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STABILITY WITH RESPECT TO INITIAL TIME DIFFERENCE FOR GENERALIZED DELAY DIFFERENTIAL EQUATIONS

RAVI AGARWAL, SNEZHANA HRISTOVA, DONAL O'REGAN

Abstract. Stability with initial data difference for nonlinear delay differential equations is introduced. This type of stability generalizes the known concept of stability in the literature. It gives us the opportunity to compare the behavior of two nonzero solutions when both initial values and initial intervals are different. Several sufficient conditions for stability and for asymptotic stability with initial time difference are obtained. Lyapunov functions as well as comparison results for scalar ordinary differential equations are employed. Several examples are given to illustrate the theory.

1. Introduction

One of the main problems in the qualitative theory of differential equations is stability of the solutions. Stability gives us the opportunity to compare the behavior of solutions starting at different points. Often in real situations it may be impossible to have only a change in the space variable and to keep the initial time or the initial time interval unchanged. This situation requires introducing and studying a new generalization of the classical concept of stability which involve the change of both the initial time/interval and the initial points/functions. The concept of stability with initial time difference is a generalization of the classical concept of stability of a solution.

Recently, various types of stability with initial time difference were studied for

- ordinary differential equations ([3], [14]-[19], [22], [23]);
- fuzzy differential equations ([21]);
- fractional differential equations ([20]).

We note that stability with initial time difference for delay differential equations was initiated recently and some initial results were published in [10], [13].

In the present paper, we study the stability with initial data difference for delay differential equations based on the application of Lyapunov's functions and the Razumikhin method. The derivative of Lyapunov functions with respect to the given equations and initial time difference is defined in an appropriate way. Comparison results for ordinary differential equation with a parameter are employed. Several examples are given to illustrate the theoretical results

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2. Preliminary notes and results

Let $r_k > 0$, $(k = 1, 2, \dots, m)$, be given finite numbers, $R_+ = [0, \infty)$. Define delay operators $G_k : C([-r_k, \infty), \mathbb{R}^n) \to \mathbb{R}^n$, $(k = 1, 2, \dots, m)$, such that for any function $x \in C([-r_k, \infty), \mathbb{R}^n)$, and any point $t \in \mathbb{R}_+$ and $k = 1, 2, \dots, m$ there exists a point $\xi \in [t - r_k, t]$, $\xi = \xi(x, t, k)$, such that $G_k(x)(t) = p_k(t)x(\xi)$ where $p_k \in C(\mathbb{R}_+, \mathbb{R})$. Let $r = \max\{r_k : k = 1, 2, \dots, m\}$.

Consider the nonlinear generalized delay functional differential equations with bounded delays

$$x' = f(t, x(t), G_1(x)(t), G_2(x)(t), \dots, G_m(x)(t)) \quad \text{for } t \geq t_0, \tag{2.1}$$

with initial condition

$$x(t + t_0) = \varphi(t) \quad \text{for } t \in [-r, 0], \tag{2.2}$$

where $x \in \mathbb{R}^n$, $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R}^n$, $t_0 \in \mathbb{R}_+$, $\varphi : [-r, 0] \to \mathbb{R}^n$.

Shortly we will denote the initial value problem by IVP. We would like to note some partial cases of (2.1):

- if $G_k(x)(t) = x(t - r_k)$ for $t \in \mathbb{R}_+$ then (2.1) reduces to delay differential equations with several constant delays $x' = f(t, x(t), x(t - r_1), x(t - r_2), \dots, x(t - r_m))$ (for example, see [5] and the cited references therein);
- if $G_k(x)(t) = \max_{s \in [t - r_k, t]} x(s)$ for $t \in \mathbb{R}_+$ then (2.1) reduces to differential equations with maxima (see, for example, [1, 4, 2, 6, 11, 12])
$$x' = f(t, x(t), \max_{s \in [t - r_1, t]} x(s), \max_{s \in [t - r_2, t]} x(s), \dots, \max_{s \in [t - r_m, t]} x(s))$$
- if $G_k(x)(t) = x(t - r_k(t))$ for $t \in \mathbb{R}_+$, where $r_k : \mathbb{R}_+ \to [0, r]$, then (2.1) reduces to delay differential equations with variable bounded delays $x' = f(t, x(t), x(t - r_1(t)), x(t - r_2(t)), \dots, x(t - r_m(t)))$ (for example, $r(t) = C|sin(t)|$ or $r(t) = \frac{Ct}{t+1}$ for $t \in \mathbb{R}_+$, where $C = const$); see [5] and the cited references therein;
- let $r > 0$ and $G(x)(t) = \int_{t-r}^{t} x(s)ds$ for $t \in \mathbb{R}_+$. Then equation (2.1) reduces to delay differential equations with distributed delay.

Denote the solution of the initial value problem (2.1), (2.2) by $x(t; t_0, \varphi)$. Consider also the initial value problem for (2.1) at a different initial data, i.e.

$$x(t + \tau_0) = \psi(t) \quad \text{for } t \in [-r, 0]. \tag{2.3}$$

where $\tau_0 \in R_+, \tau_0 \neq t_0$, $\psi \in C([-r, 0], \mathbb{R}^n)$, $\psi \not\equiv \varphi$.

Denote the solution of (2.1), (2.3) by $x(t; \tau_0, \psi)$. Both functions $x(t; t_0, \varphi)$ and $x(t; \tau_0, \psi)$ differ not only on the initial functions but also on the initial intervals.

In our work we will assume that IVP (2.1), (2.2) has a solution $x(t; t_0, \varphi)$ defined on $[t_0 - r, \infty)$ for any $t_0 \in \mathbb{R}_+$ and any $\varphi \in C([-r, 0], \mathbb{R}^n)$.

The main purpose of the paper is comparing the behavior of two solutions $x(t; t_0, \varphi)$ and $x(t; \tau_0, \psi)$ of (2.1) with initial time difference.

Let $\rho, \lambda > 0$ be given constants and consider the sets:

$$K = \{a \in C[\mathbb{R}_+, \mathbb{R}_+] : a(s) \text{ is strictly increasing and } a(0) = 0\};$$
$$S(\rho) = \{x \in \mathbb{R}^n : \|x\| \leq \rho\};$$
$$KS(\rho) = \{a \in C[[0, \rho], \mathbb{R}_+] : a(s) \text{ is strictly increasing } a(0) = 0\};$$

$$\tilde{KS}(\rho,\lambda)=\big\{a\in C[[0,\rho]\times[0,\lambda],\mathbb{R}_+]a \text{ is strictly increasing in its first argument}, \\ a(0,0)=0\big\}.$$

We introduce the notation

$$\|\phi\|_0=\max\{\|\phi(s)\|: s\in[-r,0]\},$$

where $\phi\in C(\,[-r,0],\mathbb{R}^n)$.

Definition 2.1. Let $x^*(t)=x(t;t_0,\varphi)$ be a given solution of (2.1), (2.2). The solution $x^*(t)$ is said to be

- *stable with initial time difference* if for every $\epsilon>0$ there exist $\delta=\delta(\epsilon,t_0)>0$ and $\Delta=\Delta(\epsilon,t_0)>0$ such that for any $\psi\in C([-r,0],\mathbb{R}^n)$ and any $\tau_0\in\mathbb{R}_+$, the inequalities $\|\varphi-\psi\|_0<\delta$ and $|\tau_0-t_0|<\Delta$ imply $\|x(t+\eta;\tau_0,\psi)-x^*(t)\|<\epsilon$ for $t\geq t_0$ where $\eta=\tau_0-t_0$;
- *attractive with initial time difference* if there exists $\beta>0$ such that for every $\epsilon>0$ there exist $T=T(\epsilon,t_0)>0$ such that for any $\tau_0\in\mathbb{R}_+$ and any $\psi\in C([-r,0],\mathbb{R}^n)$ the inequalities $\|\varphi-\psi\|_0<\beta$ and $|\tau_0-t_0|<\Delta$ imply $\|x(t+\eta;\tau_0,\psi)-x^*(t)\|<\epsilon$ for $t\geq t_0+T$ where $\eta=\tau_0-t_0$.
- *asymptotically stable with initial time difference* if the solution $x^*(t)$ is stable with initial time difference and attractive with initial time difference.

Definition 2.2. The generalized delay differential equation (2.1) is said to be

- *uniformly stable with initial time difference* if for any solution $x^*(t)=x(t;t_0,\varphi)$ of (2.1), (2.2) and for every $\epsilon>0$ there exist $\delta=\delta(\epsilon)>0$ and $\Delta=\Delta(\epsilon)>0$ such that for any $\psi\in C([-r,0],\mathbb{R}^n)$ and any $\tau_0\in\mathbb{R}_+$, the inequalities $\|\varphi-\psi\|_0<\delta$ and $|\tau_0-t_0|<\Delta$ imply $\|x(t+\eta;\tau_0,\psi)-x^*(t)\|<\epsilon$ for $t\geq t_0$ where $\eta=\tau_0-t_0$;
- *uniformly attractive with initial time difference* if there exist $\beta>0$ and $\Delta>0$ such that for any solution $x^*(t)=x(t;t_0,\varphi)$ of (2.1), (2.2) and for every $\epsilon>0$ there exist $T=T(\epsilon)>0$ such that for any $\tau_0\in\mathbb{R}_+$ and any $\psi\in C([-r,0],\mathbb{R}^n)$ the inequalities $\|\varphi-\psi\|_0<\beta$ and $|\tau_0-t_0|<\Delta$ imply $\|x(t+\eta;\tau_0,\psi)-x^*(t)\|<\epsilon$ for $t\geq t_0+T$ where $\eta=\tau_0-t_0$.
- *uniformly asymptotically stable with initial time difference* if (2.1) is uniformly stable with initial time difference and uniformly attractive with initial time difference.

Remark 2.3. Without loss of generality we will consider the case when $\tau_0>t_0$.

Remark 2.4. If $t_0=\tau_0$ and $x^*(t)\equiv 0$ then the introduced in Definition 2.1 stability with initial time difference is reduced to stability of zero solution (see, for example, [5] and cited therein references)

We will give a brief overview of both concepts of stability: the known in the literature stability and the introduced stability with initial time difference.

Case 1. (*Stability of a nonzero solution*). Consider the solution $x^*(t)=x(t;t_0,\varphi)$ of (2.1), (2.2). To study the stability of $x^*(t)$ we get another solution $\tilde{x}(t)=x(t;t_0,\psi)$ of (2.1), with initial condition $\tilde{x}(t+t_0)=\psi(t)$ for $t\in[-r,0]$ where the initial function $\psi\in C([-r,0],\mathbb{R}^n):\psi\not\equiv\varphi$. Now define the difference between both solutions $z(t)=\tilde{x}(t)-x^*(t)$. The function $z(t)$ is a solution of the following initial value problem for the generalized delay differential equation

$$\begin{gathered} z'=\tilde{f}(t,z(t),G_1(z)(t),G_2(z)(t),\dots,G_m(z)(t)),\quad t\geq t_0\\ z(t+t_0)=\phi(t),\quad t\in[-r,0], \end{gathered}\tag{2.4}$$

where

$$\begin{aligned}&\tilde{f}(t, z, G_1(z)(t), G_2(z)(t), \dots, G_m(z)(t))\\&= f(t, z + x^*(t), G_1(z + x^*)(t), G_2(z + x^*)(t), \dots, G_m(z + x^*)(t))\\&\quad - f(t, x^*(t), G_1(x^*)(t), G_2(x^*)(t), \dots, G_m(x^*)(t))\end{aligned}$$

and $\phi(t) = \psi(t) - \varphi(t)$, $t \in [-r, 0]$.

The initial value problem (2.4) has a zero solution. Therefore, the study of stability properties of the nonzero solution $x^*(t)$ of (2.1) is equivalent to the study of stability of the zero solution of (2.4).

Case 2. (*Stability with initial data difference*). Consider the solution $x^*(t) = x(t; t_0, \varphi)$ of (2.1), (2.2). To study the stability with initial time difference of $x^*(t)$ we get another solution $\tilde{x}(t) = x(t; \tau_0, \psi)$ of (2.1), with initial condition $\tilde{x}(t + \tau_0) = \psi(t)$ for $t \in [-r, 0]$ where the initial function $\psi \in C([-r, 0], \mathbb{R}^n) : \psi \not\equiv \varphi$ and the point $\tau_0 \neq t_0$. Similarly to Case 1 we consider the difference between both solutions $z(t) = \tilde{x}(t + \eta) - x^*(t)$ where $\eta = \tau_0 - t_0$. The function $z(t)$ depends significantly on η and it is a solution of the initial value problem for the generalized delay differential equation

$$\begin{gathered}z' = \tilde{f}(t, z, G_1(z)(t), G_2(z)(t), \dots, G_m(z)(t), \eta), \quad t \geq t_0\\z(t + t_0) = \phi(t), \quad t \in [-r, 0],\end{gathered} \tag{2.5}$$

where

$$\begin{aligned}&\tilde{f}(t, z(t), G_1(z)(t), G_2(z)(t), \dots, G_m(z)(t), \eta)\\&= f(t + \eta, z + x^*(t), G_1(z + x^*)(t), G_2(z + x^*)(t), \dots, G_m(z + x^*)(t))\\&\quad - f(t, x^*(t), G_1(x^*)(t), G_2(x^*)(t), \dots, G_m(x^*)(t))\end{aligned}$$

and $\phi(t) = \psi(t) - \varphi(t)$, $t \in [-r, 0]$.

In the nonauthonomous case the initial value problem (2.5) has no zero solution. Therefore, in this case the study of stability with initial data difference of $x^*(t)$ could not be reduced to the study of stability of the zero solution of an appropriate delay differential equation.

Now we give some examples to illustrate the concepts of stability with initial time difference.

Example 2.5. Consider the delay differential equation:

$$x'(t) = x(t)(2 - x(t-1)) \quad \text{for } t \geq t_0 \tag{2.6}$$

with an initial condition

$$x(t + t_0) = t^2 \quad \text{for } t \in [-1, 0], \tag{2.7}$$

where $x \in \mathbb{R}$.

Denote the solution of the initial value problem (2.6), (2.7) for $t_0 = 1$ by $x(t)$ and the solutions of (2.6), (2.7) for $t_0 = 5$ by $y(t)$. From Figure 1 it is seen that both solutions differ only by shifting. Therefore, the stability with initial time difference for time invariant delay differential equations reduces to stability of a nonzero solution in the literature.

Example 2.6. Consider the delay differential equation:

$$x'(t) = \frac{x(t)(10 - x(t-1))}{t} \quad \text{for } t \geq t_0 \tag{2.8}$$

with an initial condition

$$x(t+t_0) = \varphi(t) \quad \text{for } t \in [-1, 0], \tag{2.9}$$

where $x \in \mathbb{R}$, $t_0 > 0$.

Consider the initial value problem (2.8), (2.9) for various initial points t_0 and initial functions $\varphi(t)$:

- $t_0 = 3$, $\varphi(t) = t^2$ and denote its solution by $x(t)$;
- $t_0 = 5$, $\varphi(t) = t^2$ and denote its solution by $y(t)$;
- $t_0 = 3.5$, $\varphi(t) = t^2 + 0.1$ and denote its solution by $u(t)$;
- $t_0 = 2.5$, $\varphi(t) = t^2 + 0.001$ and denote its solution by $v(t)$.

We graph the shifted solutions $y(t+2)$, $u(t+0.5)$, $v(t-0.5)$ and the fixed solution $x(t)$. From Figure 2 it can be seen these solutions are closer to the solution $x(t)$ when t increases. It seems the solution $x(t)$ could be stable with initial time difference.

Both examples prove that for nonautonomous differential equations the stability with initial time difference differs from types of stability in the literature.

Remark 2.7. The concept of stability with initial time difference is important in the nonautonomous case.

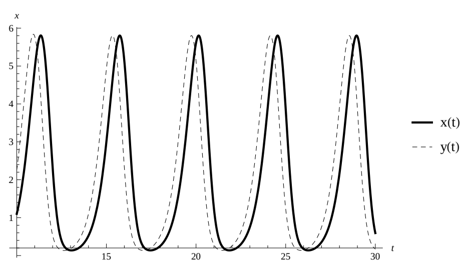

Figure 1. Graph of solutions $y(t)$ and $x(t)$ of (2.6).

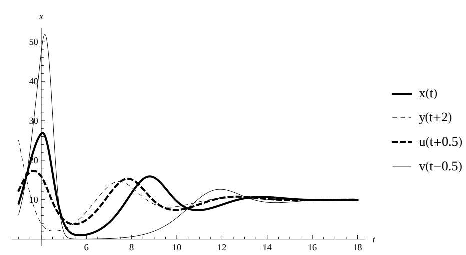

Figure 2. Graph of solutions $x(t)$, $y(t+2)$, $u(t+0.5)$ and $v(t-0.5)$ of (2.8).

Also, in Example 2.6, the equation (2.8) has an equilibrium 10 which is stable. Now we consider an equation which solution is unbounded.

Example 2.8. Consider the delay differential equation:

$$x'(t) = -x(t) + tx(t-1)) \quad \text{for } t \geq t_0 \tag{2.10}$$

with an initial condition

$$x(t+t_0) = \varphi(t) \quad \text{for } t \in [-1, 0], \tag{2.11}$$

where $x \in \mathbb{R}$, $t_0 > 0$.

Consider the initial value problem (2.10), (2.11) for various initial points t_0 and initial functions $\varphi(t)$:

- $t_0 = 3$, $\varphi(t) = t$ and denote its solution by $\tilde{x}(t)$;
- $t_0 = 3.5$, $\varphi(t) = t + 1$ and denote its solution by $\tilde{y}(t)$;
- $t_0 = 3.1$, $\varphi(t) = t - 1$ and denote its solution by $\tilde{u}(t)$;
- $t_0 = 2.8$, $\varphi(t) = t + 0.11$ and denote its solution by $\tilde{v}(t)$.

We graph the shifted solutions $\tilde{y}(t+0.5)$, $\tilde{u}(t+0.1)$, $\tilde{v}(t-0.2)$ and the fixed solution $x(t)$ on both intervals $[3,5]$ and $[98,100]$ on Figure 3 and Figure 4, respectively. The fixed solution $\tilde{x}(t)$ is unbounded. Also, the solutions $\tilde{u}(t+0.1)$ and $\tilde{v}(t-0.2)$ are closer to $\tilde{x}(t)$ comparatively with $\tilde{y}(t+0.5)$ for $t \to \infty$. Therefore, closer initial data could guarantee closeness of the solutions.

We need some sufficient conditions for stability with initial time difference.

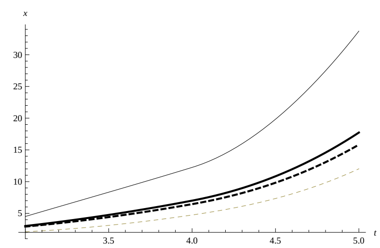

Figure 3. Graph of solutions $\tilde{x}(t)$, $\tilde{y}(t+0.5)$, $\tilde{u}(t+0.1)$, $\tilde{v}(t-0.2)$ of (2.10) for $t \in [3,5]$.

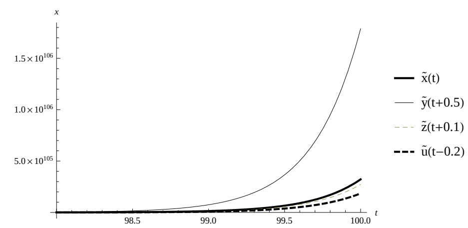

Figure 4. Graph of solutions $\tilde{x}(t)$, $\tilde{y}(t+0.5)$, $\tilde{u}(t+0.1)$, $\tilde{v}(t-0.2)$ of (2.10) for $t \in [98,100]$.

Let $J \subset \mathbb{R}_+$, $\Delta \subset \mathbb{R}^n$ and $I \subset \mathbb{R}_+$. Consider the class $\Lambda(J,\Delta)$ of functions $V(t,x) \in C(J \times \Delta, R_+) : V(t,x)$ is Lipschitz with respect to its second argument.

We will study the stability with initial time difference by Lyapunov functions from the class Λ and a modification of the Razumikhin method. Note if $x(t)$ is a solution of $x' = f(t,x)$ then $x(t+\eta)$ is a solution of $x' = f(t+\eta, x)$. It requires a new definition of the derivative of Lyapunov functions along the trajectories of the given differential equations.

We will define *a derivative* of the function $V(t,x) \in \Lambda(J,\Delta)$ along trajectory of the solutions of (2.1) *with respect to initial time difference*. Let $t \in J$, $\eta \in I$ and $\phi, \psi \in C([-r,0],\mathbb{R}^n) : \phi(0) - \psi(0) \in \Delta$. Then define

$$
\begin{aligned}
&D^-_{(2.1)} V(t,\phi(0),\psi(0),\eta)\\
&= \limsup_{\epsilon \to 0-} \frac{1}{\epsilon} \Big\{ V\Big(t+\epsilon, \phi(0)-\psi(0) + \epsilon\Big(f(t,\phi(0),G_1(\phi)(0),G_2(\phi)(0),\dots,\\
&\quad G_m(\phi)(0)) - f(t+\eta,\psi(0),G_1(\psi)(0),G_2(\psi)(0),\dots,G_m(\psi)(0))\Big)\Big)\\
&\quad - V(t,\phi(0)-\psi(0))\Big\}.
\end{aligned}
$$

Note that $V(t,x)$ is a scalar valued function, but the derivative with initial time difference $D^-_{(2.1)} V(t,\phi(0),\psi(0),\eta)$ is a functional.

Remark 2.9. The above definition of a derivative of the function $V(t,x)$ along trajectories of solutions of (2.1) with respect to initial time difference generalizes the derivative of the function $V(t,x)$ along trajectories of solutions of (2.1) used for studying the stability of the zero solution (the case when $\eta = 0$, $G_k(0)(t) \equiv 0$, $k = 1,2,\dots,m$ and $f(t,0,0,\dots,0) \equiv 0$).

Now we prove some comparison results giving us the relationship between Lyapunov functions, generalized delay differential equation (2.1) and a scalar ordinary differential equation with a parameter.

Consider the scalar ordinary differential equation with a parameter:

$$u' = g(t, u, \eta), \quad u(t_0) = u_0 \tag{2.12}$$

where $u \in \mathbb{R}$, $g \in C(\mathbb{R}_+ \times \mathbb{R} \times [0, \rho], \mathbb{R})$, $g(t, 0, 0) \equiv 0$, $\eta \in [0, \rho]$ is a parameter, $\rho > 0$ is a given number.

Also, for any fixed natural number n we will consider the initial value problem

$$u' = g(t, u, \eta) + \frac{1}{n}, \quad u(t_0) = u_0 + \frac{1}{n}. \tag{2.13}$$

We will assume that for any $(t_0, u_0) \in \mathbb{R}_+ \times \mathbb{R}$ and any $\eta \in [0, \rho]$ the initial value problems (2.12) and (2.13) have solutions $u(t; t_0, u_0, \eta)$ and $u_n(t; t_0, u_0, \eta)$, respectively, which are defined on $[t_0, \infty)$. In the case of non-uniqueness of the solution we will assume the existence of a maximal one. Note the existence of solutions of nonlinear ordinary differential equations with small parameters are studied in Chapter 1 of the book [16].

We will use the stability of the zero solution of (2.12) with respect to a parameter defined by the following definition:

Definition 2.10. The zero solution of (2.12) is said to be

- stable with respect to the parameter if for any $\epsilon > 0$ and any $t_0 \geq 0$ there exist $\delta = \delta(t_0, \epsilon) > 0$ and $\Delta = \Delta(t_0, \epsilon) > 0$ such that for any $u_0 \in \mathbb{R}$: $|u_0| < \delta$ and any $\eta \in \mathbb{R} : |\eta| < \Delta$ the inequality $|u(t; t_0, u_0, \eta)| < \epsilon$ for $t \geq t_0$ holds, where $u(t; t_0, u_0, \eta)$ is a solution of (2.12) for the given u_0 and η;
- uniformly stable with respect to the parameter if above $\delta = \delta(\epsilon) > 0$ and $\Delta = \Delta(\epsilon) > 0$, i.e. they do not depend on t_0.

Remark 2.11. Note if $g(t, u, \eta) \equiv 0$ then the zero solution of (2.12) is uniformly stable with respect to the parameter.

Lemma 2.12 (Comparison result)**.** *Assume the following conditions are satisfied:*

1. The functions $x^*(t) = x(t; t_0, \varphi)$ and $\tilde{x}(t) = x(t; \tau_0, \psi)$ are solutions of initial value problems (2.1), (2.2), *and* (2.1), (2.3) *defined on $[t_0 - r, T]$ and $[\tau_0 - r, T + \eta^*]$, respectively, where $t_0, \tau_0 \in \mathbb{R}_+$, $t_0 \neq \tau_0$, $\eta^* = \tau_0 - t_0 \in (0, \rho]$, $\rho > 0$, $T > t_0$ are given numbers.*

2. The function $g \in C([t_0, T] \times \mathbb{R} \times [0, \rho], \mathbb{R}_+)$.

3. The function $V \in \Lambda([t_0 - r, T], \mathbb{R}^n)$ and for any point $t \in [t_0, T]$ such that $V(t+s, x^*(t+s) - \tilde{x}(t+s+\eta)) < V(t, x^*(t) - \tilde{x}(t+\eta))$ for $s \in [-r, 0)$ the inequality

$$D^-_{(2.1)} V(t, x^*(t), \tilde{x}(t + \eta^*), \eta^*) \leq g\big(t, V\big(t, x^*(t) - \tilde{x}(t + \eta^*)\big), \eta^*\big)$$

holds.

4. The function $u^*(t) = u(t; t_0, u_0, \eta^*)$ is the maximal solution of the initial value problem (2.12) *defined on $[t_0, T]$, where $u_0 \in \mathbb{R}$ is such that $\max_{s \in [-r, 0]} V(t_0 + s, \varphi(s) - \psi(s)) \leq u_0$.*

Then the inequality $V(t, x^*(t) - \tilde{x}(t + \eta^*)) \leq u^*(t)$ holds for $t \in [t_0, T]$.

Proof. Let $u_n(t) = u_n(t; t_0, u_0, \eta^*)$, $t \in [t_0, T]$, be the maximal solution of the initial value problem (2.13) where $\eta = \eta^*$ and n is a natural number.

Define the function $m(t) \in C([t_0 - r, T], \mathbb{R}_+)$ by $m(t) = V(t, x^*(t) - \tilde{x}(t + \eta^*))$. We will prove that for any natural number n,

$$m(t) \leq u_n(t) \quad \text{for } t \in [t_0, T]. \tag{2.14}$$

Note that for any natural number n the inequalities $m(s) \leq u_0 < u_n(t_0)$, $s \in [t_0 - r, t_0]$ hold, i.e. (2.14) holds for $t = t_0$.

Assume that inequality (2.14) is not true. Let n be a natural number such that there exists a point $\xi > t_0 : m(\xi) > u_n(\xi)$. Let $t_n^* = \max\{t > t_0 : m(s) < u_n(s) \text{ for } s \in [t_0, t)\}$, $t_n^* < T$. Then $m(t_n^*) = u_n(t_n^*)$, $m(t) < u_n(t)$ for $t \in [t_0 - r, t_n^*)$, and

$$\begin{aligned} D_- m(t_n^*) &= \lim_{h \to 0-} \frac{m(t_n^* + h) - m(t_n^*)}{h} \geq \lim_{h \to 0-} \frac{u_n(t_n^* + h) - u(t_n^*)}{h} \\ &= u_n'(t_n^*) = g(t, u_n(t_n^*), \eta^*) + \frac{1}{n} > g(t_n^*, m(t_n^*), \eta^*). \end{aligned}$$

From $g(t, u, \eta^*) + \frac{1}{n} > 0$ on $[t_n^* - r, t_n^*]$ it follows that the function $u_n(t)$ is nondecreasing on $[t_n^* - r, t_n^*]$ and $m(t_n^*) = u_n(t_n^*) \geq u_n(s) > m(s)$ for $s \in [t_n^* - r, t_n^*)$, i.e. the inequality

$$V(t_n^* + s, x^*(t_n^* + s) - \tilde{x}(t_n^* + s + \eta^*)) < V(t_n^*, x^*(t_n^*) - \tilde{x}(t_n^* + \eta^*)) \tag{2.15}$$

holds for $s \in [-r, 0)$.

According to condition 3 of Lemma 2.12, for the point t_n^* we have that

$$\begin{aligned} &D_- m(t_n^*) \\ &= \lim_{h \to 0-} \frac{V(t_n^* + h, x^*(t_n^* + h) - \tilde{x}(t_n^* + \eta^* + h)) - V(t_n^*, x^*(t_n^*) - \tilde{x}(t_n^* + \eta^*)}{h} \\ &= \lim_{h \to 0-} \frac{1}{h} \Big\{ \Big[V(t_n^* + h, x^*(t_n^* + h) - \tilde{x}(t_n^* + \eta^* + h)) \\ &\quad - V(t_n^* + h, x^*(t_n^*) - \tilde{x}(t_n^* + \eta^*) \\ &\quad + h \Big(f(t_n^*, x^*(t_n^*), G_1(x^*)(t_n^*), \dots, G_m(x^*)(t_n^*)) \\ &\quad - f(t_n^* + \eta^*, \tilde{x}(t_n^* + \eta^*), G_1(\tilde{x})(t_n^* + \eta^*), \dots, G_m(\tilde{x})(t_n^* + \eta^*)) \Big) \Big] \\ &\quad + \Big[V(t_n^* + h, x^*(t_n^*) - \tilde{x}(t_n^* + \eta^*) \\ &\quad + h \Big(f(t_n^*, x^*(t_n^*), G_1(x^*)(t_n^*), \dots, G_m(x^*)(t_n^*)) \\ &\quad - f(t_n^* + \eta^*, \tilde{x}(t_n^* + \eta^*), G_1(\tilde{x})(t_n^* + \eta^*), \dots, G_m(\tilde{x})(t_n^* + \eta^*)) \Big) \\ &\quad - V(t_n^*, x^*(t_n^*) - \tilde{x}(t_n^* + \eta^*) \Big] \Big\} \\ &\leq D^-_{(2.1)} V(t_n^*, x^*(t_n^*), \tilde{x}(t_n^* + \eta^*), \eta^*) \\ &\leq g(t_n^*, V(t_n^*, x^*(t_n^*) - \tilde{x}(t_n^* + \eta^*)), \eta^*) = g(t_n^*, m(t_n^*), \eta^*). \end{aligned}$$

The obtained contradiction proves inequality (2.14) for any natural number n. Let $\lim_{n \to \infty} u_n(t) = \tilde{u}(t)$. It is clear $\tilde{u}(t)$ is a solution of IVP(2.12). Since $u^*(t)$ is the maximal solution of IVP(2.12) we obtain from (2.14) $V(t, x^*(t) - \tilde{x}(t + \eta)) = m(t) \leq \tilde{u}(t) \leq u^*(t)$, $t \in [t_0, T]$. □

Remark 2.13. Note that the claim of Lemma 2.12 is true if the inequality

$$\max_{s \in [-r, 0]} V(t_0 + s, \varphi(s) - \psi(s)) \leq u_0$$

in Condition 4 is replaced by $V(t_0, \varphi(0) - \psi(0)) \leq u_0$.

In the case when $g(t, x, \eta) \equiv 0$ in Lemma 2.12, we obtain the following result.

Corollary 2.14. *Assume the following conditions are satisfied:*

1. The functions $x^*(t) = x(t; t_0, \varphi)$ and $\tilde{x}(t) = x(t; \tau_0, \psi)$ are solutions of initial value problems (2.1), (2.3)*, and* (2.1), (2.2) *defined on $[t_0 - r, T]$ and $[\tau_0 - r, T + \eta^*]$, respectively. where $t_0, \tau_0 \in \mathbb{R}_+ : \eta^* = \tau_0 - t_0$, $T > t_0$.*

2. The function $V \in \Lambda([t_0 - r, T], \mathbb{R}^n)$ and for any point $t \in [t_0, T]$ such that $V(t + s, x^*(t + s) - \tilde{x}(t + s + \eta^*)) < V(t, x^*(t) - \tilde{x}(t + \eta^*))$ for $s \in [-r, 0)$ the inequality

$$D^-_{(2.1)} V(t, x^*(t), \tilde{x}(t + \eta^*), \eta^*) \le 0$$

holds.

Then for $t \in [t_0, T]$ the inequality $V(t, x^*(t) - \tilde{x}(t + \eta^*)) \le \max_{s \in [-r,0]} V(t_0 + s, \varphi(s) - \psi(s))$ holds.

The result of Lemma 2.12 is true on the half line. The idea is to fix $T > t_0$ and once again we have (2.14). Then $\tilde{u}(t) = \lim_{n\to\infty} u_n(t)$ satisfies IVP(2.12) for $t \in [t_0, T]$. We can do this argument for each $T < \infty$. Thus yields the following result.

Corollary 2.15. *Assume the following conditions are satisfied:*

1. The functions $x^*(t) = x(t; t_0, \varphi)$ and $\tilde{x}(t) = x(t; \tau_0, \psi)$ are solutions of initial value problems (2.1), (2.2)*, and* (2.1), (2.3) *defined on $[t_0 - r, \infty)$ and $[\tau_0 - r, \infty)$, respectively, where $t_0, \tau_0 \in \mathbb{R}_+$, $t_0 \ne \tau_0$, $\eta^* = \tau_0 - t_0 \in (0, \rho]$, $\rho > 0$, $T > t_0$ are given number.*

2. The function $g \in C([t_0, \infty) \times \mathbb{R} \times [0, \rho], \mathbb{R}_+)$.

3. The function $V \in \Lambda([t_0 - r, \infty), \mathbb{R}^n)$ and for any point $t \ge t_0$ such that $V(t + s, x^*(t + s) - \tilde{x}(t + s + \eta^*)) < V(t, x^*(t) - \tilde{x}(t + \eta^*))$ for $s \in [-r, 0)$ the inequality

$$D^-_{(2.1)} V(t, x^*(t), \tilde{x}(t + \eta^*), \eta^*) \le g(t, V(t, x^*(t) - \tilde{x}(t + \eta^*), \eta^*)$$

holds.

4. The function $u^*(t) = u(t; t_0, u_0, \eta^*)$ is the maximal solution of the initial value problem (2.12) *defined on $[t_0, \infty)$, where $u_0 \in \mathbb{R}$ is such that $\max_{s\in[-r,0]} V(t_0 + s, \varphi(s) - \psi(s)) \le u_0$.*

Then the inequality $V(t, x^*(t) - \tilde{x}(t + \eta^*)) \le u^*(t)$ holds for $t \ge t_0$.

Remark 2.16. Note the claim of Corollary 2.15 is true if the inequality

$$\max_{s\in[-r,0]} V(t_0 + s, \varphi(s) - \psi(s)) \le u_0$$

in Condition 4 is replaced by $V(t_0, \varphi(0) - \psi(0)) \le u_0$.

When in Condition 4 of Lemma 2.12 the derivative of the Lyapunov function is negative, the following result is true.

Lemma 2.17. *Assume the following conditions are satisfied:*

1. The functions $x^*(t) = x(t; t_0, \varphi)$ and $\tilde{x}(t) = x(t; \tau_0, \psi)$ are solutions of initial value problems (2.1), (2.2)*, and* (2.1), (2.3) *defined on $[t_0 - r, T]$ and $[\tau_0 - r, T + \eta^*]$, respectively, and $\|x^*(t) - \tilde{x}(t + \eta^*)\| \le \lambda$ for $t \in [t_0 - r, T]$, where $t_0, \tau_0 \in \mathbb{R}_+$, $\eta^* = \tau_0 - t_0 \in (0, \rho]$, $\rho, \lambda > 0$, $T > t_0$ are given numbers.*

2. The function $V \in \Lambda([t_0 - r, T], S(\lambda))$ and for any point $t \in [t_0, T]$ such that $V(t + s, x^*(t + s) - \tilde{x}(t + s + \eta^*)) < V(t, x^*(t) - \tilde{x}(t + \eta^*))$ for $s \in [-r, 0)$ the inequality

$$D^-_{(2.1)} V(t, x^*(t), \tilde{x}(t + \eta^*), \eta^*) < -c(\|x^*(t) - \tilde{x}(t + \eta^*)\|, \eta)$$

holds where $c \in \tilde{KS}(\lambda,\rho)$.

Then for $t \in [t_0, T]$ *the inequality*

$$V(t, x^*(t) - \tilde{x}(t+\eta)) \le V(t_0, \varphi(0) - \psi(0)) - \int_{t_0}^{t} c(\|x^*(s) - \tilde{x}(s+\eta^*)\|, \eta^*) ds$$

holds.

Proof. Define the function $m(t) \in C([t_0-r, T], \mathbb{R}_+)$ by $m(t) = V(t, x^*(t) - \tilde{x}(t+\eta^*))$ and the function $p \in C([t_0, T], \mathbb{R}_+)$ by $p(t) = c(\|x^*(t) - \tilde{x}(t+\eta^*)\|, \eta^*)$. Let $\epsilon > 0$ be an arbitrary number. We will prove that

$$m(t) < m(t_0) - \int_{t_0}^{t} p(s) ds + \epsilon, \quad t \in [t_0, T]. \tag{2.16}$$

Assume the contrary and let $t^* = \inf\{t \in [t_0, T] : m(t) \ge m(t_0) - \int_{t_0}^{t} p(s) ds + \epsilon\}$. It is clear $t^* \in (t_0, T]$ and

$$\begin{gathered} m(t^*) = m(t_0) - \int_{t_0}^{t^*} p(s) ds + \epsilon, \\ m(t) < m(t_0) - \int_{t_0}^{t} p(s) ds + \epsilon \quad \text{for } t \in [t_0, t^*). \end{gathered} \tag{2.17}$$

Therefore,

$$D^- m(t^*) \ge -p(t^*). \tag{2.18}$$

From (2.17) it follows that $m(t^*) > m(t^* - s)$ for $s \in [-r, 0)$, i.e. $V(t^* + s, x^*(t^* + s) - \tilde{x}(t^* + s + \eta^*)) < V(t^*, x^*(t^*) - \tilde{x}(t^* + \eta^*))$ for $s \in [-r, 0)$. Then from condition 2 we obtain

$$\begin{aligned} D_- m(t^*) &\le D^-_{(2.1)} V(t^*, x^*(t^*), \tilde{x}(t^* + \eta^*), \eta^*) \\ &< -c(\|x^*(t^*) - \tilde{x}(t^* + \eta^*)\|, \eta^*) \\ &= -p(t^*). \end{aligned}$$

This inequality contradicts (2.18). Therefore, inequality (2.16) is satisfied. Since $\epsilon > 0$ is an arbitrary number, inequality (2.16) proves the result. ☐

3. Main results

We obtain some sufficient conditions for the stability with initial time difference. We will start with stability for a given solution.

Theorem 3.1 (Stability with initial time difference of a solution)**.** *Assume:*

1. The function $x^*(t) = x(t; t_0, \varphi)$, $t \ge t_0 - r$, *is a solution of* (2.1),(2.2), *where* $\varphi \in C([-r, 0], \mathbb{R}^n)$, $t_0 \in \mathbb{R}_+$.

2. The function $g \in C([t_0, \infty) \times \mathbb{R} \times [0, \rho], \mathbb{R}_+)$, $g(t, 0, 0) \equiv 0$, $\rho > 0$ *is a given number.*

3. There exists a function $V \in \Lambda([t_0 - r, \infty), \mathbb{R}^n)$ *such that* $V(t_0, 0) = 0$ *and*

(i) *for any point* $t \ge t_0$ *and any function* $\psi \in C([-r, 0], \mathbb{R}^n)$ *such that* $V(t + s, x^*(t + s) - \psi(s)) < V(t, x^*(t) - \psi(0))$ *for* $s \in [-r, 0)$ *the inequality*

$$D^-_{(2.1)} V(t, x^*(t), \psi(0)), \eta) \le g(t, V(t, x^*(t) - \psi(0)), \eta) \tag{3.1}$$

holds for $\eta \in [0, \rho]$.

(ii) $b(\|x\|) \le V(t, x)$ *for* $t \ge t_0$, $x \in \mathbb{R}^n$, *where* $b \in K$.

4. The zero solution of (2.12) *is stable with respect to the parameter.*
Then the solution $x^*(t)$ is stable with initial time difference.

Proof. Let $\epsilon \in (0, \rho]$ be a positive number. From condition 4 there exist $\Delta = \Delta(t_0, \epsilon) > 0$ and $\delta_1 = \delta_1(t_0, \epsilon) > 0$ such that the inequalities $|\eta| < \Delta$ and $|u_0| < \delta_1$ imply

$$|u(t; t_0, u_0, \eta)| < b(\epsilon), \quad t \geq t_0, \tag{3.2}$$

where $u(t; t_0, u_0, \eta)$ is a solution of IVP (2.12).

Since $V(t_0, 0) = 0$ there exists $\delta_2 = \delta_2(t_0, \delta_1) > 0$ such that $V(t_0, x) < \delta_1$ for $\|x\| < \delta_2$. Let $\psi \in C([-r, 0], \mathbb{R}^n) : |||\varphi - \psi|||_0 < \delta_2$ and $\tau_0 : 0 < \eta = \tau_0 - t_0 < \Delta$. Denote by $\tilde{x}(t) = x(t; \tau_0, \psi),\ t \geq \tau_0 - r$ the solution of the initial value problem (2.1), (2.3).

From the choice of the initial function ψ we have $\|\varphi(0) - \psi(0)\| < \delta_2$ and $V(t_0, \varphi(0) - \psi(0)) < \delta_1$.

Now let $u_0 = V(t_0, \varphi(0) - \psi(0))$. Then $u_0 < \delta_1$ and inequality (3.2) holds. Then from condition 3, Corollary 2.15 and Remark 2.16 we have

$$V(t, x^*(t) - \tilde{x}(t + \eta)) \leq u(t; t_0, u_0, \eta) < b(\epsilon), \quad t \geq t_0. \tag{3.3}$$

Then for any $t \geq t_0$ from condition (ii) we obtain $b(\|x^*(t) - \tilde{x}(t+\eta)\|) \leq V(t, x^*(t) - \tilde{x}(t + \eta)) \leq |u(t; t_0, u_0, \eta)| < b(\epsilon)$, so the result follows. $\square$

Corollary 3.2. *Let $x^*(t) = x(t; t_0, \varphi)$, $t \geq t_0 - r$, be a solution of* (2.1), (2.2), *where $\varphi \in C([-r, 0], \mathbb{R}^n)$, $t_0 \in \mathbb{R}_+$ and suppose there exist a function $V \in \Lambda([t_0 - r, \infty), \mathbb{R}^n)$ such that $V(t_0, 0) \equiv 0$ and*

- (i) *for any point $t \geq t_0$ and any function $\psi \in C([-r, 0], \mathbb{R}^n)$ such that $V(t + s, x^*(t + s) - \psi(s)) < V(t, x^*(0) - \psi(0))$ for $s \in [-r, 0)$ the inequality*
$$D^-_{(2.1)} V(t, x^*(0), \psi(0)), \eta) \leq 0 \tag{3.4}$$
holds.
- (ii) *$b(\|x\|) \leq V(t, x)$ for $t \geq t_0,\ x \in \mathbb{R}^n$, where $b \in K$.*

Then the solution $x^*(t)$ is stable with initial time difference.

Now we obtain some sufficient conditions for the stability with initial time difference for the given generalized system of delay differential equations.

Theorem 3.3 (Uniform stability with initial time difference)**.** *Assume:*

1. The function $g \in C(\mathbb{R}_+ \times \mathbb{R} \times [0, \rho], \mathbb{R}_+)$, $g(t, 0, 0) \equiv 0$, $\rho > 0$ is a given number.

2. There exists a function $V \in \Lambda([-r, \infty), \mathbb{R}^n)$ such that

- (i) *for any point $t \geq 0$, any parameter $\eta \in (0, \rho]$ and any functions $\varphi, \psi \in C([-r, 0], \mathbb{R}^n)$ such that $V(t + s, \varphi(s) - \psi(s)) < V(t, \varphi(0) - \psi(0))$ for $s \in [-r, 0)$ the inequality*
$$D^-_{(2.1)} V(t, \varphi(0), \psi(0)), \eta) \leq g(t, V(t, \varphi(0) - \psi(0)), \eta) \tag{3.5}$$
holds;
- (ii) *$b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ for $t \geq -r$, $x \in \mathbb{R}^n$, where $a, b \in K$.*

3. The zero solution of (2.12) *is uniformly stable with respect to the parameter.*

Then the generalized system of delay differential equations (2.1) *is uniformly stable with initial time difference.*

Proof. Let $\epsilon \in (0, \rho]$ be a positive number and $x^*(t) = x(t; t_0, \varphi)$, $t \geq t_0 - r$, be a solution of (2.1),(2.2), where $\varphi \in C([-r, 0], \mathbb{R}^n)$, $t_0 \in \mathbb{R}_+$.

From condition 3, there exist $\Delta = \Delta(\epsilon) > 0$ and $\delta_1 = \delta_1(\epsilon) > 0$ such that the inequalities $|\eta| < \Delta$ and $|u_0| < \delta_1$ imply

$$|u(t; t_0, u_0, \eta)| < b(\epsilon), \quad t \geq t_0, \tag{3.6}$$

where $u(t; t_0, u_0, \eta)$ is a solution of IVP (2.12).

Choose $\delta_2 = \delta_2(\delta_1) > 0$ such that $a(\delta_2) < \delta_1$. Let $\psi \in C([-r, 0], \mathbb{R}^n) : \|\|\varphi - \psi\|\|_0 < \delta_2$ and $\tau_0 : 0 < \eta = \tau_0 - t_0 < \Delta$. Denote by $\tilde{x}(t) = x(t; \tau_0, \psi)$, $t \geq \tau_0 - r$, the solution of the initial value problem (2.1), (2.3).

Now let $u_0 = \max_{s \in [-r,0]} V(t_0 + s, \varphi(s) - \psi(s))$. Then for every $s \in [-r, 0]$ from condition (ii) we get $V(t_0 + s, \varphi(s) - \psi(s)) \leq a(\|\varphi(s) - \psi(s)\|) \leq a(\|\|\varphi - \psi\|\|_0) \leq a(\delta_2) < \delta_1$ and therefore $u_0 < \delta_1$. Then inequality (3.6) holds for $t \geq t_0$.

From condition 2 and Corollary 2.15 we have

$$V(t, x^*(t) - \tilde{x}(t + \eta)) \leq u(t; t_0, u_0, \eta), \quad t \geq t_0. \tag{3.7}$$

Then for any $t \geq t_0$ from condition (ii) we obtain $b(\|x^*(t) - \tilde{x}(t+\eta)\|) \leq V(t, x^*(t) - \tilde{x}(t + \eta)) \leq |u(t; t_0, u_0, \eta)| < b(\epsilon)$, so the result follows. $\square$

When the derivative of the Lyapunov function is nonpositive we obtain the following result for the uniform stability with initial time difference.

Corollary 3.4. *Suppose there exist a function $V \in \Lambda([-r, \infty), \mathbb{R}^n)$ such that*

(i) *for any point $t \geq 0$ and any functions $\varphi, \psi \in C([-r, 0], \mathbb{R}^n)$ such that $V(t + s, \varphi(s) - \psi(s)) < V(t, \varphi(0) - \psi(0))$ for $s \in [-r, 0)$ the inequality*

$$D^-_{(2.1)} V(t, \varphi(0), \psi(0)), \eta) \leq 0 \tag{3.8}$$

holds.

(ii) $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ *for* $t \geq -r$, $x \in \mathbb{R}^n$, *where* $a, b \in K$.

Then the generalized system of delay differential equations (2.1) *is uniformly stable with initial time difference.*

Theorem 3.5 (Uniform stability with initial time difference of a delay system)**.** *Assume:*

1. The function $g \in C([\mathbb{R}_+ \times \mathbb{R} \times [0, \rho], \mathbb{R}_+)$, $g(t, 0, 0) \equiv 0$ where $\rho > 0$ is a fixed number.

2. There exists a function $V \in \Lambda([-r, \infty), S(\lambda))$ such that

(i) *for any point $t \geq 0$ and any functions $\varphi, \psi \in C([-r, 0], \mathbb{R}^n)$ such that $\|\|\varphi - \psi\|\|_0 < \lambda$ and $V(t + s, \varphi(s) - \psi(s)) < V(t, \varphi(0) - \psi(0))$ for $s \in [-r, 0)$ the inequality*

$$D^-_{(2.1)} V(t, \varphi(0), \psi(0)), \eta) \leq g(t, V(t, \varphi(0) - \psi(0)), \eta), \tag{3.9}$$

holds for $\eta \in [0, \rho]$ where $\lambda > 0$ is a given number.

(ii) $b(\|x\|) \leq V(t, x) \leq a(\|x\|)$ *for* $t \geq -r$, $x \in S(\lambda)$, *where* $a, b \in K$.

3. The zero solution of (2.12) *is uniformly stable with respect to the parameter.*

Then the generalized system of delay differential equations (2.1) *is uniformly stable with initial time difference.*

Proof. Let $\epsilon \in (0,\lambda]$ be a positive number and $x^*(t) = x(t;t_0,\varphi)$, $t \geq t_0 - r$, be a solution of (2.1),(2.2), where $\varphi \in C([-r,0],\mathbb{R}^n)$, $t_0 \in \mathbb{R}_+$.

From condition 3 of Theorem 3.3 there exist $\Delta = \Delta(\epsilon) > 0$ and $\delta_1 = \delta_1(\epsilon) > 0$ such that for any $\tilde{t}_0 \geq 0$ the inequalities $|\eta| < \Delta$ and $|u_0| < \delta_1$ imply

$$|u(t;\tilde{t}_0,u_0,\eta)| < b(\epsilon), \quad t \geq \tilde{t}_0, \tag{3.10}$$

where $u(t;\tilde{t}_0,u_0,\eta)$ is a solution of the equation (2.12). Let $\delta_1 < \min\{\epsilon, b(\epsilon)\}$ and $\Delta \leq \rho$.

From $a \in K$ there exists $\delta_2 = \delta_2(\epsilon) > 0$: if $s < \delta_2$ then $a(s) < \delta_1$.

Let $\delta = \min(\delta_1,\delta_2)$. Choose the initial function $\psi \in C([-r,0],\mathbb{R}^n)$ such that $\|\varphi - \psi\|_0 < \delta$ and the initial point $\tau_0 > t_0$ such that $\eta_0 = \tau_0 - t_0 < \Delta$. Denote by $\tilde{x}(t) = x(t;\tau_0,\psi)$, $t \geq \tau_0 - r$ the solution of the initial value problem (2.1), (2.3).

We will prove that

$$\|x^*(t) - \tilde{x}(t+\eta_0)\| < \epsilon, \quad t \geq t_0 - r. \tag{3.11}$$

This inequality holds on $[t_0 - r, t_0]$. Assume inequality (3.11) is not true for all $t > t_0$ and let

$$t^* = \inf\{t > t_0 : \|x^*(t) - \tilde{x}(t+\eta_0)\| \geq \epsilon\}.$$

Then

$$\|x^*(t^*) - \tilde{x}(t^*+\eta_0)\| = \epsilon, \quad \text{and} \quad \|x^*(t) - \tilde{x}(t+\eta_0)\| < \epsilon, \quad t \in [t_0, t^*). \tag{3.12}$$

From the choice of the initial function ψ, inequalities $\delta \leq \epsilon$ and (3.12) it follows there exists a point $t_0^* \in (t_0, t^*)$ such that $\|x^*(t) - \tilde{x}(t+\eta_0)\| < \delta \leq \delta_2$ for $t \in [t_0 - r, t_0^*]$.

Now let $u_0 = \max_{s\in[-r,0]} V(t_0^* + s, x^*(t_0^*+s) - \tilde{x}(t_0^*+s+\eta_0))$. From the choice of the point t_0^* it follows that $\max_{s\in[-r,0]} \|x^*(t_0^*+s) - \tilde{x}(t_0^*+s+\eta_0)\| < \epsilon \leq \lambda$. Then from Lemma 2.12 for the interval $[t_0^*, t^*]$ and $\eta^* = \eta_0$ we have

$$V(t, x^*(t) - \tilde{x}(t+\eta_0)) \leq u^*(t;t_0^*,u_0,\eta_0), \quad t \in [t_0^*, t^*] \tag{3.13}$$

where $u^*(t;t_0^*,u_0,\eta_0), t \geq t_0^*$ is the maximal solution of initial value problem for the scalar differential equation (2.12) for the parameter $\eta_0 = \tau_0 - t_0$ and initial point (t_0^*, u_0).

Since $[t_0^* - r, t_0^*] \subset [t_0 - r, t_0^*]$ we get $\|x^*(t_0^*+s) - \tilde{x}(t_0^*+s+\eta)\| < \rho$ for $s \in [-r,0]$ and therefore,

$$V(t_0^*+s, x^*(t_0^*+s) - \tilde{x}(t_0^*+s+\eta)) \leq a(\|x^*(t_0^*+s) - \tilde{x}(t_0^*+s+\eta)\|) < \delta_1, \;\; s \in [-r,0],$$

or $u_0 < \delta_1$. Therefore, the solution $u^*(t;t_0^*,u_0,\eta)$ satisfies the inequality (3.10) for $t \geq t_0^*$ and $\eta = \tau_0 - t_0$.

From inequalities (3.10), (3.13), the choice of the point t^*, and condition (ii) of Theorem 3.5 we obtain $b(\epsilon) > |u^*(t^*;t_0^*,u_0,\eta)| \geq V(t^*, x^*(t^*) - \tilde{x}(t^*+\eta)) \geq b(\|x^*(t^*) - \tilde{x}(t^*+\eta)\|) = b(\epsilon)$. The contradiction proves inequality (3.11) and the result follows. □

Corollary 3.6. *Suppose there exist a function $V \in \Lambda([-r,\infty), S(\lambda))$ such that*

(i) *for any point $t \geq 0$ and any functions $\varphi, \psi \in C([-r,0],\mathbb{R}^n)$ such that $|||\varphi - \psi|||_0 < \lambda$ and $V(t+s, \varphi(s) - \psi(s)) < V(t, \varphi(0) - \psi(0))$ for $s \in [-r,0)$ the inequality*

$$D^-_{(2.1)} V(t, \varphi(0), \psi(0)), \eta) \leq 0, \tag{3.14}$$

holds where $\lambda > 0$ is a given number;

(ii) *$b(\|x\|) \leq V(t,x) \leq a(\|x\|)$ for $t \geq -r$, $x \in S(\lambda)$, where $a, b \in K$.*

Then the generalized system of delay differential equations (2.1) *is uniformly stable with initial time difference.*

Now consider the derivative of Lyapunov function which is widely used for the stability of the zero solution:

$$D^-_{(2.1)}V(t,\phi(0)) = \limsup_{\epsilon\to 0-}\frac{1}{\epsilon}\Big\{V(t+\epsilon,\phi(0)+\epsilon\Big(f(t,\phi(0),G_1(\phi)(0),G_2(\phi)(0),\\ \dots,G_m(\phi)(0)) - V(t,\phi(0))\Big)\Big\},$$

where $t\in J$ and $\phi\in C([-r,0],\mathbb{R}^n):\phi(0)\in\Delta$.

Now we give a relationship between both derivatives of Lyapunov functions, the first one guaranteeing the stability of the zero solution and the second one guaranteeing the stability with initial time difference. First we have the stability of zero solution and stability with initial time difference:

Theorem 3.7. *Assume:*

1. The function $\tilde g\in C(\mathbb{R}_+\times\mathbb{R},\mathbb{R}_+)$, $\tilde g(t,0)\equiv 0$.

2. The function $f\in C(\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^{nm},\mathbb{R}^n)$ and for any $(t,x,U)\in\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}^{nm}$ and any $\eta\in[0,\rho]$ the inequality

$$|f(t+\eta,x,U)-f(t,x,U)|\le\lambda(t)|\eta|$$

holds, where $\rho>0$ and $\lambda\in C(\mathbb{R}_+,\mathbb{R}_+)$ is a bounded function.

3. There exists a function $V\in\Lambda([-r,\infty),\mathbb{R}^n)$ such that

(i) *for any point $t\ge 0$ and any function $\varphi\in C([-r,0],\mathbb{R}^n)$ such that $V(t+s,\varphi(s))<V(t,\varphi(0))$ for $s\in[-r,0)$ the inequality*

$$D^-_{(2.1)}V(t,\varphi(0))\le\tilde g(t,V(t,\varphi(0))) \tag{3.15}$$

holds;

(ii) *$b(\|x\|)\le V(t,x)\le a(\|x\|)$ for $t\ge -r$, $x\in\mathbb{R}^n$, where $a,b\in K$.*

4. The zero solution of $u'=g(t,u)$ is uniformly stable.

Then the generalized system of delay differential equations (2.1) *is uniformly stable with initial time difference.*

Proof. Note

$$\begin{aligned}
&D^-_{(2.1)}V(t,\varphi(0))-\psi(0),\eta)\\
&=\lim_{h\to 0-}\frac{1}{h}\Big(\Big(V(t+h,\varphi(0)-\psi(0))+h\big(f(t,\varphi(0),\dots,G_m(\varphi)(0))\\
&\quad - f(t,\psi(0),G_1(\psi)(0),\dots,G_m(\psi)(0))\big)\Big)-V(t,\varphi(0))-\psi(0))\Big)\\
&\quad + \Big(V(t+h,\varphi(0)-\psi(0))+h\big(f(t,\varphi(0),\dots,G_m(\varphi)(0))\\
&\quad - f(t+\eta,\psi(0),G_1(\psi)(0),\dots,G_m(\psi)(0))\big)\\
&\quad - V(t+h,\varphi(0)-\psi(0))+h\big(f(t,\varphi(0),\dots,G_m(\varphi)(0))\\
&\quad - f(t,\psi(0),G_1(\psi)(0),\dots,G_m(\psi)(0))\big)\Big).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&D^-_{(2.1)}V(t,\varphi(0))-\psi(0),\eta)\\
&\le D^-_{(2.1)}V(t,\varphi(0))-\psi(0))+L\|f(t+\eta,\psi(0),G_1(\psi)(0),\dots,G_m(\psi)(0))
\end{aligned}$$

$$\begin{aligned}&- f(t,\psi(0),G_1(\psi)(0),\dots,G_m(\psi)(0))\| \\ &\le D^-_{(2.1)}V(t,\varphi(0))-\psi(0))+L\lambda(t)|\eta| \\ &\le \tilde{g}(t,\varphi(0))-\psi(0))+L\lambda(t)|\eta|.\end{aligned}$$

Define the function $g(t,u,\eta)=\tilde{g}(t,u)+L\lambda(t)|\eta|$ for which the inequality (26) in Theorem 3.5 is satisfied and for which the zero solution of equation (2.12) is uniformly stable with respect to the parameter. □

Now we have the uniform asymptotic stability with initial time difference of a system:

Theorem 3.8. *Assume:*

1. There exists a function $V\in\Lambda([-r,\infty),S(\lambda))$ such that

- (i) *for any point $t\ge 0$, any parameter $\eta\in[0,\rho]$ and any functions φ,ψ in $C([-r,0],\mathbb{R}^n)$ such that $|||\varphi-\psi|||_0<\lambda$ and $V(t+s,\varphi(s)-\psi(s))<V(t,\varphi(0)-\psi(0))$ for $s\in[-r,0)$ the inequality*

$$D^-_{(2.1)}V(t,\varphi(0),\psi(0)),\eta)<-c(\|\varphi(0)-\psi(0)\|,\eta), \tag{3.16}$$

holds where $\rho,\lambda>0$ is are given constants, function $c\in\tilde{KS}(\lambda,\rho)$;

- (ii) *$b(\|x\|)\le V(t,x)\le a(\|x\|)$ for $t\ge -r$, $x\in S(\lambda)$, where $a,b\in KS(\lambda)$.*

Then the generalized system of delay differential equations (2.1) *is uniformly asymptotically stable with initial time difference.*

Proof. According to Corollary 3.6 the generalized system of delay differential equations (2.1) is uniformly stable with initial time difference. Therefore, for λ and any solution $x^*(t)=x(t;t_0,\varphi)$ of (2.1),(2.2) there exist numbers $\alpha=\alpha(\lambda)\in(0,\lambda)$ and $\Delta=\Delta(\lambda)\in(0,\rho]$ such that for any $\psi\in C([-r,0],\mathbb{R}^n)$ and any $\tau_0\in\mathbb{R}_+$, the inequalities $|||\varphi-\psi|||_0<\alpha$ and $|\tau_0-t_0|<\Delta$ imply $\|x(t+\eta;\tau_0,\psi)-x^*(t)\|<\lambda$ for $t\ge t_0$. where $\eta=\tau_0-t_0$.

Now we prove the generalized system of delay differential equations (2.1) is uniformly attractive with initial time difference.

Consider the constant $\beta\in(0,\alpha]$ such that $a(\beta)\le b(\alpha)$. Let $\epsilon\in(0,\lambda]$ be an arbitrary number and $x^*(t)=x(t;t_0,\varphi)$ be a solution of (2.1),(2.2).

Now choose the initial data τ_0,ψ such that $|||\varphi-\psi|||_0<\beta$ and $\eta_0=\tau_0-t_0<\Delta$ and consider the solution $\tilde{x}(t)=x(t;\tau_0,\psi)$ of (2.1), (2.3). Then $|||\varphi-\psi|||_0<\alpha$ and therefore the inequality

$$\|\tilde{x}(t+\eta_0)-x^*(t)\|<\lambda\quad\text{for } t\ge t_0 \tag{3.17}$$

holds.

Choose the constant $\gamma=\gamma(\epsilon)\in(0,\epsilon]$ such that $a(\gamma)<b(\epsilon)$. Let $T>\frac{a(\alpha)}{c(\gamma,\eta_0)}$, $T=T(\epsilon)>0$. We will prove that

$$\|x^*(t)-\tilde{x}(t+\eta_0)\|<\epsilon\quad\text{for } t\ge t_0+T. \tag{3.18}$$

Assume

$$\|x^*(t)-\tilde{x}(t+\eta_0)\|\ge\gamma\quad\text{for every } t\in[t_0,t_0+T]. \tag{3.19}$$

Then according to Lemma 2.17 applied to the interval $[t_0, t_0 + T]$, we obtain

$$\begin{aligned} &V(t_0 + T, x^*(t_0 + T) - \tilde{x}(t_0 + T + \eta_0)) \\ &\leq V(t_0, \varphi(0) - \psi(0)) - \int_{t_0}^{t_0+T} c(\|x^*(s) - \tilde{x}(s + \eta_0)\|, \eta_0) ds \\ &\leq a(\|\varphi(0) - \psi(0)\|) - c(\gamma, \eta_0)T \leq a(\||\varphi - \psi\||_0) - c(\gamma, \eta_0)T \\ &< a(\alpha) - c(\gamma, \eta_0)T < 0. \end{aligned} \tag{3.20}$$

The obtained contradiction proves that there exists $t^* \in [t_0, t_0 + T]$ such that $\|x^*(t^*) - \tilde{x}(t^* + \eta_0)\| < \gamma$. Then for any $t \geq t^*$ the inequalities

$$\begin{aligned} b(\|x^*(t) - \tilde{x}(t + \eta_0)\|) &\leq V(t, x^*(t) - \tilde{x}(t^* + \eta_0)) \\ &\leq V(t^*, x^*(t^*) - \tilde{x}(t^* + \eta_0)) \\ &\leq a(\|x^*(t^*) - \tilde{x}((t^* + \eta_0)\|) \\ &\leq a(\gamma) < b(\epsilon) \end{aligned} \tag{3.21}$$

hold. Therefore, the inequality (3.18) holds for all $t \geq t^*$ (hence for $t \geq t_0 + T$). $\square$

4. Applications

Example 4.1. Consider the system of delay differential equations with bounded variable delay,

$$\begin{aligned} x_1'(t) &= -1.5x_1(t) + x_2(\tau(t)) + h_1(t) \\ x_2'(t) &= -1.5x_2(t) + x_1(\tau(t)) + h_2(t) \quad \text{for } t \geq t_0 \end{aligned} \tag{4.1}$$

with an initial condition

$$x_1(t + t_0) = \varphi_1(t), \quad x_2(t + t_0) = \varphi_2(t) \quad \text{for } t \in [-1, 0], \tag{4.2}$$

where $(x, y) \in \mathbb{R}^2$, $t_0 \geq 0$, $\tau \in C(\mathbb{R}_+, [-1, \infty)) : t - 1 \leq \tau(t) \leq t$. Note that $\tau(t) = t - |\sin(t)|$ is an example of the delay argument in (4.1).

Let there exist constants $L_1, L_2 > 0$ such that $|h_i(t_1) - h_i(t_2)| \leq L_i|t_1 - t_2|$, $i = 1, 2$. Consider the Lyapunov function $V(t, x_1, x_2) = 0.5(x_1^2 + x_2^2)$. Let $\varphi, \psi \in C([-1, 0], \mathbb{R}^2)$, $\varphi = (\varphi_1, \varphi_2)$, $\psi = (\psi_1, \psi_2)$, be such that $(\varphi_1(0) - \psi_1(0), \varphi_2(0) - \psi_2(0)) \in S(\lambda)$, $\lambda > 0$, and for every $s \in [-1, 0)$ the inequality

$$(\varphi_1(0) - \psi_1(0))^2 + (\varphi_2(0) - \psi_2(0))^2 > (\varphi_1(s) - \psi_1(s))^2 + (\varphi_2(s) - \psi_2(s))^2 \tag{4.3}$$

holds. Define $f_1(t, \varphi) = -1.5\varphi_1(t) + \varphi_2(\tau(t)) + h_1(t)$ and $f_2(t, \varphi) = -1.5\varphi_2(t) + \varphi_1(\tau(t)) + h_2(t)$.

From the definition of the derivative of Lyapunov function we obtain

$$\begin{aligned} &D^-_{(4.1)} V(t, \varphi(0), \psi(0), \eta) \\ &= \lim_{h \to 0^-} \sup \frac{1}{h} \Big[V(t + h, \varphi_1(0) - \psi_1(0) + h(f_1(t, \varphi) \\ &\quad - f_1(t + \eta, \psi)), \varphi_2(0) - \psi_2(0) + h(f_2(t, \varphi) - f_2(t + \eta, \psi))) \\ &\quad - V(t, \varphi_1(0) - \psi_1(0), \varphi_2(0) - \psi_2(0)) \Big] \\ &= 0.5 \lim_{h \to 0^-} \sup \frac{1}{h} \Big[\Big(\varphi_1(0) - \psi_1(0) + h(f_1(t, \varphi) - f_1(t + \eta, \psi)) \Big)^2 \\ &\quad - \Big(\varphi_1(0) - \psi_1(0) \Big)^2 \end{aligned}$$

$$
\begin{aligned}
&+ \Big(\varphi_2(0) - \psi_2(0) + h(f_2(t,\varphi) - f_2(t+\eta,\psi)\Big)^2 - \Big(\varphi_2(0) - \psi_2(0)\Big)\Big] \\
&= 0.5 \lim_{h\to 0^-} \sup \Big[\Big(2\varphi_1(0) - 2\psi_1(0) + h(f_1(t,\varphi) - f_1(t+\eta,\psi))\Big) \\
&\quad \times (f_1(t,\varphi) - f_1(t+\eta,\psi)) \\
&\quad + \Big(2\varphi_2(0) - 2\psi_2(0) + h(f_2(t,\varphi) - f_2(t+\eta,\psi)\Big)(f_2(t,\varphi) - f_2(t+\eta,\psi))\Big] \\
&= \Big(\varphi_1(0) - \psi_1(0)\Big)(f_1(t,\varphi) - f_1(t+\eta,\psi)) \\
&\quad + \Big(\varphi_2(0) - \psi_2(0)\Big)(f_2(t,\varphi) - f_2(t+\eta,\psi)) \\
&= -1.5\Big(\Big(\varphi_1(0) - \psi_1(0)\Big)^2 + \Big(\varphi_2(0) - \psi_2(0)\Big)^2\Big) \\
&\quad + \Big(\varphi_1(0) - \psi_1(0)\Big)\Big(\varphi_1(\tau(0)) - \psi_1(\tau(0))\Big) \\
&\quad + \Big(\varphi_2(0) - \psi_2(0)\Big)\Big(\varphi_2(\tau(0)) - \psi_2(\tau(0))\Big) \\
&\quad + (h_1(t+\eta) - h_1(t)) + (h_2(t+\eta) - h_2(t)).
\end{aligned}
$$

Applying the properties of functions $h_i(t), i = 1, 2$, inequalities $2ab \le a^2 + b^2$ and (4.3) we obtain

$$
\begin{aligned}
&D^-_{(4.1)} V(t, \varphi(0), \psi(0), \eta) \\
&\le -\Big(\Big(\varphi_1(0) - \psi_1(0)\Big)^2 + \Big(\varphi_2(0) - \psi_2(0)\Big)^2\Big) \\
&\quad + 0.5\Big(\Big(\varphi_1(\tau(0)) - \psi_1(\tau(0))\Big)^2 + \Big(\varphi_2(\tau(0)) - \psi_2(\tau(0))\Big)^2\Big) \\
&\quad + L|\eta|\Big(|\varphi_1(0) - \psi_1(0)| + |\varphi_2(0) - \psi_2(0)|\Big) \\
&\le -0.5\Big(\Big(\varphi_1(0) - \psi_1(0)\Big)^2 + \Big(\varphi_2(0) - \psi_2(0)\Big)^2\Big) \\
&\quad + L|\eta|\Big(|\varphi_1(0) - \psi_1(0)| + |\varphi_2(0) - \psi_2(0)|\Big),
\end{aligned}
$$

where $L = \max\{L_1, L_2\}$.

Therefore, using $|\varphi_1(0) - \psi_1(0)| + |\varphi_2(0) - \psi_2(0)| \le \lambda$ we obtain

$$
D^-_{(4.1)} V(t, \varphi(0), \psi(0), \eta) \le -V(t, \varphi_1(0) - \psi_1(0), \varphi_2(0) - \psi_2(0)) + L\lambda|\eta|. \tag{4.4}
$$

The comparison scalar equation in this case is

$$
u'(t) = -u + L\lambda|\eta|. \tag{4.5}
$$

The solution of (3.18) is $u(t) = (-L\lambda|\eta| + u_0)e^{-(t-t_0)} + L\lambda|\eta|$. Therefore, $|u(t)| \le |u_0| + 2L\lambda|\eta|$ which shows the zero solution of (4.5) is uniformly stable with respect to the parameter η and according to Theorem 3.5 the system (4.1) is uniformly stable with initial time difference.

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References

[1] R. Agarwal, S. Hristova; Strict stability in terms of two measures for impulsive differential equations with "supremum", *Appl. Analysis*, **91**, 7 (2012), 1379-1392.

[2] D. Bainov, S. Hristova; *Differential Equations with "Maxima"*, Francis & Taylor, CRC Press, 2011.

[3] M. Cicek, C. Yakar, M. Bayram Gucen; Boundedness and Lagrange stability of fractional order perturbed system related to unperturbed systems with initial time difference in Caputo's sense, *Adv. Difference Eq.*, 2011:54

[4] A. Dishliev, S. Hristova; Stability on a cone in terms of two measures for differential equations with "maxima", *Ann. Funct. Anal.* **1**, 1 (2010), 133-143.

[5] J. Hale; *Functional Differential Equations*, 1971 , Springer.

[6] J. Henderson, S. Hristova, Eventual Practical Stability and cone valued Lyapunov functions for differential equations with "Maxima", *Commun. Appl. Anal.*, **14**, 4 (2010), 515-524.

[7] S. Hristova; Razumikhin method and cone valued Lyapunov functions for impulsive differential equations with 'supremum', *Intern. J. Dynam. Sys. Diff. Eq.*, **2**3 (2009), 223-236.

[8] S. Hristova, Lipschitz stability for impulsive differential equations with "supremum", *Intern. Electr. J. Pure Appl. Math*, **1** 4 (2010), 345-358.

[9] S. Hristova; Integral stability in terms of two measures for impulsive functional differential equations, *Math. Comput. Model.*, **51** 1 (2010), 100-108

[10] S. Hristova; Generalization of practical stability for delay differential equations with respect to initial time difference, *AIP Conf. Proc.* 1570, 313 (2013); http://dx.doi.org/10.1063/1.4854771

[11] S. Hristova; Stability on a cone in terms of two measures for impulsive differential equations with "supremum", *Applied Mathematics Letters*, **23** 5 (2010), 508-511.

[12] S. Hristova, K. Stefanova; Practical stability of impulsive differential equations with "supremum" by integral inequalities, *European Journal of Pure and Applied Mathematics*, **5** 1 (2012), 30-44.

[13] S. Hristova, I. Pahleva; Stability with initial data difference for delaay differential equations, *Proc. 42 Spring Conf.UBM*, Borovetz, April 26 (2013), 228-233.

[14] V. Lakshmikantham, S. Leela, J. V. Devi; Stability criteria for solutions of differential equations relative to initial time difference, *Int. J. Nonlinear Diff. Eqns.* **5** (1999), 109-114.

[15] F. A. McRae; Perturbing Lyapunov functions and stability criteria for initial time difference, *Apll. Math. Comput.* **117** (2001), 313-320.

[16] E. Mishchenko, N. Rozov; *Differential equations with small parameters and relaxation oscillations*, Plenum Press, New York, 1980, ISBN-13: 978-1-4615-9049-1.

[17] M. D. Shaw, C. Yakar; Stability criteria and slowly growing motions with initial time difference, *Probl. Nonl. Anal. Eng. Sys.* **1** (2000), 5066.

[18] X. Song , A. Li, Z. Wang; Study on the stability of nonlinear differential equations with initial time difference, *Nonlinear Anal.: Real World Appl.* **11** (2010) 1304-1311.

[19] X. Song, S. Li, A. Li; Practical stability of nonlinear differential equation with initial time difference, *Appl. Math. Comput.* **203** (2008) 157162

[20] C. Yakar; Fractional Differential Equations in Terms of Comparison Results and Lyapunov Stability with Initial Time Difference, *Abstr. Appl. Anal.*, **2010** (2010), Article ID 762857.

[21] C. Yakar, M. Cicek, M. B. Gucen; Practical stability, boundedness criteria and Lagrange stability of fuzzy differential systems, *Comput. Math. Appl.* **64**, 6 (2012), 21182127

[22] C. Yakar, M. Shaw; A Comparison result and Lyapunov stability criteria with initial time difference, *Dynam. Cont., Discr. Impuls. Sys.*, series A: Mathematical Analysis **12** (2005), 731–737.

[23] C. Yakar, M. D. Shaw; Initial time difference stability in terms of two measures and variational comparison result, *Dynamics of Continuous, Discrete and Impulsive Systems. Series A: Mathematical Analysis* **15** (3) (2008), 417-425.

Ravi Agarwal

Department of Mathematics, Texas A& M University-Kingsville, Kingsville, TX 78363, USA

E-mail address: agarwal@tamuk.edu

Snezhana Hristova

Plovdiv University, Tzar Asen 24, 4000 Plovdiv, Bulgaria

E-mail address: snehri@gmail.bg

Donal O'Regan
School of Mathematics, Statistics and Applied Mathematics, National University of Ireland, Galway, Ireland
E-mail address: donal.oregan@nuigalway.ie