leq \int_{t_{n+1}}^{t_{n+p+1}-\tau} |X(t_{n+p+1}, \xi+\tau)||A(\xi+\tau)|d\xi \\ &\quad + \int_{t_n}^{t_{n+1}} |X(t_{n+p+1}, \xi+\tau)||A(\xi+\tau) - A(\xi)|d\xi \\ &\quad + \int_u^{t_{n+1}} |\Delta_n(\xi)||A(\xi)|d\xi.\end{aligned}$$

So, by Gronwall's inequality the result follows.

Similarly, assume without loss of generality that $t_{n+1} \geq t_{n+p} - \tau$. If $\Delta_n^*(t) = X(t+\tau, t_{n+p}) - X(t, t_n)$, then

$$\begin{aligned}|\Delta_n^*(t)| &\leq \Big|\int_{t_{n+p}-\tau}^{t_n} |A(\xi)||X(\xi+\tau, t_{n+p})|d\xi\Big| \\ &\quad + \int_{t_{n+p}-\tau}^{t_{n+1}} |A(\xi+\tau) - A(\xi)||X(\xi+\tau, t_{n+p})|d\xi + \int_{t_n}^{t} |A(\xi)||\Delta_n^*(\xi)|d\xi,\end{aligned}$$

for $t \in [t_n, t_{n+1}]$. So, by Gronwall's inequality, (c) is obtained. To prove part (d), proceed as in the proof of [22, Proposition 8]. To prove (e), note that

$$\begin{aligned}X(t_{n+p+1}, t_{n+p}) - X(t_{n+1}, t_n) &= X(t_{n+p+1}, u+\tau)[X(u+\tau, t_{n+p}) - X(u, t_n)] \\ &\quad + [X(t_{n+p+1}, u+\tau) - X(t_{n+p}, u)]X(u, t_n)\end{aligned}$$

and apply the previous results. ☐

By Lemma 2.5, the following result is obtained.

Lemma 2.6. *Consider θ defined in (2.4). Let $\varepsilon > 0$, $\tau \in \Gamma_\varepsilon \cap T(A, \varepsilon)$ and $p \in P_\tau(\varepsilon)$. Then there is $K' > 0$ such that for all $n \in \mathbb{Z}$,*

(a) $|X(t_{n+p+1}, u+\tau) - X(t_{n+1}, u)| \leq K'\varepsilon$, *for all* $u \in [t_n, t_{n+1}]$;
(b) $|X(t+\tau, t_{n+p}) - X(t, t_n)| \leq K'\varepsilon$, *for all* $t \in [t_n, t_{n+1}]$;
(c) $|X(t+\tau, s+\tau) - X(t, s)| \leq K'\varepsilon$, *for all* $s, t \in \mathbb{R} : |t-s| \leq \theta$;
(d) $|X(t_{n+p+1}, t_{n+p}) - X(t_{n+1}, t_n)| \leq K'\varepsilon$.

Notice that the existence of $p \in P_\tau(\varepsilon)$ is given by Lemma 2.2 and the existence of $\tau \in \Gamma_\varepsilon \cap T(A, \varepsilon)$ is given by Lemma 2.3.

Lemma 2.7. *The sequence $H = (H(n))_{n=-\infty}^{+\infty}$ given by (1.8) and the sequence $h = (h(n))_{n=-\infty}^{+\infty}$ given by (1.11) are almost periodic.*

Proof. Firstly, notice that $H(n) = X(t_{n+1}, t_n) + \psi(n)$, for all $n \in \mathbb{Z}$, where

$$\psi(n) = \int_{t_n}^{t_{n+1}} X(t_{n+1}, u)B(u)du.$$

From Lemma 2.6 (d), it is not hard to see that $(X(t_{n+1},t_n))_{n=-\infty}^{+\infty}$ is almost periodic. ψ is also almost periodic. In fact, let $\varepsilon > 0$. From Lemma 2.4, $\Gamma = T(A,\varepsilon)\cap T(B,\varepsilon)\cap\Gamma_\varepsilon$ is relatively dense. Let $p \in P = \cup_{\tau\in\Gamma}P_\tau(\varepsilon)$, so there is $\tau\in\Gamma$ such that $p\in P_\tau(\varepsilon)$. Then, for all $n\in\mathbb{Z}$ it is obtained

$$
\begin{aligned}
&\psi(n+p)-\psi(n)\\
&=\int_{t_{n+p}}^{t_{n+p+1}} X(t_{n+p+1},u)B(u)du - \int_{t_n}^{t_{n+1}} X(t_{n+1},u)B(u)du\\
&=\int_{t_{n+p}}^{t_{n+p+1}} X(t_{n+p+1},u)B(u)du - \int_{t_n+\tau}^{t_{n+p+1}} X(t_{n+p+1},u)B(u)du\\
&\quad+\int_{t_n+\tau}^{t_{n+p+1}} X(t_{n+p+1},u)B(u)du - \int_{t_n}^{t_{n+1}} X(t_{n+p+1},u+\tau)B(u+\tau)du\\
&\quad+\int_{t_n}^{t_{n+1}} X(t_{n+p+1},u+\tau)B(u+\tau)du - \int_{t_n}^{t_{n+1}} X(t_{n+1},u)B(u)du,\\
&=\int_{t_{n+p}}^{t_n+\tau} X(t_{n+p+1},u)B(u)du + \int_{t_{n+1}+\tau}^{t_{n+p+1}} X(t_{n+p+1},u)B(u)du\\
&\quad+\int_{t_n}^{t_{n+1}} [X(t_{n+p+1},u+\tau)B(u+\tau) - X(t_{n+1},u)B(u)]du.
\end{aligned}
$$

By Lemmas 2.5 and 2.6, there are positive constants M and K' such that

$$|\psi(n+p)-\psi(n)| \le |t_n^{(p)}-\tau|M + |t_{n+1}^{(p)}-\tau|M + K'\varepsilon \le [2M+K']\varepsilon$$

for all $n\in\mathbb{Z}$. So, $p\in T(\psi,[2M+K']\varepsilon)$. Since p was taken arbitrarily in P, $P\subseteq T(\psi,[2M+K']\varepsilon)$. By Lemma 2.1, P is relatively dense. So, $T(\psi,[2M+K']\varepsilon)$ is relatively dense. Since $\varepsilon>0$ is arbitrary, ψ is almost periodic. Therefore, $H=(H(n))_{n=-\infty}^{+\infty}$ is almost periodic.

In the similar way, h is almost periodic. □

3. Inhomogeneous linear DEPCAG

To study the existence of an almost periodic solution of (1.1), recall that $f\in\mathcal{AP}(\mathbb{R},\mathbb{C}^q)$.

By the variation constants formula [4, 27],

$$y(t) = Z(t,k(t))c(k(t)) + \int_{\gamma^0(t)}^{t} X(t,u)f(u)du, \tag{3.1}$$

is obtained, for all $t\in\mathbb{R}$, where c is solution of the discrete system (1.12). By taking $t\to t_{n+1}^-$, it is obtained a solution y for (1.1) such that $y(t_n)=c(n)$ for all $n\in\mathbb{Z}$. It will be proved that y is almost periodic.

If c is the bounded solution of equation (1.12) then

$$c(n) = \sum_{k=-\infty}^{+\infty}\mathcal{G}(n,k)h(k), \tag{3.2}$$

where the Green matrix $\mathcal{G}(n,k)$ is given by (2.2) and h is given by (1.11).

From (3.1) and (3.2), y is the bounded solution of (1.1) and satisfies (2.5). This relation shows y as a bounded linear function of f.

By using the equivalent definition of almost periodicity due to Bochner, two important facts are obtained.

Lemma 3.1 ([22, Proposition 7] and [37]). *A sequence $x = (x(n))_{n=-\infty}^{+\infty}$ is almost periodic if and only if for any integer sequences $(k'_j)_{j=1}^{+\infty}$ and $(\ell'_j)_{j=1}^{+\infty}$ there are subsequences $k = (k_j)_{j=1}^{+\infty}$ and $\ell = (\ell_j)_{j=1}^{+\infty}$ of $(k'_j)_{n=1}^{+\infty}$ and $(\ell'_j)_{n=1}^{+\infty}$ respectively, such that*

$$T_k T_\ell x = T_{k+\ell} x,$$

uniformly on $\mathbb{Z}$, where $k+\ell = (k_j+\ell_j)_{j=1}^{+\infty}$, $T_m x(n) = \lim_{j\to+\infty} x(n+m_j)$ and $m = (m_j)_{j=1}^{+\infty} \in \{k, \ell, k+\ell\}$, for all $n \in \mathbb{Z}$.

Theorem 3.2. *Assume that hypotheses* (H1), (H3) *and* (H6) *are satisfied. If c is given by* (3.2)*, then c is the unique almost periodic solution of the linear inhomogeneous difference system* (1.12)*. Moreover,*

$$|c|_\infty \le \frac{2K}{1-\rho}|h|_\infty. \tag{3.3}$$

Proof. By Lemmas 2.7 and 3.1, for any integer sequences $(k'_j)_{j=1}^{+\infty}$ and $(\ell'_j)_{j=1}^{+\infty}$ there are subsequences $k = (k_j)_{j=1}^{+\infty}$ and $\ell = (\ell_j)_{j=1}^{+\infty}$ of $(k'_j)_{n=1}^{+\infty}$ and $(\ell'_j)_{n=1}^{+\infty}$ respectively, such that $T_{k+\ell}H = T_k T_\ell H$ and $T_{k+\ell}h = T_k T_\ell h$, uniformly on $\mathbb{Z}$.

Now, notice that c given by (3.2) is the only solution of (1.12) which is bounded. Moreover, $z = T_{k+\ell}c$ and $z = T_k T_\ell c$ are bounded solutions of

$$\begin{aligned} z(n+1) &= T_{k+\ell}H(n)z(n) + T_{k+\ell}h(n),\\ z(n+1) &= T_k T_\ell H(n)z(n) + T_k T_\ell h(n), \end{aligned}$$

respectively. By uniqueness $T_{k+\ell}c = T_k T_\ell c$. So, $c = (c(n))_{n=-\infty}^{+\infty}$ is an almost periodic sequence. Since c is given by (3.2), it is the only bounded solution of (1.12) and satisfies (3.3). □

Theorem 3.3. *Consider θ defined in* (2.4)*. Assume that hypotheses* (H1), (H3) *and* (H6) *are satisfied. Then* (1.1) *has a unique almost periodic solution. Moreover,*

$$|y|_\infty \le K_3 |f|_\infty, \tag{3.4}$$

where $K_3 = [\sqrt{q}K_0(1+|B|_\infty\theta)\frac{2K}{1-\rho}+1]\sqrt{q}K_0\theta$.

Proof. Let $\varepsilon > 0$. By Lemma 2.4, there is $\tau \in T(A,\varepsilon)\cap T(B,\varepsilon)\cap T(f,\varepsilon)$ and $p \in P_\varepsilon \cap T(c,\varepsilon)$. Let y be the solution of (1.1). Fix $t \in \mathbb{R}$ and let $n \in \mathbb{Z}$ such that $t \in J_n$. Then

$$\begin{aligned} &y(t+\tau) - y(t)\\ &= [X(t+\tau, t_{n+p}) - X(t,t_n)]c(n+p) + X(t,t_n)[c(n+p) - c(n)]\\ &\quad + \int_{t_{n+p}-\tau}^{t} [X(t+\tau, u+\tau) - X(t,u)]B(u+\tau)du \cdot c(n+p)\\ &\quad + \int_{t_{n+p}-\tau}^{t} X(t,u)B(u+\tau)du \cdot [c(n+p) - c(n)]\\ &\quad + \int_{t_{n+p}-\tau}^{t} X(t,u)[B(u+\tau) - B(u)]du \cdot c(n) + \int_{t_{n+p}-\tau}^{t_n} X(t,u)B(u)du \cdot c(n) \end{aligned}$$

$$\begin{aligned}&+\int_{t_{n+p}-\tau}^{t}[X(t+\tau,u+\tau)-X(t,u)]f(u+\tau)du\\&+\int_{t_{n+p}-\tau}^{t}X(t,u)[f(u+\tau)-f(u)]du+\int_{t_{n+p}-\tau}^{t_n}X(t,u)f(u)du\end{aligned}$$

So, by Lemmas 2.5 and 2.6, there is $K'>0$ large enough such that $|y(t+\tau)-y(t)|\le\varepsilon K'$ for all $t\in\mathbb{R}$. Since $\tau>0$ was taken arbitrarily in $T(A,\varepsilon)\cap T(B,\varepsilon)\cap T(f,\varepsilon)$, this set is contained in $T(x,\varepsilon K')$. By Lema 2.4, $T(x,\varepsilon K')$ is relatively dense. Since $\varepsilon>0$ was taken arbitrarily, y is an almost periodic solution of (1.1). From (2.5), it can be noticed that y is the unique bounded solution of DEPCAG (1.1). So, y is the unique almost periodic solution of DEPCAG (1.1).

Since $Z(t,s)$ is bounded and the relations (2.4), (2.5) and (3.3) are satisfied, we have inequality (3.4). □

4. The nonlinear equation (1.2)

To study the existence of an almost periodic solution of (1.2), recall that $W\subseteq(\mathbb{C}^q)^\ell$ is not empty and the set

$$T(F,\varepsilon,W)=\{\tau\in\mathbb{R}:|F(t+\tau,w)-F(t,w)|\le\varepsilon,\ \text{for all }(t,w)\in\mathbb{R}\times W\}$$

is relatively dense for all$\varepsilon>0$.

Lemma 4.1. *Let $y:\mathbb{R}\to\mathbb{C}^q$ an almost periodic function. Assume that* (H2) *is satisfied and F satisfies* (H4)*. Then $F(t,y_\gamma(t))$ satisfies* (H3)*.*

Proof. Let $\varepsilon>0$ and $\tau\in T(y,\varepsilon)\cap T(F,\varepsilon,W)$. Since y is almost periodic, it is uniformly continuous. So, there is $\delta>0$ such that $s,t\in\mathbb{R}:|s-t|\le\delta$ implies that $|y(t)-y(s)|\le\varepsilon$. Since $P_\tau(\delta)\ne\phi$, $|\gamma^{p_j}(t+\tau)-(\gamma^{p_j}(t)+\tau)|\le\delta$, for $j=1,\dots,\ell$. Moreover,

$$\begin{aligned}&|F(t+\tau,y_\gamma(t+\tau))-F(t,y_\gamma(t))|\\&\le|F(t+\tau,y_\gamma(t+\tau))-F(t,y_\gamma(t+\tau))|+|F(t,y_\gamma(t+\tau))-F(t,y_\gamma(t))|\\&\le\varepsilon+L\ell\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ was taken arbitrarily, $F(t,y_\gamma(t))$ satisfies (H3). □

Theorem 4.2. *Assume that* (H1), (H2) *and* (H6) *hold. Assume that F satisfies* (H4)*. If*

$$2\frac{KL\ell}{1-\rho}<1, \tag{4.1}$$

then (1.2) *has an almost periodic solution.*

Proof. Let

$$(\mathcal{T}c)(n)=\sum_{k=-\infty}^{+\infty}\mathcal{G}(n,k)h(k,\hat{c}(k)), \tag{4.2}$$

where $h(n,\hat{c}(n))=\int_{t_n}^{t_{n+1}}X(t_{n+1},s)F(s,\hat{c}(n))ds$ and $\mathcal{G}(n,k)$ is given in (2.2) and $\hat{c}(n)=(c(n-p_1),\dots,c(n-p_\ell))$.

If c is a fixed point of the operator defined by (4.2) then c is solution of the difference equation

$$c(n+1)=H(n)c(n)+h(n,\hat{c}(n)). \tag{4.3}$$

If c is almost periodic then $h(n, \hat{c}(n))$ is almost periodic. In that case, $\mathcal{T}c$ is almost periodic. So, $\mathcal{T}(\mathcal{AP}(\mathbb{Z}, \mathbb{C}^q)) \subseteq \mathcal{AP}(\mathbb{Z}, \mathbb{C}^q)$. Moreover,

$$|(\mathcal{T}c_1)(n) - (\mathcal{T}c_2)(n)| \le 2\frac{KL\ell}{1-\rho}|c_1 - c_2|_\infty.$$

If (4.1) holds, $\mathcal{T} : \mathcal{AP}(\mathbb{Z}, \mathbb{C}^q) \to \mathcal{AP}(\mathbb{Z}, \mathbb{C}^q)$ is a contracting mapping. By the Banach fixed point theorem, there is $c \in \mathcal{AP}(\mathbb{Z}, \mathbb{C}^q)$ a unique fixed point for $\mathcal{T}$. Therefore, equation (4.3) has an almost periodic solution c. By Theorem 3.3, it can be constructed a solution y of (1.2) which is almost periodic. □

The ret of this article is devoted to the exponential stability of the almost periodic solution of (1.2), whose existence was proved in the previous section. First, we say what we understand by exponential stability.

Assume that $p_j > 0$ for $j = 1, \dots, \ell$. Let $p = \max_{j=1,\dots,\ell} p_j$. A solution y of (1.2), is *exponentially stable* as $t \to +\infty$ if there is $\alpha \in]0, 1[$ such that given $\varepsilon > 0$, there exists $\delta > 0$ such that $\tilde{y} = \tilde{y}(t)$ is a solution of (1.2) defined for $t \ge t_0$ then

$$\max_{j=0,1,\dots,p} |y(t_{-j}) - \tilde{y}(t_{-j})| \le \delta$$

implies

$$|\tilde{y}(t) - y(t)| \le \varepsilon\alpha^t, \quad \text{for all } t \ge t_0. \tag{4.4}$$

This kind of stability is in the half axis although the solution being exponentially stable is defined on the whole axis.

This definition is independent on the choice of t_0. Any other value could be chosen.

Let $\Phi(n, k) = \Phi(n)\Phi(k)^{-1}$, for all $(n, k) \in \mathbb{Z}^2$. Assume that the difference system (1.10) is *exponentially stable* as $n \to +\infty$, i.e., assume that there are positive constants ρ, K with $\rho < 1$ and $K \ge 1$ such that

$$|\Phi(n, k + 1)| \le K\rho^{n-k}, \tag{4.5}$$

for all $n, k \in \mathbb{Z} : n \ge k$.

By Theorem 4.2 and the exponential stability, the condition

$$\frac{KL\ell}{1-\rho} < 1, \tag{4.6}$$

ensures the existence of a unique almost periodic solution $y = y(t)$ of (1.2) defined for all $t \in \mathbb{R}$.

For (1.4), notice that an exponential stability for (1.10) implies a direct notion on exponential stability on $Z(t, s)$. In fact, from (1.9) and (4.5), it is obtained, for $n > k$, $t \in J_n$ and $s \in]t_k, t_{k+1}]$, that

$$|Z(t, s)| \le K_4\rho^{n-k},$$

where $K_4 = K\sqrt{q}K_0^2[1 + \sqrt{q}K_0|B|_\infty\theta]^2$ and θ is given in (2.4). Since $t - s \le t_{n+1} - t_k \le \theta(n - k + 2)$,

$$|Z(t, s)| \le K_4\rho^{-2}\rho^{\frac{t-s}{\theta}}.$$

If $\eta_0, \eta_1, \dots, \eta_p \in \mathbb{C}^q$, it is not hard to see that the difference system (4.3) has a solution $\tilde{c} = \tilde{c}(n)$ defined for $n \ge 0$ with the initial conditions $\tilde{c}(-j) = \eta_j \in \mathbb{C}^q$ for

$j = 0, 1, \dots, p$. Let

$$\begin{aligned}\tilde{y}(t) = Z_{k(t)}(t)\Big(&\Phi(n,0)\tilde{c}(0) + \sum_{k=0}^{n-1}\Phi(n,k+1)\int_{t_k}^{t_{k+1}} X(t_{k+1},u)\\ &\times F(u,\tilde{c}(k-p_1),\dots,\tilde{c}(k-p_\ell))du\Big)\\ &+\int_{\gamma^0(t)}^{t} X(t,u)F(u,\tilde{c}(n-p_1),\dots,\tilde{c}(n-p_\ell))du,\end{aligned}\tag{4.7}$$

where $t \geq t_0$. Then, $\tilde{y} = \tilde{y}(t)$ is the unique solution of (1.2) with $t \geq t_0$ and fixed initial conditions $\tilde{y}(t_{-j}) = \eta_j$ for $j = 0, 1, \dots, p$.

Theorem 4.3. *Assume that* (H1), (H2), (H4) *hold and that the difference system* (1.10) *has an exponential stability as $n \to +\infty$. Assume that* (4.5) *and* (4.6) *hold. If y is the almost periodic solution of* (1.2) *and $\tilde{y}$ is solution of* (1.2) *for $t \geq t_0$ with initial conditions $\tilde{y}(t_{-j}) = \eta_j$ for $j = 0, 1, \dots, p$, then there is $\tilde{K} > 0$ such that*

$$|y(t) - \tilde{y}(t)| \leq \tilde{K}\left(\rho(1 + KL\ell\rho^{-p})\right)^n \max_{j=0,1,\dots,p}|c(-j) - \eta_j|,\tag{4.8}$$

where $t \geq t_0$. Hence, if

$$\frac{KL\ell}{1-\rho} < \rho^{p-1}\tag{4.9}$$

then y is exponentially stable.

Proof. Consider that $c(n) = y(t_n)$ and $\tilde{c}(n) = \tilde{y}(t_n)$ for all integer $n \geq n_0$. Let $u(n) = c(n) - \tilde{c}(n)$ for all $n \in \mathbb{Z}$. Then, for $n_0 \in \mathbb{Z}$,

$$\begin{aligned}|u(n)| &\leq |\Phi(n,0)||u(0)| + \sum_{k=0}^{n-1}|\Phi(n,k+1)||F(k,\hat{c})(n) - F(k,\hat{\tilde{c}}(n))|\\ &\leq K\rho^n|u(0)| + KL\sum_{k=0}^{n-1}\rho^{n-k}\sum_{j=1}^{\ell}|u(k-p_j)|.\end{aligned}$$

Let $\omega(n) = \rho^{-n}\sum_{j=1}^{\ell}|u(n-p_j)|$ and $v(n) = \rho^{-n}|u(n)|$. Then

$$\rho^{-n}u(n) \leq K|u(0)| + KL\sum_{k=0}^{n-1}\omega(k)$$

Note that

$$\omega(n) = \sum_{j=1}^{\ell}\rho^{-p_j}\rho^{(-n-p_j)}|u(n-p_j)| \leq \rho^{-p}\sum_{j=1}^{\ell}v(n-p_j).$$

For $n \geq 0$,

$$v(n) \leq Kv(0) + KL\sum_{k=0}^{n-1}\omega(k).$$

Let $z_n = \max\{|v(m)| : m = -p, -p+1, \dots, n\}$. Then, $\omega(n) \leq \rho^{-p}\ell z_n$, for all $n \geq 0$. Hence,

$$v(n) \leq Kv(0) + KL\rho^{-p}\ell\sum_{k=0}^{n-1}z_k.$$

Let $m_n \in \{-p, n-p+1, \dots, n\}$ such that $z_n = v(m_n)$.

If $m_n \geq 0$, then $z_n \leq Kv(0) + KL\ell\rho^{-p}\sum_{k=0}^{m_n-1} z_k$. Hence,

$$z_n \leq Kv(0) + KL\ell\rho^{-p}\sum_{k=0}^{n-1} z_k.$$

If $m_n < 0$, then there is $j_0 \in \{1, \dots, p\}$ such that $m_n = n - j_0$. Since $K \geq 1$, $z_n \leq Kz_0$. So,

$$z_n \leq Kz_0 + KL\ell\rho^{-p}\sum_{k=0}^{n-1} z_k,$$

for all $n \geq 0$. By Gronwall's inequality,

$$z_n \leq (1 + KL\ell\rho^{-p})^n z_0.$$

So, for all $n \geq 0$,

$$|c(n) - \tilde{c}(n)| \leq K\rho^n(1 + KL\ell\rho^{-p})^n \max_{j=0,1,\dots,p} |c(-j) - \tilde{c}(-j)|. \tag{4.10}$$

By Lemma 2.5, there is a positive constant K_0 such that $|X(t,u)| \leq \sqrt{q}K_0$, for all $u \in J_{k(t)}$ and $|Z(t,\gamma^0(t))| \leq \sqrt{q}K_0(1 + \sqrt{q}K_0|B|_\infty\theta)$ for all $t \geq t_0$. By relation (4.7), for $t \geq t_0$, there is a positive constant K' such that

$$|y(t) - \tilde{y}(t)| \leq K'|c(n) - \tilde{c}(n)|. \tag{4.11}$$

This inequality show a Lipschitz continuous relation $\tilde{c} \mapsto \tilde{y}$.

By combining (4.10) and (4.11), this result is proved with $\tilde{K} = K'K$.

Notice that (4.9) implies (4.6). Then Theorem 4.2 ensures the existence of the unique almost periodic solution of (1.2) which is exponentially stable. In fact, let $\alpha = \rho(1 + KL\ell\rho^{-p})$. By (4.9), $\alpha < 1$. For $\varepsilon > 0$ consider $\delta = \frac{\varepsilon}{\tilde{K}}$. By (4.8), (4.4) is satisfied and y is exponentially stable. □

In the previous theorem, condition (4.9) is impled and is slightly stronger than the condition of existence (4.6).

5. Examples

5.1. **Exponential Dichotomy.** It is not obvious how to extend the exponential dichotomy from the difference equation (1.10) to (1.4). Akhmeth studied this topic in [6]; other helpful references are [13, 23, 27, 29]. We could consider an intuitively direct definition given by the existence of a projection Π_* and positive constants M and α such that

$$\begin{aligned} |Z(t,t_0)\Pi_* Z(s,t_0)^{-1}| &\leq Me^{-\alpha(t-s)}, \quad \text{if } t \geq s \\ |Z(t,t_0)(I - \Pi_*)Z(s,t_0)^{-1}| &\leq Me^{\alpha(t-s)}, \quad \text{if } t \leq s. \end{aligned} \tag{5.1}$$

However, if we take

(1) $t_n = \nu n + q_n$, where ν is a positive constant and $(q_n)_{n=1}^{+\infty}$ is an almost periodic sequence in $[0,\nu[$ such that $\Delta_n = q_{n+1} - q_n \to 0$ as $n \to \pm\infty$,
(2) $A(t) = 0$
(3) and $B(t) = \operatorname{diag}(\lambda_0(t), \lambda_1(t))$, where $\lambda_0(t) = -\frac{2}{\pi} + \sin\big(\frac{2\pi}{\nu}(t - q_n)\big)$ and $\lambda_1(t) = -\lambda_0(t)$ for all $t \in [t_n, t_{n+1}[$, $n \in \mathbb{Z}$,

then, for

$$g_0(\delta) = \int_{t_n}^{t_n+\delta} \lambda_0(\xi)d\xi = -\frac{\nu}{2\pi}\Big(4\frac{\delta}{\nu} - 1 + \cos\big(2\pi\frac{\delta}{\nu}\big)\Big),$$

we have that $g_0\big(\frac{\pi}{4}\nu\big) = 0$ and

$$\lim_{\delta\to(t_{n+1}-t_n)^-} g_0(\delta) = \frac{\nu}{2\pi}\Big(-4\big(1+\frac{\Delta_n}{\nu}\big) - 1 + \cos\big(\frac{\Delta_n}{\nu}\big)\Big) = -\frac{2\nu}{\pi} + \beta_n,$$

where $\beta_n = \frac{\nu}{2\pi}\big(-\frac{4\Delta_n}{\nu} - 1 + \cos\big(\frac{\Delta_n}{\nu}\big)\big) \to 0$ as $n \to \pm\infty$. Note that for n large enough, there is $\alpha > 0$ such that $-\frac{2\nu}{\pi} + \beta_n \le -\alpha$. This is equivalent to,

$$\int_{t_n}^{t_n+\frac{\pi}{4}\nu} \lambda_1(\xi)d\xi = -g_0\big(\frac{\pi}{4}\nu\big) = 0$$

and $\lim_{t\to t_{n+1}^-} -g_0(t) \ge \frac{2\nu}{\pi} - \beta_n \ge \alpha$ for n large enough. So, the exponential dichotomy on the difference equation (1.10) can be written as (2.1) for $\Pi = \mathrm{diag}(1,0)$ but there is no Π_* such that condition (5.1) is satisfied.

Notice that a dichotomy condition on the ordinary differential equation (1.6) implies an exponential dichotomy on the difference equation (1.10) [23, Proposition 2] when $|B(t)|$ is small enough. However, an exponential dichotomy for the difference equation on (1.10) is not a necessary condition for an exponential dichotomy for the ordinary differential system (1.6). In fact, let's consider $A(t) = 0$ and $B(t) = \mathrm{diag}\,\big(-\frac{3}{2}, \frac{1}{2}\big)$. Then the exponential dichotomy for difference system (1.10) is satisfied, with no exponential dichotomy for the ordinary differential system (1.6).

5.2. **Constant coefficients.** Assume that in (1.2), $A(t) = A_0$ and $B(t) = B_0$ are constants matrices and $F(t,\cdot)$ is almost periodic. Then (1.2) becomes

$$y'(t) = A_0y(t) + B_0y(\gamma^0(t)) + F(t, y_\gamma(t)),\ t \in \mathbb{R}. \tag{5.2}$$

Assume that $t_{n+1} - t_n = \nu + \Delta_n$, where $\Delta_n \to 0$ as $n \to \pm\infty$,that A_0 and

$$H(n) = e^{(\nu+\Delta_n)A_0}[I + A_0^{-1}(I - e^{-(\nu+\Delta_n)A_0})B_0]$$

are invertible matrices, for all $n \in \mathbb{Z}$.

By using $\sigma(H(n))$ as the usual notation for the spectrum of the matrix $H(n)$, assume that

$$\sigma(H(n)) \subseteq \{z \in \mathbb{C} : |z| < R \text{ or } 1 + R < |z|\},$$

for all $n \in \mathbb{Z}$, where $R < 1$ and that L in (1.5) satisfies (4.6). Then, (5.2) has an almost periodic solution. In particular, it is obtained when the elements of $\sigma(A_0)$ have non zero real part and $|B_0|$ is small enough.

Now, assume that

$$\sigma(H(n)) \subseteq \{z \in \mathbb{C} : |z| < R\}$$

where $R < 1$ and that L in (1.5) satisfies the condition (4.9). Then, (5.2) has an almost periodic solution which is exponentially stable. In particular, it is obtained when the elements of $\sigma(A_0)$ have negative real part and $|B_0|$ is small enough.

Assume that $A_0 = 0$, that $H(n) = I + (\nu + \Delta_n)B_0$ is invertible for all $n \in \mathbb{Z}$, that $\sigma(B_0) \subseteq \{z \in \mathbb{C} : |z| < 1/(\nu + r)\}$, where $r = \max \Delta_n$ and that $\nu + \Delta_n$ and L in (1.5) satisfies (4.6). Then (5.2) has an almost periodic solution. We can notice that it behaves as a difference equation.

Acknowledgments. S. Castillo was supported by DIUBB 110908 2/R. M. Pinto was supported by FONDECYT 1120709 and DGI MATH UNAP 2009.

References

[1] E. Ait Dads, L. Lhachimi; Pseudo almost periodic solutions for equation with piecewise constant argument. *J. of Math. Analysis and Applications.* **371** (2010), 842–854.

[2] M. U. Akhmet; Integral manifolds of differential equations with piecewise constant argument of generalized type, *Nonlinear Analysis TMA* **66** (2) (2007), 367–383.

[3] M.U. Akhmet; Almost periodic solutions of differential equations with piecewise constant argument of generalized type, *Nonlinear Analysis: Hybrid Systems* **2** (2008) 456–467.

[4] M. U. Akhmet; Stability of differential equations with piecewise constant arguments of generalized type, *Nonlinear Analysis: TMA* **68** (2008) 794–803.

[5] M. U. Akhmet, C. Buyukadali; Differential equations with state-dependent piecewise constant argument, *Nonlinear Analysis: TMA* **72** (2010) 4200–4210.

[6] M. U. Akhmet; Exponentially Dichotomous Linear systems of Differential Equations with Piecewise Constant Argument, *Discontinuity, Nonlinearity, and Complexity*, **1** 4 (2012) 337-352.

[7] A. S. Besicovitch; *Almost Periodic Functions.* Dover Publications, Inc., New York, 1955.

[8] S. Bochner. Beitrage zur Theorie der fastperiodische Funktionen. I: Funktionen einer Variablen, *Math. Ann.* **96** (1927) 119–147.

[9] H. Bohr; Zur Theorie der fastperiodischen Funktionen I, *Acta Math.* **45** (1925) 29–127.

[10] H. Bohr; *Almost Periodic Functions.* Chelsea Publishing Company, New York, NY, 1947.

[11] J. Cao, M. Lin, Y. Xia; The existence of almost periodic solutions of certain perturbation systems, *J. of Math. Analysis and Applications*, **310** (2005), 81–96.

[12] J. Cao, Y. Xia; Almost periodicity in an ecological model with M-predators and N-preys by "pure-delay type" system, *Nonlinear Dynamics* **39** (2005), 275–304.

[13] A. Chávez, S. Castillo, M. Pinto; Discontinuous almost periodic type functions, almost automorphy of solutions of differential equations with discontinuous delay and applications. *Electronic Journal of Qualitative Theory of Differential Equations* **2014** (2015), no. 75, 1-17.

[14] K. Chiu; Stability of oscillatory solutions of differential equations with a general piecewise constant argument, *Electron. J. Qual. Theory Differ. Equ.* (2011) **88**, 15 pp.

[15] K. Chiu, M. Pinto; Periodic solutions of differential equations with a general piecewise constant argument and applications, *Electron. J. Qual. Theory Differ. Equ.* (2010) **46**, 19 pp.

[16] K. S. Chiu, M. Pinto, J.C. Jeng; Existence and global convergence of periodic solutions in recurrent neural network models with a general piecewise alternately advanced and retarded argument, *Acta Applicandae Mathematicae* (2013) doi: 10.1007/s10440-013-9863-y

[17] K. L. Cooke, J. Wiener; Retarded differential equations with piecewise constant delays, *J. of Math. Analysis and Applications*, **99** (1984), 265–297.

[18] W. A. Coppel; *Dichotomies in Stability Theory.* Lecture Notes in Mathematics, Vol. 629. Springer-Verlag, Berlin-New York, 1978.

[19] C. Cuevas, M. Pinto; Existence and uniqueness of pseudo almost periodic solutions of semilinear Cauchy problems with non dense domain, *Nonlinear Analysis TMA.* **45** (8) (2001) 73–83.

[20] A. M. Fink; *Almost Periodic Differential Equations.* Lecture Notes in Mathematics **377** Springer, Berlin, 1974.

[21] M. Han, Z. Huang, Y. Xia; Existence of almost periodic solutions for forced perturbed systems with piecewise constant argument, *J. of Math. Analysis and Applications.* **333** (2007), 798–816.

[22] J. Hong, R. Yuan; The existence of almost periodic solutions for a class of differential equations with piecewise constant argument, *Nonlinear Analysis TMA.* **28** (8) (1997) 1439–1450.

[23] G. Papaschinopoulos; Exponential dichotomy, topological equivalence and structural stability for differential equations with piecewise constant argument. *Analysis* **14** n 2-3 (1994) 239–247.

[24] N. A. Perestyuk and A. M. Samoilenko; *Impulsive Differential Equations.* World Scientific, 1995.

[25] M. Pinto; Asymptotic equivalence of nonlinear and quasi linear differential equations with piecewise constant arguments, *Mathematical and Computer Modelling.* **49** (2009) 1750–1758.

[26] M. Pinto; Dichotomy and existence of periodic solutions of quasilinear functional differential equations. *Nonlinear Analysis TMA.* **72** (3-4) (2010) 1227–1234.

[27] M. Pinto; Cauchy and Green matrices type and stability in alternately advanced and delayed differential systems. *J. of Difference Equations and Applications.* **17** (2011), no. 2, 235–254.

[28] M. Pinto; Pseudo-almost periodic solutions of neutral integral and differential equations with applications, *Nonlinear Analysis TMA.* **72** (2010) 4377–4383.

[29] M. Pinto; Dichotomies and asymptotic equivalence in alternatively advanced and delayed systems, In preparation, 2014.

[30] M. Pinto, G. Robledo; Existence and stability of almost periodic solutions in impulsive neural network models. *Applied Mathematics and Computation* **217** (2010), no. 8, 4167–4177.

[31] M. Pinto, G. Robledo; Controllability and observability for a linear time varying system with piecewise constant delay. *Acta Applicandae Mathematicae* (2014) doi: 10.1007/s10440-014-9954-4.

[32] S. M. Shah, J. Wiener; Advanced differential equations with piecewise constant argument deviations. *Internat. J. Math. and Math. Sci.* **6** n 4 (1983) 671–703.

[33] N. Van Minh; Almost periodic solutions of C-well-posed evolution equations, *Math. J. Okayama Univ.* **48** (2006), 145–157.

[34] J. Wiener; Differential equations with piecewise constant delays, *Trends in theory and practice of nonlinear differential equations (Arlington, Tex., 1982)*, *Lecture Notes in Pure and Appl. Math.* **90** Dekker, New York, (1984) 547–552.

[35] J. Wiener; *Generalized Solutions of Functional Differential Equations.* World Scientific, 1993.

[36] R. Yuan; The existence of almost periodic solutions of retarded differential equations with piecewise constant argument, *Nonlinear Analysis TMA.* **48** (7) (2002), 1013–1032.

[37] C. Zhang; *Almost Periodic Type Functions and Ergodicity.* Science Press, Beijing; Kluwer Academic Publishers, Dordrecht, 2003.

Samuel Castillo

Departamento de Matemática, Facultad de Ciencias, Universidad del Bío-Bío, Casilla 5-C, Concepciíon, Chile

E-mail address: `scastill@ubiobio.cl`

Manuel Pinto

Departamento de Matemática, Facultad de Ciencias, Universidad de Chile, Casilla 653, Santiago, Chile

E-mail address: `pintoj@uchile.cl`