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EXISTENCE OF SOLUTIONS TO A PARABOLIC $p(x)$-LAPLACE EQUATION WITH CONVECTION TERM VIA L^∞ ESTIMATES

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Abstract. This article is devoted to the study of the existence of weak solutions to an initial and boundary value problem for a parabolic $p(x)$-Laplace equation with convection term. Using the De Giorgi iteration technique, the authors establish the critical a priori L^∞-estimates and thus prove the existence of weak solutions.

1. Introduction

In this article, we consider the initial and boundary value problem for parabolic $p(x)$-Laplace equation

$$\begin{gathered}\frac{\partial u}{\partial t} - \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = B(x,t)|\nabla u|^{p(x)} - \operatorname{div}\overrightarrow{F}(x,t), \quad (x,t)\in Q_T,\\ u(x,t)=0, \quad (x,t)\in\Gamma_T,\\ u(x,0)=u_0(x)\in L^\infty(\Omega), \quad x\in\Omega.\end{gathered} \tag{1.1}$$

Here, $\Omega\subset\mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $Q_T=\Omega\times(0,T)$, $\Gamma_T=\partial\Omega\times(0,T)$, $T>0$ is finite, and $p(x)$, $B(x,t)$, $\overrightarrow{F}(x,t)$ are given quantities satisfying conditions to be specified later.

Recently, partial differential equations involving variable exponents, such as the $p(x)$-Laplace equation in (1.1), have been extensively investigated, owing to their physical importance and powerful application. The mathematical model of Problem (1.1) originates from heat and mass transfer in nonhomogeneous media and non-Newtonian fluids with thermo-convective effects [2]. Equations of this type also appear in the study of digital image recovery [4] and electrorheological fluids [16]. It describes the evolution diffusion and filtration process. In particular, the models like (1.1) with variable exponent provide a good mathematical interpretation for the mechanical properties of certain viscous electrorheological fluids characterized by their abilities to undergo significant changes when an electric field is applied.

We focus on mathematical analysis concerning the existence of solutions to Problem (1.1). Similar problems with constant exponents or L^1 data have been studied by many authors; see, e.g., [3, 5, 13, 14, 15, 18, 21, 24]. To study our problem,

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we encounter several difficulties arising from the variable exponents. To deal with (1.1), one must face the typical difficulty of how to define the solution space to (1.1). When $p(x)=p$ is a constant, it is well known that $L^p(0,T;W_0^{1,p}(\Omega))$ can be taken as the solution space. However, in the nonconstant case and $p^-=\inf p(x)>1$, if the solution space is defined to be $L^{p(x)}(0,T;W_0^{1,p(x)}(\Omega))$, or $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$, etc., then it leads to an unfavorable fact that the $p(x)$-Laplace operator is not bounded and not continuous from this space into its dual. To conquer this difficulty, we adopt the appropriate solution space V as defined below, which helps us to define a weak solution to (1.1). However, other difficulties arise from it at the same time. On one hand, one must verify the chain rule in the variable exponent space, as given in Lemma 2.2 with its proof in the Appendix, even if this is an obvious fact in the case when p is a constant [5, 13]. On the other hand, we will get the existence result for Problem (1.1) through a limit process in which Simon's compactness theorem [17] plays a crucial role. Nevertheless, the solution space V prevents from directly employing the theorem. We take into account the properties associated with V and surmount this difficulty. There are other differences between the variable exponent case and the constant exponent case. Some important properties and inequalities are no longer valid. For example, the variable exponent spaces are not translation invariant, Young's inequality with convolution $\|f*g\|_{p(\cdot)}\le C\|f\|_{p(\cdot)}\|g\|_1$ holds if and only if p is constant, and for $u\in W_0^{1,p(x)}(\Omega)$, $\int_\Omega |u|^{p(x)}dx\le C\int_\Omega |\nabla u|^{p(x)}dx$ is not valid for the variable exponent p, etc.; we refer to monograph [7] for details and more references.

To define an appropriate solution space for Problem (1.1), we make the following hypotheses on the quantities appearing in (1.1).

- (H1) $p\in C(\overline{\Omega})$, and $p^+:=\max_{\overline{\Omega}}p(x)$, $p^-:=\min_{\overline{\Omega}}p(x)$ satisfy $1<p^-\le p^+<+\infty$; furthermore, there exists a positive constant C such that the following log-Hölder continuous condition holds:

$$|p(x)-p(y)|\le \frac{-C}{\log|x-y|}\quad \text{for every } x,y\in\Omega \text{ satisfying } |x-y|\le\frac{1}{2}. \tag{1.2}$$

- (H2) $B\in L^\infty(Q_T)$ satisfies $0\le B(x,t)\le b$, where $b>0$ is a constant, and $\overrightarrow{F}$ is a vector field satisfying $|\overrightarrow{F}|^{(p^-)'}\in L^r(Q_T)$, where $(p^-)'=\frac{p^-}{p^--1}$ and $r>\frac{N+p^-}{p^-}$. Hence, $\overrightarrow{F}\in\big(L^{p'(x)}(Q_T)\big)^N$ as $|\overrightarrow{F}|\in L^{(p^-)'}(Q_T)\hookrightarrow L^{p'(x)}(Q_T)$; see the relevant definitions below.

We remark that, when p is a constant, it is well known that $W_0^{1,p}(\Omega)$ (the closure of $C_0^\infty(\Omega)$ in $W^{1,p}(\Omega)$) is identical to $H_0^{1,p}(\Omega):=\{f\in L^p(\Omega):|\nabla f|\in L^p(\Omega)\text{ with } f|_{\partial\Omega}=0\}$. However, when p is a function, there exists an interesting Lavrentiev phenomenon [22], which shows that the above two space are not equivalent. The log-Hölder continuous condition (1.2) above guarantees an important fact that $C_0^\infty(\Omega)$ is dense in $W^{1,p(x)}(\Omega)$ [23]. Under this condition, one can define variable Sobolev spaces with homogeneous boundary values, $W_0^{1,p(x)}(\Omega)$, as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$; moreover, the condition makes $p(x)$-Poincaré's inequality hold [1, 10, 21].

We introduce the function space

$$V=\{v\in L^{p^-}(0,T;W_0^{1,p(x)}(\Omega)):|\nabla v|\in L^{p(x)}(Q_T)\},$$

endowed with the norm $\|u\|_V = |\nabla u|_{L^{p(x)}(Q_T)}$, or the equivalent norm $\|u\|_V = |u|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} + |\nabla u|_{L^{p(x)}(Q_T)}$; the equivalence follows from $p(x)$-Poincaré's inequality. Then V is a separable and reflexive Banach space (see [3, 21]).

We now give the definition of weak solutions to Problem (1.1).

Definition 1.1. We say that $u \in V \cap L^\infty(Q_T)$ is a weak solution to (1.1), provided that $u_t \in V^* + L^1(Q_T)$, $u(x,0) = u_0(x)$ in $L^{p^-}(\Omega)$, and

$$\begin{aligned} &\int_0^T \langle u_t, \phi\rangle dt + \int_0^T \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla\phi \, dx\, dt \\ &= \int_0^T \int_\Omega B|\nabla u|^{p(x)}\phi \, dx\, dt + \int_0^T \int_\Omega \nabla\phi \cdot \overrightarrow{F} \, dx\, dt \end{aligned} \tag{1.3}$$

holds for every $\phi(x,t) \in V \cap L^\infty(Q_T)$. Here, with $u_t = \alpha^{(1)} + \alpha^{(2)} \in V^* + L^1(Q_T)$, it is understood that

$$\int_0^T \langle u_t, \phi\rangle dt := \langle u_t, \phi\rangle_{V^*+L^1(Q_T), V\cap L^\infty(Q_T)} = \langle \alpha^{(1)}, \phi\rangle_{V^*,V} + \int_0^T \int_\Omega \alpha^{(2)}\phi \, dx\, dt.$$

When $p(x) = p$ is a constant, sup-/sub-solution method is powerful and direct to the existence results (see [13]). Nevertheless, it is not suitable to our problem because, due to the complicated nonlinearities of $p(x)$-Laplace, it may be quite difficult to construct a supsolution $\overline{u}$ and a subsolution $\underline{u}$ in V which simultaneously satisfy $\underline{u} \le \overline{u}$. Roughly speaking, in Equation (1.1), the growth power of $|\nabla u|^{p(x)-2}\nabla u$ at the left-hand side of (1.1) is less than that of the convection term $|\nabla u|^{p(x)}$ at the right-hand side, which leads us not to directly utilizing pseudo-monotone operator method [12]. Instead, to obtain the existence of weak solutions to Problem (1.1), we will employ the L^∞ estimate method and get the solution through a limit process to the approximate equations. We carry out the De Giorgi iteration, different from the classical constant exponent case (see [24, 5, 14] and the excellent and elegant argument therein), in the setting of variable exponent. We first give a general form of [5, Theorem 5.1] or [24, Lemma 1], as stated in (2.5), by which we obtain the L^∞ regularity under the classification when $p^- \ge 2$ and when $1 < p^- < 2$, other than the classification appeared in [24]. It should be remarked that, we employ the infimum of $p(x)$, which facilitates this iteration, however, on the other side of the coin, it makes the iteration process more technical and complexity. By the way, our result in Theorem 2.3 shows an interesting phenomenon: the uniformly L^∞ bound of u can depend on p^- other than $p(x)$ itself as in the constant exponent case [24]. In the limit process, the properties of solution space V and its related variable exponent space will be frequently used, which is one of the features in the equation with variable exponent.

The plan of this paper is as follows. In section 2, we apply the De Giorgi iteration to Problem (1.1) to obtain a uniform bound for the *bounded* weak solution $u \in V$; this a priori L^∞-assumption is crucial for such a uniform bound, as in [5, 14]. In section 3, we construct an approximation equation to Problem (1.1). Based on the uniform bound of u_n, we obtain the strong convergence of u_n in the solution space V, by virtue of which we establish the existence of solutions. Section 4 is an Appendix in which we give some brief proofs to some lemmas in the paper.

To conclude this section, we recall some preliminary results on the Lebesgue and Sobolev spaces with variable exponents; for more details, see [9, 10] or monograph [7, 16]. Let p be a continuous function defined in $\overline{\Omega}$, $p(x) > 1$, for any $x \in \overline{\Omega}$.

1. The space

$$L^{p(x)}(\Omega) := \big\{u : u \text{ is measurable in } \Omega \text{ and } \int_\Omega |u(x)|^{p(x)}dx < \infty\big\}.$$

This space is equipped with the Luxemburg's norm

$$|u|_{L^{p(x)}(\Omega)} := \inf\big\{\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)}dx \le 1\big\}.$$

The space $\big(L^{p(x)}(\Omega), |\cdot|_{L^{p(x)}(\Omega)}\big)$ is a separable, uniformly convex Banach space.

2. The space

$$W^{1,p(x)}(\Omega) := \Big\{u \in L^{p(x)}(\Omega) : |\nabla u| \in L^{p(x)}(\Omega)\Big\},$$

endowed with the norm

$$|u|_{W^{1,p(x)}(\Omega)} := |\nabla u|_{L^{p(x)}(\Omega)} + |u|_{L^{p(x)}(\Omega)}.$$

We denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W^{1,p(x)}(\Omega)$. In fact, the norm $|\nabla u|_{L^{p(x)}(\Omega)}$ and $|u|_{W^{1,p(x)}(\Omega)}$ are equivalent norms in $W_0^{1,p(x)}(\Omega)$. $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.

3. Frequently used relationships for the estimates.

$$\min\big\{|u|^{p^-}_{L^{p(x)}(\Omega)},\ |u|^{p^+}_{L^{p(x)}(\Omega)}\big\} \le \int_\Omega |u(x)|^{p(x)}dx \le \max\big\{|u|^{p^-}_{L^{p(x)}(\Omega)},\ |u|^{p^+}_{L^{p(x)}(\Omega)}\big\}.$$

Consequently,

$$|u_k - u|_{L^{p(x)}(\Omega)} \to 0 \Longleftrightarrow \int_\Omega |u_k - u|^{p(x)}dx \to 0.$$

4. $p(x)$-Hölder's inequality: For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, with $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$, we have

$$\Big|\int_\Omega uv\,dx\Big| \le \big(\frac{1}{p^-} + \frac{1}{(p')^-}\big)|u|_{L^{p(x)}(\Omega)}|v|_{L^{p'(x)}(\Omega)} \le 2|u|_{L^{p(x)}(\Omega)}|v|_{L^{p'(x)}(\Omega)}.$$

5. Embedding relationships: If p_1 and p_2 are in $C(\overline{\Omega})$, and $1 \le p_1(x) \le p_2(x)$, for any $x \in \overline{\Omega}$, then there exists a positive constant $C_{p_1(x),p_2(x)}$ such that

$$|u|_{L^{p_1(x)}(\Omega)} \le C_{p_1(x),p_2(x)}|u|_{L^{p_2(x)}(\Omega)}.$$

i.e. the embedding $L^{p_2(x)}(\Omega) \hookrightarrow L^{p_1(x)}(\Omega)$ is continuous. If $q \in C(\overline{\Omega})$ and $1 \le q(x) < p^*(x)$, for any $x \in \overline{\Omega}$, then the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact, where

$$p^*(x) := \begin{cases} \frac{Np(x)}{N-p(x)}, & p(x) < N, \\ +\infty, & p(x) \ge N. \end{cases}$$

6. $p(x)$-Poincaré's inequality: Under the condition (1.2), there exists a positive constant C_p such that

$$|u|_{L^{p(x)}(\Omega)} \le C_p|\nabla u|_{L^{p(x)}(\Omega)}, \quad \text{for all } u \in W_0^{1,p(x)}(\Omega).$$

2. A priori bounds

First of all, we give some technical lemmas frequently used in the process of De Giorgi iteration. In particular, (2.5) can be seen as a general form of [5, Theorem 5.1] or [24, Lemma 1]. Their proofs will be given in the Appendix for the convenience of the readers.

Lemma 2.1. *Assume that a, b, λ are positive constants, with $\lambda \geq \frac{1}{2} + \frac{b}{a}$. Define*

$$\varphi(s) = \begin{cases} e^{\lambda s} - 1, & s \geq 0, \\ -e^{-\lambda s} + 1, & s \leq 0. \end{cases} \tag{2.1}$$

Then the following properties hold:

(1) *For all $s \in \mathbb{R}$,*

$$|\varphi(s)| \geq \lambda |s|, \quad a\varphi'(s) - b|\varphi(s)| \geq \frac{a}{2} e^{\lambda |s|}. \tag{2.2}$$

(2) *There exist constants $d \geq 0$ and $M > 1$ such that, for all $s \geq d$,*

$$\varphi'(s) \leq \lambda M \big[\varphi\big(\frac{s}{p^-}\big)\big]^{p^-}, \quad \varphi(s) \leq M \big[\varphi\big(\frac{s}{p^-}\big)\big]^{p^-}. \tag{2.3}$$

(3) *Let $\Phi(s) = \int_0^s \varphi(\sigma) d\sigma$. If $p^- \geq 2$, then there exists a positive constant c^* such that*

$$\Phi(s) \geq c^* \big[\varphi\big(\frac{s}{p^-}\big)\big]^{p^-}, \quad \forall s \geq 0; \tag{2.4}$$

if $1 < p^- < 2$, then there exist $d \geq 0$ and $c^* = c^*(p^-, d)$ such that

$$\begin{gathered} \Phi(s) \geq c^* \big[\varphi\big(\frac{s}{p^-}\big)\big]^{p^-}, \quad \forall s \geq d, \\ \Phi(s) \geq c^* \big[\varphi\big(\frac{s}{p^-}\big)\big]^{2}, \quad \forall 0 \leq s \leq d. \end{gathered} \tag{2.5}$$

Lemma 2.2. *Assume that function $\pi : \mathbb{R} \to \mathbb{R}$ is piecewise C^1 with $\pi(0) = 0$ and $\pi' = 0$ outside a compact set. Let $\Pi(s) = \int_0^s \pi(\sigma) d\sigma$. If $u \in V$ with $u_t \in V^* + L^1(Q_T)$, then*

$$\int_0^T \langle u_t, \pi(u) \rangle dt = \langle u_t, \pi(u) \rangle_{V^* + L^1(Q_T), V \cap L^\infty(Q_T)} = \int_\Omega \Pi(u(T)) dx - \int_\Omega \Pi(u(0)) dx. \tag{2.6}$$

Using the lemmas above, we begin the De Giorgi iteration to get the a priori L^∞ estimate.

Theorem 2.3. *Let $u \in L^\infty(Q_T) \cap V$ be a weak solution to Problem (1.1). Then*

$$\|u\|_{L^\infty(Q_T)} \leq \|u_0\|_{L^\infty(\Omega)} + C,$$

where C is a constant depending on $p^-, N, T, r, b, \Omega, \||\overrightarrow{F}|^{(p^-)'}\|_{L^r(Q_T)}$, but independent of u.

Proof. Let k be a real number such that $k > \|u_0\|_{L^\infty(\Omega)}$ and let φ be the function defined in (2.1) with constant $\lambda \geq \frac{1}{2} + 2b$, where $b > 0$ is the constant in Hypothesis (H2). (We shall use (2.2) with $a = 1$ and $a = 1/2$ below.) Define

$$G_k(u) = \begin{cases} u - k, & \text{if } u > k, \\ u + k, & \text{if } u < -k, \\ 0, & \text{if } |u| \leq k. \end{cases}$$

Note that $u \in L^\infty(Q_T) \cap V$; so does $\varphi(G_k(u))$. Then, for each $\tau \in [0,T]$, one may choose $v = \varphi(G_k(u))\chi_{[0,\tau]}$ as a test function in (1.3) (where χ_A is the characteristic function on the set A). Noting that $\nabla v = \chi_{[0,\tau]}\chi\{|u|>k\}\varphi'(G_k(u))\nabla u$, we have

$$\begin{aligned}&\int_0^\tau \langle u_t, \varphi(G_k(u))\rangle dt + \int_0^\tau\int_\Omega |\nabla u|^{p(x)}\varphi'(G_k(u))\chi\{|u|>k\}\,dx\,dt\\ &= \int_0^\tau\int_\Omega B|\nabla u|^{p(x)}\varphi(G_k(u))\,dx\,dt + \int_0^\tau\int_\Omega \chi\{|u|>k\}\varphi'(G_k(u))\nabla u\cdot\overrightarrow{F}\,dx\,dt.\end{aligned}\tag{2.7}$$

Denote $A_k(t) = \{x\in\Omega : |u(x,t)|>k\}$. In what follows, we write $\varphi = \varphi(G_k(u))$ and $\varphi' = \varphi'(G_k(u))$ for simplicity. Thanks to the choice of k, one has

$$\begin{aligned}\int_0^\tau \langle u_t, \varphi(G_k(u))\rangle dt &= \int_\Omega \Phi(G_k(u))(\tau)dx - \int_\Omega \Phi(G_k(u_0))dx\\ &= \int_{A_k(\tau)} \Phi(G_k(u))(\tau)dx - \int_{A_k(0)}\Phi(G_k(u_0))dx\\ &= \int_{A_k(\tau)} \Phi(G_k(u))(\tau)dx.\end{aligned}\tag{2.8}$$

From Young's inequality with ϵ, it follows that

$$\begin{aligned}&\int_0^\tau\int_{A_k(t)} \varphi'\nabla u\cdot\overrightarrow{F}\,dx\,dt\\ &\le \epsilon\int_0^\tau\int_{A_k(t)}|\nabla u|^{p^-}\varphi'\,dx\,dt + C(\epsilon)\int_0^\tau\int_{A_k(t)}|\overrightarrow{F}|^{(p^-)'}\varphi'\,dx\,dt.\end{aligned}\tag{2.9}$$

Substituting (2.8) and (2.9) in (2.7) yields

$$\begin{aligned}&\int_{A_k(\tau)}\Phi(G_k(u))(\tau)dx + \int_0^\tau\int_{A_k(t)}|\nabla u|^{p(x)}\left(\varphi' - B|\varphi|\right)\,dx\,dt\\ &\le \epsilon\int_0^\tau\int_{A_k(t)}|\nabla u|^{p^-}\varphi'\,dx\,dt + C(\epsilon)\int_0^\tau\int_{A_k(t)}|\overrightarrow{F}|^{(p^-)'}\varphi'\,dx\,dt.\end{aligned}\tag{2.10}$$

Note that $\varphi' - B|\varphi| \ge \varphi' - b|\varphi| \ge \frac{1}{2}e^{\lambda|G_k(u)|} > 0$ by (2.2) (with $a=1$). By utilizing $|\nabla u|^{p(x)} \ge |\nabla u|^{p^-} - 1$ and choosing $\epsilon = \frac{1}{2}$, we get from (2.10) that

$$\begin{aligned}&\int_{A_k(\tau)}\Phi(G_k(u))(\tau)dx + \int_0^\tau\int_{A_k(t)}|\nabla u|^{p^-}\big(\frac{1}{2}\varphi' - B|\varphi|\big)\,dx\,dt\\ &\le C\int_0^\tau\int_{A_k(t)}|\overrightarrow{F}|^{(p^-)'}\varphi'\,dx\,dt + \int_0^\tau\int_{A_k(t)}\left(\varphi' - B|\varphi|\right)\,dx\,dt\\ &\le \int_0^\tau\int_{A_k(t)}\Big(C|\overrightarrow{F}|^{(p^-)'}+1\Big)\varphi'\,dx\,dt.\end{aligned}\tag{2.11}$$

Using (2.2) with $a=\frac{1}{2}$, we have $\frac{1}{2}\varphi' - B|\varphi| \ge \frac{1}{2}\varphi' - b|\varphi| \ge \frac{1}{4}e^{\lambda|G_k(u)|} > 0$. Denoting $w_k = \varphi\left(\frac{|G_k(u)|}{p^-}\right)$, we proceed to estimate (2.11),

$$\begin{aligned}\int_0^\tau\int_{A_k(t)}|\nabla u|^{p^-}\Big(\frac{1}{2}\varphi' - B|\varphi|\Big)\,dx\,dt &\ge \frac{1}{4}\int_0^\tau\int_{A_k(t)}|e^{\lambda\frac{|G_k(u)|}{p^-}}\nabla u|^{p^-}\,dx\,dt\\ &\ge \frac{1}{4}\big(\frac{1}{\lambda}\big)^{p^-}\int_0^\tau\int_{A_k(t)}|\nabla w_k|^{p^-}\,dx\,dt.\end{aligned}\tag{2.12}$$

By definition, $A_k(t) \setminus A_{k+d}(t) = t\{x \in \Omega : k < |u(x,t)| \le k+d\}$; hence $0 < |G_k(u)| \le d$ and $\varphi'(G_k(u)) = \lambda e^{\lambda|G_k(u)|} \le \lambda e^{\lambda d}$ on $A_k(t) \setminus A_{k+d}(t)$. So, from (2.3), it follows that

$$
\begin{aligned}
&\int_0^\tau \int_{A_k(t)} \Big(C|\overrightarrow{F}|^{(p^-)'} + 1\Big)\varphi'\,dx\,dt \\
&\le \lambda M \int_0^\tau \int_{A_{k+d}(t)} \Big(C|\overrightarrow{F}|^{(p^-)'} + 1\Big)|w_k|^{p^-}\,dx\,dt \\
&\quad + \int_0^\tau \int_{A_k(t)\setminus A_{k+d}(t)} \Big(C|\overrightarrow{F}|^{(p^-)'} + 1\Big)\varphi'\,dx\,dt \\
&\le \lambda M \int_0^\tau \int_{A_{k+d}(t)} h|w_k|^{p^-}\,dx\,dt + \lambda e^{\lambda d}\int_0^\tau \int_{A_k(t)\setminus A_{k+d}(t)} h\,dx\,dt,
\end{aligned} \tag{2.13}
$$

where $h = C|\overrightarrow{F}|^{(p^-)'} + 1$. Putting (2.11), (2.12) and (2.13) together, we deduce

$$
\begin{aligned}
&\int_{A_k(\tau)} \Phi(G_k(u))(\tau)dx + \frac{1}{4}\big(\frac{1}{\lambda}\big)^{p^-} \int_0^\tau \int_{A_k(t)} |\nabla w_k|^{p^-}\,dx\,dt \\
&\le \lambda M \int_0^\tau \int_{A_{k+d}(t)} h|w_k|^{p^-}\,dx\,dt + \lambda e^{\lambda d}\int_0^\tau \int_{A_k(t)\setminus A_{k+d}(t)} h\,dx\,dt.
\end{aligned} \tag{2.14}
$$

Case 1. $p^- \ge 2$. In this case, by (2.4), one has

$$
\int_{A_k(\tau)} \Phi(G_k(u))(\tau)dx \ge c^* \int_{A_k(\tau)} |w_k|^{p^-}\,dx. \tag{2.15}
$$

Substituting (2.15) in (2.14) and taking the supremum for $\tau \in [0,t_1]$, with $t_1 \le T$ to be determined later, we have

$$
\begin{aligned}
&c^* \sup_{\tau\in[0,t_1]} \int_{A_k(\tau)} |w_k|^{p^-}\,dx + \frac{1}{4}\big(\frac{1}{\lambda}\big)^{p^-} \int_0^{t_1} \int_{A_k(t)} |\nabla w_k|^{p^-}\,dx\,dt \\
&\le \lambda M \int_0^{t_1} \int_{A_k(t)} h|w_k|^{p^-}\,dx\,dt + \lambda e^{\lambda d}\int_0^{t_1} \int_{A_k(t)\setminus A_{k+d}(t)} h\,dx\,dt.
\end{aligned} \tag{2.16}
$$

By the embedding inequality (see [6, 11]), we have

$$
\begin{aligned}
&\Big(\int_0^{t_1} \int_{A_k(t)} |w_k|^{p^- \frac{N+p^-}{N}}\,dx\,dt\Big)^{\frac{N}{N+p^-}} \\
&\le \gamma\Big(\sup_{\tau\in[0,t_1]} \int_{A_k(\tau)} |w_k|^{p^-}\,dx + \int_0^{t_1} \int_{A_k(t)} |\nabla w_k|^{p^-}\,dx\,dt\Big),
\end{aligned} \tag{2.17}
$$

where γ is a constant depending on N, p^-, but independent of $t_1 \le T$. Hence, from (2.16), it follows that

$$
\begin{aligned}
J_{k_{t_1}} &:= \Big(\int_0^{t_1} \int_{A_k(t)} |w_k|^{p^- \frac{N+p^-}{N}}\,dx\,dt\Big)^{\frac{N}{N+p^-}} \\
&\le C\Big(\int_0^{t_1} \int_{A_k(t)} h|w_k|^{p^-}\,dx\,dt + \int_0^{t_1} \int_{A_k(t)\setminus A_{k+d}(t)} h\,dx\,dt\Big),
\end{aligned}
$$

where C is a constant independent of t_1. Consequently, by Hölder's inequality (thanks to the assumption $|\overrightarrow{F}|^{(p^-)'} \in L^r(Q_T)$ with $r > \frac{N+p^-}{p^-}$), we deduce

$$\begin{aligned}
J_{k_{t_1}} &\le C\Big(\int_0^{t_1}\int_{A_k(t)} |w_k|^{p^- \frac{N+p^-}{N}}\,dx\,dt\Big)^{\frac{N}{N+p^-}} \Big(\int_0^{t_1}\int_{A_k(t)} h^{\frac{N+p^-}{p^-}}\,dx\,dt\Big)^{\frac{p^-}{N+p^-}} \\
&\quad + C\Big(\int_0^{t_1}\int_{A_k(t)} h^r\,dx\,dt\Big)^{1/r}\Big(\int_0^{t_1}\mu(A_k(t))dt\Big)^{1-\frac{1}{r}} \\
&\le C\Big(\int_0^{t_1}\int_{A_k(t)} |w_k|^{p^- \frac{N+p^-}{N}}\,dx\,dt\Big)^{\frac{N}{N+p^-}} \|h\|_{L^r(Q_{t_1})}\big(t_1\mu(\Omega)\big)^{\frac{p^-}{N+p^-}-\frac{1}{r}} \\
&\quad + C\|h\|_{L^r(Q_{t_1})}\Big(\int_0^{t_1}\mu(A_k(t))dt\Big)^{1-\frac{1}{r}},
\end{aligned}$$

where $\mu(\Omega)$ represents the Lebesgue measure of Ω. Choosing t_1 small enough such that

$$C\|h\|_{L^r(Q_{t_1})}\,(t_1\mu(\Omega))^{\frac{p^-}{N+p^-}-\frac{1}{r}} \le \frac{1}{2} \tag{2.18}$$

and we obtain

$$J_{k_{t_1}} \le C\|h\|_{L^r(Q_T)}\Big(\int_0^{t_1}\mu(A_k(t))dt\Big)^{1-\frac{1}{r}}. \tag{2.19}$$

For any $l > k \ge \|u_0\|_{L^\infty(\Omega)}$, using (2.2), we conclude that

$$\begin{aligned}
J_{k_{t_1}} &\ge \Big(\int_0^{t_1}\int_{A_k(t)} \big|\frac{\lambda G_k(u)}{p^-}\big|^{p^- \frac{N+p^-}{N}}\,dx\,dt\Big)^{\frac{N}{N+p^-}} \\
&\ge \big(\frac{\lambda}{p^-}\big)^{p^-}\Big(\int_0^{t_1}\int_{A_k(t)} \big(|u|-k\big)^{p^- \frac{N+p^-}{N}}\,dx\,dt\Big)^{\frac{N}{N+p^-}} \\
&\ge \big(\frac{\lambda}{p^-}\big)^{p^-}(l-k)^{p^-}\Big(\int_0^{t_1}\mu(A_l(t))dt\Big)^{\frac{N}{N+p^-}}.
\end{aligned} \tag{2.20}$$

Let $\psi_k = \int_0^{t_1}\mu(A_k(t))dt$. It follows from (2.19) and (2.20) that

$$\psi_l \le \frac{C}{(l-k)^{\frac{p^-(N+p^-)}{N}}}\psi_k^{(1-\frac{1}{r})\frac{N+p^-}{N}}. \tag{2.21}$$

Case 2. $1 < p^- < 2$. In this case, from (2.5) (it should be remarked that the constant d in (2.3) and (2.5) could be the same if we choose d suitably large), we have

$$\int_{A_k(\tau)} \Phi(G_k(u))(\tau)dx \ge c^*\int_{A_{k+d}(\tau)} |w_k|^{p^-}dx + c^*\int_{A_k(\tau)\setminus A_{k+d}(\tau)} |w_k|^2 dx. \tag{2.22}$$

Substituting (2.22) into (2.14) and taking the supremum for $\tau \in [0, t_1]$, where $t_1 \leq T$ to be chosen later, we derive

$$\begin{aligned} & c^* \sup_{\tau\in[0,t_1]} \int_{A_{k+d}(\tau)} |w_k|^{p^-} dx + \frac{1}{4}\big(\frac{1}{\lambda}\big)^{p^-} \int_0^{t_1} \int_{A_{k+d}(t)} |\nabla w_k|^{p^-}\, dx\, dt \\ & + c^* \sup_{\tau\in[0,t_1]} \int_{A_k(\tau)\setminus A_{k+d}(\tau)} |w_k|^2 dx + \frac{1}{4}\big(\frac{1}{\lambda}\big)^{p^-} \int_0^{t_1} \int_{A_k(t)\setminus A_{k+d}(t)} |\nabla w_k|^{p^-}\, dx\, dt \\ & \leq \lambda M \int_0^{t_1} \int_{A_{k+d}(t)} h |w_k|^{p^-}\, dx\, dt + \lambda e^{\lambda d} \int_0^{t_1} \int_{A_k(t)\setminus A_{k+d}(t)} h\, dx\, dt. \end{aligned} \tag{2.23}$$

Again, recall the following embedding estimates [6, 11]:

$$\begin{aligned} & \int_0^{t_1} \int_{A_{k+d}(t)} |w_k|^{p^- \frac{N+p^-}{N}}\, dx\, dt \\ & \leq \gamma^{p^- \frac{N+p^-}{N}} \Big(\sup_{\tau\in[0,t_1]} \int_{A_{k+d}(\tau)} |w_k|^{p^-} dx + \int_0^{t_1} \int_{A_{k+d}(t)} |\nabla w_k|^{p^-}\, dx\, dt \Big)^{1+\frac{p^-}{N}}, \end{aligned} \tag{2.24}$$

$$\begin{aligned} & \int_0^{t_1} \int_{A_k(t)\setminus A_{k+d}(t)} |w_k|^{p^- \frac{N+2}{N}}\, dx\, dt \\ & \leq \gamma^{p^- \frac{N+2}{N}} \Big(\sup_{\tau\in[0,t_1]} \int_{A_k(\tau)\setminus A_{k+d}(\tau)} |w_k|^2 dx \\ & \quad + \int_0^{t_1} \int_{A_k(t)\setminus A_{k+d}(t)} |\nabla w_k|^{p^-}\, dx\, dt \Big)^{1+\frac{p^-}{N}}. \end{aligned} \tag{2.25}$$

Combining (2.24), (2.25) with (2.23), we obtain

$$\begin{aligned} J^{(1)}_{k_{t_1}} & := \Big(\int_0^{t_1} \int_{A_{k+d}(t)} |w_k|^{p^- \frac{N+p^-}{N}}\, dx\, dt \Big)^{\frac{N}{N+p^-}} \\ & \quad + \Big(\int_0^{t_1} \int_{A_k(t)\setminus A_{k+d}(t)} |w_k|^{p^- \frac{N+2}{N}}\, dx\, dt \Big)^{\frac{N}{N+p^-}} \\ & \leq C \int_0^{t_1} \int_{A_{k+d}(t)} h |w_k|^{p^-}\, dx\, dt + C \int_0^{t_1} \int_{A_k(t)\setminus A_{k+d}(t)} h |w_k|^{p^-}\, dx\, dt \\ & \quad + C \int_0^{t_1} \int_{A_k(t)} h\, dx\, dt := (E1) + (E2) + (E3). \end{aligned}$$

We estimate $(E1)$ as follows.

$$\begin{aligned} & (E1) \\ & \leq C \Big(\int_0^{t_1} \int_{A_{k+d}(t)} |w_k|^{p^- \frac{N+p^-}{N}}\, dx\, dt \Big)^{\frac{N}{N+p^-}} \Big(\int_0^{t_1} \int_{A_{k+d}(t)} h^{\frac{N+p^-}{p^-}}\, dx\, dt \Big)^{\frac{p^-}{N+p^-}} \\ & \leq C \Big(\int_0^{t_1} \int_{A_{k+d}(t)} |w_k|^{p^- \frac{N+p^-}{N}}\, dx\, dt \Big)^{\frac{N}{N+p^-}} \|h\|_{L^r(Q_{t_1})} \left(t_1 \mu(\Omega)\right)^{\frac{p^-}{N+p^-} - \frac{1}{r}}. \end{aligned}$$

Using Hölder's inequality and Young's inequality with ϵ, we have

$(E2)$

$$
\begin{aligned}
&\le C\Big(\int_0^{t_1}\int_{A_k(t)\setminus A_{k+d}(t)} |w_k|^{p^- \frac{N+2}{N}}\,dx\,dt\Big)^{\frac{N}{N+2}}\Big(\int_0^{t_1}\int_{A_k(t)\setminus A_{k+d}(t)} h^{\frac{N+2}{2}}\,dx\,dt\Big)^{\frac{2}{N+2}}\\
&\le \frac{1}{2}\Big(\int_0^{t_1}\int_{A_k(t)\setminus A_{k+d}(t)} |w_k|^{p^- \frac{N+2}{N}}\,dx\,dt\Big)^{\frac{N}{N+p^-}}\\
&\quad + C\Big(\int_0^{t_1}\int_{A_k(t)\setminus A_{k+d}(t)} h^{\frac{N+2}{2}}\,dx\,dt\Big)^{\frac{2}{2-p^-}}\\
&\le \frac{1}{2}\Big(\int_0^{t_1}\int_{A_k(t)\setminus A_{k+d}(t)} |w_k|^{p^- \frac{N+2}{N}}\,dx\,dt\Big)^{\frac{N}{N+p^-}}\\
&\quad + C\|h\|_{L^r(Q_{t_1})}^{\frac{N+2}{2-p^-}}\Big(\int_0^{t_1}\mu(A_k(t))dt\Big)^{\frac{2}{2-p^-}(1-\frac{N+2}{2r})}.
\end{aligned}
$$

For $(E3)$, we have

$$
(E3)\le C\|h\|_{L^r(Q_{t_1})}\Big(\int_0^{t_1}\mu(A_k(t))dt\Big)^{1-\frac{1}{r}}.
$$

Now select $t_1\in(0,(\mu(\Omega))^{-1}]$ sufficiently small so that

$$
C\|h\|_{L^r(Q_{t_1})}\,(t_1\mu(\Omega))^{\frac{p^-}{N+p^-}-\frac{1}{r}}\le\frac{1}{2}. \tag{2.26}
$$

From the above estimates, we have

$$
\begin{aligned}
J_{k_{t_1}}^{(1)} &\le C\|h\|_{L^r(Q_{t_1})}\Big(\int_0^{t_1}\mu(A_k(t))dt\Big)^{1-\frac{1}{r}}\\
&\quad + C\|h\|_{L^r(Q_{t_1})}^{\frac{N+2}{2-p^-}}\Big(\int_0^{t_1}\mu(A_k(t))dt\Big)^{\frac{2}{2-p^-}(1-\frac{N+2}{2r})}.
\end{aligned} \tag{2.27}
$$

Noticing that $r>\frac{N+p^-}{p^-}$, after a straightforward computation, we have $\frac{2}{2-p^-}(1-\frac{N+2}{2r})>1-\frac{1}{r}$. Meanwhile, the choice of t_1 ensures $\psi_k\le t_1\mu(\Omega)\le 1$. As a result, (2.27) becomes

$$
J_{k_{t_1}}^{(1)}\le C\Big(\int_0^{t_1}\mu(A_k(t))dt\Big)^{1-\frac{1}{r}}. \tag{2.28}
$$

For any $l>k\ge\|u_0\|_{L^\infty(\Omega)}$, using (2.2), we deduce that

$$
\begin{aligned}
J_{k_{t_1}}^{(1)} &\ge \Big(\int_0^{t_1}\int_{A_{k+d}(t)}|\frac{\lambda G_k(u)}{p^-}|^{p^-\frac{N+p^-}{N}}\,dx\,dt\Big)^{\frac{N}{N+p^-}}\\
&\quad + \Big(\int_0^{t_1}\int_{A_k(t)\setminus A_{k+d}(t)}|\frac{\lambda G_k(u)}{p^-}|^{p^-\frac{N+2}{N}}\,dx\,dt\Big)^{\frac{N}{N+p^-}}\\
&\ge (\frac{\lambda}{p^-})^{p^-}\Big(\int_0^{t_1}\int_{A_{k+d}(t)}(|u|-k)^{p^-\frac{N+p^-}{N}}\,dx\,dt\Big)^{\frac{N}{N+p^-}}\\
&\quad + (\frac{\lambda}{p^-})^{p^-\frac{N+2}{N+p^-}}\Big(\int_0^{t_1}\int_{A_k(t)\setminus A_{k+d}(t)}(|u|-k)^{p^-\frac{N+2}{N}}\,dx\,dt\Big)^{\frac{N}{N+p^-}}\\
&\ge (\frac{\lambda}{p^-})^{p^-}(l-k)^{p^-}\Big(\int_0^{t_1}\mu\left(A_l(t)\cap A_{k+d}(t)\right)dt\Big)^{\frac{N}{N+p^-}}\\
&\quad + (\frac{\lambda}{p^-})^{p^-\frac{N+2}{N+p^-}}(l-k)^{p^-\frac{N+2}{N+p^-}}\Big(\int_0^{t_1}\mu\left(A_l(t)\setminus A_{k+d}(t)\right)dt\Big)^{\frac{N}{N+p^-}}.
\end{aligned}
$$

In fact, we have

$$\begin{aligned}\Big(J^{(1)}_{k_{t_1}}\Big)^{\frac{N+p^-}{N}} \geq \big(\frac{\lambda}{p^-}\big)^{p^- \frac{N+p^-}{N}} (l-k)^{p^- \frac{N+p^-}{N}} \int_0^{t_1} \mu\left(A_l(t)\cap A_{k+d}(t)\right) dt \\ + \big(\frac{\lambda}{p^-}\big)^{p^- \frac{N+2}{N}} (l-k)^{p^- \frac{N+2}{N}} \int_0^{t_1} \mu\left(A_l(t)\backslash A_{k+d}(t)\right) dt.\end{aligned} \tag{2.29}$$

Consequently, combining (2.29) and (2.28), with $\psi_k = \int_0^{t_1} \mu(A_k(t))dt$, we have again

$$\psi_l \leq \frac{C}{\min\left\{(l-k)^{\frac{p^-(N+p^-)}{N}}, (l-k)^{\frac{p^-(N+2)}{N}}\right\}} \psi_k^{(1-\frac{1}{r})\frac{N+p^-}{N}}. \tag{2.30}$$

Now we have proved (2.30) and (2.21). Our hypothesis $r > \frac{N+p^-}{p^-}$ guarantees $\left(1-\frac{1}{r}\right)\frac{N+p^-}{N} > 1$. Therefore, thanks to the iteration lemma in [24], we eventually obtain that $\psi_{(\|u_0\|_{L^\infty(\Omega)}+D)} = 0$, where $D > 0$ is a constant depending only on $p^-, N, t_1, r, b, \Omega, \||\overrightarrow{F}|^{(p^-)'}\|_{L^r(Q_{t_1})}$. This proves that, for a fixed λ validating Lemma 2.1,

$$\|u(x,t)\|_{L^\infty(Q_{t_1})} \leq \|u_0\|_{L^\infty(\Omega)} + D. \tag{2.31}$$

Finally, partition the time interval $[0,T]$ into finite subintervals $[0,t_1]$, $[t_1,t_2]\cdots$ $[t_{n-1},T]$ such that the conditions similar to those in (2.18) and (2.26) are available for each subinterval $[t_i,t_{i+1}]$; then, using the same method, we deduce an inequality of the form (2.31). Eventually, we conclude that $\|u(x,t)\|_{L^\infty(Q_T)} \leq \|u_0\|_{L^\infty(\Omega)} + C$, where the constant C depends only on $p^-, N, T, r, b, \Omega, \||\overrightarrow{F}|^{(p^-)'}\|_{L^r(Q_T)}$. □

3. Application to the existence of solutions to (1.1)

With the L^∞-estimate obtained above, we can prove the existence of solutions to Problem (1.1). First, we recall a lemma from [13], which plays an important role in our estimates.

Lemma 3.1. *Let $\theta(s) = se^{\eta s^2}$, $s\in\mathbb{R}$, where $\eta \geq \frac{b^2}{4a^2}$ is fixed, and let $\Theta(s) = \int_0^s \theta(\tau)d\tau$. Then $\theta(0)=0$ and*

$$\Theta(s) \geq 0, \quad a\theta'(s) - b|\theta(s)| \geq \frac{a}{2}, \quad \forall\, s\in\mathbb{R}. \tag{3.1}$$

We are now in a position to prove the existence of solutions to (1.1) based on the L^∞ estimate.

Theorem 3.2. *Under the hypotheses* (H1) *and* (H2)*, there exists a solution $u \in L^\infty(Q_T)\cap V$ to* (1.1).

Proof. **Step 1: The approximation equation.** We introduce the following approximation equation of Problem (1.1).

$$\begin{gathered}\frac{\partial u_n}{\partial t} - \operatorname{div}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n\Big) = B(x,t)\min\{|\nabla u_n|^{p(x)}, n\} - \operatorname{div}\overrightarrow{F}(x,t), \\ (x,t)\in Q_T, \\ u_n(x,t) = 0, \quad (x,t)\in\Gamma_T, \\ u_n(x,0) = u_0(x) \in L^\infty(\Omega), \quad x\in\Omega.\end{gathered} \tag{3.2}$$

For each fixed $n\in\mathbb{N}$, since $\min\left\{|\nabla u_n|^{p(x)}, n\right\}$ is bounded, the existence of a weak solution $u_n \in L^\infty \cap V$ to (3.2) follows from the standard methods (for instance, the pseudo-monotonicity operator theory in [12, 10, 20], or the difference and variation methods in [21]).

We write the term $B(x,t)\min\{|\nabla u_n|^{p(x)}, n\}$ in (3.2) as $B_n(x,t)|\nabla u_n|^{p(x)}$, with $B_n(x,t)$ defined by

$$B_n(x,t) = \begin{cases} 0, & \text{if } |\nabla u_n(x,t)| = 0, \\ B(x,t)\frac{\min\{|\nabla u_n(x,t)|^{p(x)}, n\}}{|\nabla u_n(x,t)|^{p(x)}}, & \text{if } |\nabla u_n(x,t)| \neq 0. \end{cases}$$

Then $B_n \in L^\infty(Q_T)$ satisfies $0 \le B_n(x,t) \le B(x,t) \le b$. Hence, by Theorem 2.3, we have the uniform bound

$$\|u_n(x,t)\|_{L^\infty(Q_T)} \le \|u_0\|_{L^\infty(\Omega)} + C, \tag{3.3}$$

where C depends only on $p^-, N, T, r, b, \Omega, \||\overrightarrow{F}|^{(p^-)'}\|_{L^r(Q_T)}$ and it is independent of n. Our goal is to show that a subsequence of the approximate solution sequence $\{u_n\}$ converges to a measurable function u, which coincides with a weak solution of Problem (1.1).

Step 2: The weak convergence $u_n \rightharpoonup u$ in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$. Choosing $\theta(u_n)$ as a testing function in (3.2), we have

$$\begin{aligned} &\int_0^T \langle \frac{\partial u_n}{\partial t}, \theta(u_n)\rangle dt + \iint_{Q_T} |\nabla u_n|^{p(x)}\theta'(u_n)\,dx\,dt \\ &= \iint_{Q_T} B\min\{|\nabla u_n|^{p(x)}, n\}\theta(u_n)\,dx\,dt + \iint_{Q_T} \theta'(u_n)\nabla u_n\cdot\overrightarrow{F}\,dx\,dt. \end{aligned} \tag{3.4}$$

Lemma 2.2 yields $\int_0^T \langle \frac{\partial u_n}{\partial t}, \theta(u_n)\rangle dt = \int_\Omega [\Theta(u_n(T)) - \Theta(u_0)]\,dx$. Using Young's inequality with ϵ in the last term of the right-hand side, (3.4) becomes

$$\begin{aligned} &\int_\Omega \Theta(u_n(T))dx + \iint_{Q_T} |\nabla u_n|^{p(x)}\theta'(u_n)\,dx\,dt \\ &\le \int_\Omega \Theta(u_0)dx + \iint_{Q_T} B|\nabla u_n|^{p(x)}|\theta(u_n)|\,dx\,dt \\ &\quad + \epsilon\iint_{Q_T} |\nabla u_n|^{p(x)}\theta'(u_n)\,dx\,dt + \iint_{Q_T} \epsilon^{-\frac{1}{p(x)-1}}|\overrightarrow{F}|^{p'(x)}\theta'(u_n)\,dx\,dt. \end{aligned}$$

Taking $\epsilon = 1/2$, we rewrite the above inequality as

$$\begin{aligned} &\int_\Omega \Theta(u_n(T))dx + \iint_{Q_T} \Big[\frac{1}{2}\theta'(u_n) - B|\theta(u_n)|\Big]|\nabla u_n|^{p(x)}\,dx\,dt \\ &\le \int_\Omega \Theta(u_0)dx + \Big(\frac{1}{2}\Big)^{-\frac{1}{p^- -1}}\iint_{Q_T} |\overrightarrow{F}|^{p'(x)}\theta'(u_n)\,dx\,dt. \end{aligned} \tag{3.5}$$

With the aid of (3.1) in Lemma 3.1 (with $a = \frac{1}{2}$, and $\frac{1}{2}\theta'(u_n) - B|\theta(u_n)| \ge \frac{1}{2}\theta'(u_n) - b|\theta(u_n)| \ge \frac{1}{4}$), we deduce that

$$\frac{1}{4}\iint_{Q_T} |\nabla u_n|^{p(x)}\,dx\,dt \le \int_\Omega \Theta(u_0)dx + \Big(\frac{1}{2}\Big)^{-\frac{1}{p^- -1}}\iint_{Q_T} |\overrightarrow{F}|^{p'(x)}\theta'(u_n)\,dx\,dt. \tag{3.6}$$

Since u_n is uniformly bounded with respect to n and $u_0 \in L^\infty(\Omega)$, it follows that

$$\iint_{Q_T} |\nabla u_n|^{p(x)}\,dx\,dt \le C\Big(|\overrightarrow{F}|_{L^{p'(x)}(Q_T)}, \|u_0\|_{L^\infty(\Omega)}, \sup_n \|u_n\|_{L^\infty(Q_T)}\Big). \tag{3.7}$$

This implies that u_n is uniformly bounded in V. By the way, obviously, the following inequality holds

$$\begin{aligned}
&|u_n|^{p^-}_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))}\\
&= \int_0^T |\nabla u_n|^{p^-}_{L^{p(x)}(\Omega)} dt\\
&\le \max\Big\{\Big(\iint_{Q_T} |\nabla u_n|^{p(x)}\,dx\,dt\Big)^{\frac{p^-}{p^+}} T^{1-\frac{p^-}{p^+}}, \iint_{Q_T} |\nabla u_n|^{p(x)}\,dx\,dt\Big\},
\end{aligned}$$

which implies

$$|u_n|_{L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))} \le C\Big(|\overrightarrow{F}|_{L^{p'(x)}(Q_T)}, \|u_0\|_{L^\infty(\Omega)}, \sup_n \|u_n\|_{L^\infty(Q_T)}, p^-, p^+, T\Big). \tag{3.8}$$

Therefore, u_n is bounded in the space $L^\infty(Q_T) \cap L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$. We can extract a subsequence of u_n, still denoted by u_n, such that $u_n \rightharpoonup u$, weakly in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$. Simultaneously, $u_n \rightharpoonup u$, weakly* in $L^\infty(Q_T)$.

Step 3: The strong convergence $u_n \to u$ in $L^{p^-}(0,T;L^{p(x)}(\Omega))$. From (3.2), we deduce that

$$\frac{\partial u_n}{\partial t} = \operatorname{div}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n - \overrightarrow{F}\Big) + B\min\{|\nabla u_n|^{p(x)}, n\} \in V^* + L^1(Q_T). \tag{3.9}$$

For each $v \in V$, by the definition of the norm on V and $p(x)$-Hölder's inequality, we have

$$\begin{aligned}
&\sup_{\|v\|_V\le 1} |\langle \operatorname{div}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n - \overrightarrow{F}\Big), v\rangle_{V^*,V}|\\
&= \sup_{\|v\|_V\le 1} \Big|\iint_{Q_T}\Big(-|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla v + \overrightarrow{F}\cdot\nabla v\Big)\,dx\,dt\Big|\\
&\le \sup_{\|v\|_V\le 1}\big[2||\nabla u_n|^{p(x)-2}\nabla u_n|_{L^{p'(x)}(Q_T)}|\nabla v|_{L^{p(x)}(Q_T)} + 2|\overrightarrow{F}|_{L^{p'(x)}(Q_T)}|\nabla v|_{L^{p(x)}(Q_T)}\big]\\
&\le 2\max\Big\{\Big(\iint_{Q_T}|\nabla u_n|^{p(x)}\,dx\,dt\Big)^{\frac{1}{(p')^+}}, \Big(\iint_{Q_T}|\nabla u_n|^{p(x)}\,dx\,dt\Big)^{\frac{1}{(p')^-}}\Big\}\\
&\quad + 2|\overrightarrow{F}|_{L^{p'(x)}(Q_T)}.
\end{aligned}$$

It follows from (3.7) that

$$\big\|\operatorname{div}\big(|\nabla u_n|^{p(x)-2}\nabla u_n - \overrightarrow{F}\big)\big\|_{V^*} \le C, \tag{3.10}$$

where C is independent of n. Thanks to the embedding relationship

$$\begin{aligned}
&L^{(p^-)'}(0,T;W^{-1,p'(x)}(\Omega)) \hookrightarrow V^*\\
&\hookrightarrow L^{(p^+)'}(0,T;W^{-1,p'(x)}(\Omega)) = L^{(p')^-}(0,T;W^{-1,p'(x)}(\Omega)),
\end{aligned} \tag{3.11}$$

from (3.10), (3.7) and (3.9), we conclude that $\frac{\partial u_n}{\partial t}$ is bounded in the space $L^{(p')^-}(0,T;W^{-1,p'(x)}(\Omega)) + L^1(Q_T)$.

For a fixed s such that $s > \frac{N}{2}+1$, the following embedding relationships hold $1^\diamond$ $s > \frac{N}{2}$, we have $H_0^s(\Omega) \hookrightarrow L^\infty(\Omega)$, and then $L^1(\Omega) \hookrightarrow H^{-s}(\Omega)$; $2^\diamond$ $s-1 > \frac{N}{2}$, one has $H_0^s(\Omega) \hookrightarrow W^{1,p(x)}(\Omega)$, consequently, $W^{-1,p'(x)}(\Omega) \hookrightarrow H^{-s}(\Omega)$. As a result, we have

$$\|\frac{\partial u_n}{\partial t}\|_{L^1(0,T;H^{-s}(\Omega))} \leq C, \tag{3.12}$$

where C is independent of n. Noticing that $W_0^{1,p(x)}(\Omega) \overset{\text{compact}}{\hookrightarrow} L^{p(x)}(\Omega) \hookrightarrow H^{-s}(\Omega)$ and by (3.8), we employ Simon's compactness theorem in [17] to obtain that $u_n \to u$, strongly in $L^{p^-}(0,T;L^{p(x)}(\Omega))$.

Step 4: The convergence $\nabla u_n \to \nabla u$ a.e. in Q_T. From the strong convergence obtained in Step 3, one may choose a subsequence of u_n, still denoted by u_n for simplicity, such that $u_n \to u$, a.e. in Q_T. We now use Egoroff's theorem (recalling $\mu(Q_T) < +\infty$) to obtain, for fixed $\delta > 0$, there exists a measurable closed subset $E_\delta \subset Q_T$ such that

(1) $\mu(Q_T - E_\delta) \leq \delta$;
(2) $u_n \rightrightarrows u$ uniformly on E_δ. It follows that $|u_n - u_m| < k$, for fixed $k > 0$, and sufficiently large m, n.

Let ζ be a cut-off function satisfying $\zeta \in C_0^\infty(Q_T)$; $\zeta = 1$ on E_δ; $0 \leq \zeta \leq 1$ on Q_T. Introduce the following truncation function

$$T_k(s) = \begin{cases} s, & \text{if } |s| < k, \\ k, & \text{if } s \geq k, \\ -k, & \text{if } s \leq -k. \end{cases}$$

Subtracting Equations (3.2) for different parameters n and m, we have

$$\begin{gathered} \frac{\partial(u_n - u_m)}{\partial t} - \operatorname{div}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_m|^{p(x)-2}\nabla u_m\Big) \\ = B\Big(\min\{|\nabla u_n|^{p(x)}, n\} - \min\{|\nabla u_m|^{p(x)}, m\}\Big), \quad (x,t) \in Q_T, \\ (u_n - u_m)(x,t) = 0, \quad (x,t) \in \Gamma_T, \\ (u_n - u_m)(x,0) = 0, \quad x \in \Omega. \end{gathered} \tag{3.13}$$

Since T_k is Lipschitz continuous, one may take $\zeta T_k(u_n - u_m)$ as a test function in (3.13); hence we have

$$\begin{aligned} &\int_0^T \langle \frac{\partial(u_n - u_m)}{\partial t}, \zeta T_k(u_n - u_m)\rangle dt \\ &+ \iint_{Q_T} \big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_m|^{p(x)-2}\nabla u_m\big) \cdot \big(\nabla u_n - \nabla u_m\big)\zeta T_k'(u_n - u_m)\, dx\, dt \\ &+ \iint_{Q_T} \big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_m|^{p(x)-2}\nabla u_m\big) \cdot \nabla\zeta T_k(u_n - u_m)\, dx\, dt \\ &= \iint_{Q_T} B\Big(\min\{|\nabla u_n|^{p(x)}, n\} - \min\Big\{|\nabla u_m|^{p(x)}, m\Big\}\Big)\zeta T_k(u_n - u_m)\, dx\, dt. \end{aligned} \tag{3.14}$$

Since $\zeta(x,0) = \zeta(x,T)$, by Lemma 2.2, we handle the first term on the left-hand side of (3.14) as follows,

$$\int_0^T \langle \frac{\partial(u_n - u_m)}{\partial t}, \zeta T_k(u_n - u_m)\rangle dt = -\int_\Omega \int_0^T \zeta_t \int_0^{u_n - u_m} T_k(s) ds dt dx.$$

Noticing that T_k is an odd function, $|T_k(s)| \le k$, we get

$$\begin{aligned}
&\iint_{Q_T} \Big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_m|^{p(x)-2}\nabla u_m\Big) \cdot (\nabla u_n - \nabla u_m)\, \zeta T_k'(u_n - u_m)\, dx\, dt \\
&\le k \iint_{Q_T} |\zeta_t||u_n - u_m|\, dx\, dt \\
&\quad + k \iint_{Q_T} ||\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_m|^{p(x)-2}\nabla u_m||\nabla \zeta|\, dx\, dt \\
&\quad + bk \iint_{Q_T} |\min\{|\nabla u_n|^{p(x)}, n\} - \min\{|\nabla u_m|^{p(x)}, m\}|\zeta\, dx\, dt \le kC(\delta).
\end{aligned}$$

Noting that $T_k' \ge 0$, $T_k'(s) = 1$ on $|s| < k$ and that u_n converges uniformly on E_δ, we obtain

$$\begin{aligned}
&\iint_{E_\delta} \big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_m|^{p(x)-2}\nabla u_m\big) \cdot (\nabla u_n - \nabla u_m)\; dx\, dt \\
&= \iint_{E_\delta} \big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_m|^{p(x)-2}\nabla u_m\big) \cdot (\nabla u_n - \nabla u_m)\, T_k'(u_n - u_m)\, dx\, dt \\
&\le \iint_{Q_T} \big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_m|^{p(x)-2}\nabla u_m\big) \cdot (\nabla u_n - \nabla u_m)\, \zeta T_k'(u_n - u_m)\, dx\, dt.
\end{aligned}$$

Hence, based on the above estimates, by (3.3), (3.7) and the arbitrariness of k, we have

$$\limsup_{n,m\to+\infty} \iint_{E_\delta} \Big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_m|^{p(x)-2}\nabla u_m\Big) \cdot (\nabla u_n - \nabla u_m)\; dx\, dt = 0. \tag{3.15}$$

From (3.15) and using the method in [19, 24] (or the method to be used in Step 5 below), we may obtain that $\iint_{E_\delta} |\nabla u_n - \nabla u_m|^{p(x)}\, dx\, dt \to 0$ (it is equivalent to $|\nabla u_n - \nabla u_m|_{L^{p(x)}(E_\delta)} \to 0$), which shows that $\{\nabla u_n\}_{n=1}^\infty$ is a Cauchy sequence in $(L^{p(x)}(E_\delta))^N$. Thus, we can extract a subsequence of u_n, still denoted by itself, such that $\nabla u_n \to \alpha$, strongly in $(L^{p^-}(E_\delta))^N$. In step 3, we know that $u_n \to u$, strongly in $L^{p^-}(0, T; L^{p(x)}(\Omega))$, it is easy to say $u_n \to u$, strongly in $L^{p^-}(E_\delta)$. It follows from above analysis that $\alpha = \nabla u$, i.e. $\nabla u_n \to \nabla u$ a.e. in E_δ. The arbitrariness of δ indicates that $\nabla u_n \to \nabla u$ a.e. in Q_T.

Step 5: The convergence $\iint_{Q_T} |\nabla u_n - \nabla u|^{p(x)}\, dx\, dt \to 0$. For the function θ defined in Lemma 3.1, it follows that $\theta(u_n - u_m) \in L^\infty(Q_T) \cap V$ since $u_n, u_m \in L^\infty(Q_T) \cap V$. Therefore, $\theta(u_n - u_m)$ can be taken as a test function in (3.13) to yield that

$$\begin{aligned}
&\int_0^T \langle \frac{\partial(u_n - u_m)}{\partial t}, \theta(u_n - u_m)\rangle dt \\
&+ \iint_{Q_T} \Big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_m|^{p(x)-2}\nabla u_m\Big) \cdot (\nabla u_n - \nabla u_m)\, \theta'(u_n - u_m)\, dx\, dt \\
&= \iint_{Q_T} B\Big(\min\Big\{|\nabla u_n|^{p(x)}, n\Big\} - \min\Big\{|\nabla u_m|^{p(x)}, m\Big\}\Big)\, \theta(u_n - u_m)\, dx\, dt.
\end{aligned} \tag{3.16}$$

Use (3.1) in Lemma 3.1 to estimate the first term on the left-hand side of (3.16) to obtain

$$\int_0^T \langle \frac{\partial(u_n - u_m)}{\partial t}, \theta(u_n - u_m)\rangle dt = \int_\Omega \Theta(u_n - u_m)(T) dx \geq 0.$$

After a simple computation, the right-hand side of (3.16) can be estimated as follows.

$$\begin{aligned}
&\iint_{Q_T} B\left(\min\left\{|\nabla u_n|^{p(x)}, n\right\} - \min\left\{|\nabla u_m|^{p(x)}, m\right\}\right)\theta(u_n - u_m)\,dx\,dt \\
&\leq b\iint_{Q_T}\left(|\nabla u_n|^{p(x)} + |\nabla u_m|^{p(x)}\right)|\theta(u_n - u_m)|\,dx\,dt \\
&= b\iint_{Q_T}\left(|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla u_m + |\nabla u_m|^{p(x)-2}\nabla u_m\cdot\nabla u_n\right)|\theta(u_n - u_m)|\,dx\,dt \\
&\quad + b\iint_{Q_T}\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_m|^{p(x)-2}\nabla u_m\right) \\
&\quad \cdot(\nabla u_n - \nabla u_m)\,|\theta(u_n - u_m)|\,dx\,dt.
\end{aligned}$$

Consequently, (3.16) can be estimated as

$$\begin{aligned}
&\iint_{Q_T}\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u_m|^{p(x)-2}\nabla u_m\right) \\
&\quad\cdot(\nabla u_n - \nabla u_m)\left[\theta'(u_n - u_m) - b|\theta(u_n - u_m)|\right]\,dx\,dt \\
&\leq b\iint_{Q_T}\left(|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla u_m + |\nabla u_m|^{p(x)-2}\nabla u_m\cdot\nabla u_n\right)|\theta(u_n - u_m)|\,dx\,dt.
\end{aligned} \tag{3.17}$$

With the help of (3.1) in Lemma 3.1 (with $a = 1$), since $\nabla u_n \to \nabla u$ a.e. in Q_T (Step 4), we may utilize Fatou's Lemma in (3.17) as $m \to +\infty$ to obtain that

$$\begin{aligned}
E(n) &:= \iint_{Q_T}\left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right)\cdot(\nabla u_n - \nabla u)\,dx\,dt \\
&\leq 2b\iint_{Q_T}\left(|\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla u + |\nabla u|^{p(x)-2}\nabla u\cdot\nabla u_n\right)|\theta(u_n - u)|\,dx\,dt \\
&\leq 4b\left||\nabla u_n|^{p(x)-2}\nabla u_n\right|_{L^{p'(x)}(Q_T)}|\theta(u_n - u)\nabla u|_{L^{p(x)}(Q_T)} \\
&\quad + 4b\left||\nabla u|^{p(x)-2}\nabla u\theta(u_n - u)\right|_{L^{p'(x)}(Q_T)}|\nabla u_n|_{L^{p(x)}(Q_T)} \\
&\leq C\max\left\{\left(\iint_{Q_T}|\nabla u_n|^{p(x)}\,dx\,dt\right)^{\frac{1}{(p')^{\pm}}}\right\} \\
&\quad\times\max\left\{\left(\iint_{Q_T}|\theta(u_n - u)|^{p(x)}|\nabla u|^{p(x)}\,dx\,dt\right)^{\frac{1}{p^{\pm}}}\right\} \\
&\quad + C\max\left\{\left(\iint_{Q_T}|\theta(u_n - u)|^{p'(x)}|\nabla u|^{p(x)}\,dx\,dt\right)^{\frac{1}{(p')^{\pm}}}\right\} \\
&\quad\times\max\left\{\left(\iint_{Q_T}|\nabla u_n|^{p(x)}\,dx\,dt\right)^{\frac{1}{p^{\pm}}}\right\} \\
&\leq C\max\left\{\left(\iint_{Q_T}|\theta(u_n - u)|^{p(x)}|\nabla u|^{p(x)}\,dx\,dt\right)^{\frac{1}{p^{\pm}}}\right\}
\end{aligned}$$

$$+ C\max\Big\{\Big(\iint_{Q_T} |\theta(u_n-u)|^{p'(x)}|\nabla u|^{p(x)}\,dx\,dt\Big)^{\frac{1}{(p')^{\pm}}}\Big\}.$$

In view of (3.7) and (3.3), $\theta(u_n-u)$ is uniformly bounded. The Lebesgue dominated convergence theorem yields

$$E(n) := \iint_{Q_T}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\Big)\cdot(\nabla u_n-\nabla u)\,dx\,dt \to 0. \tag{3.18}$$

We now estimate

$$\begin{aligned} &\iint_{Q_T}|\nabla u_n-\nabla u|^{p(x)}\,dx\,dt\\ &= \int_0^T\int_{\{x\in\Omega;p(x)\ge 2\}}|\nabla u_n-\nabla u|^{p(x)}\,dx\,dt\\ &\quad + \int_0^T\int_{\{x\in\Omega;1<p(x)<2\}}|\nabla u_n-\nabla u|^{p(x)}\,dx\,dt = I^{(1)}+I^{(2)}. \end{aligned}\tag{3.19}$$

Applying the following basic inequality, for any $y,z\in\mathbb{R}^N$,

$$\Big(|y|^{p(x)-2}y-|z|^{p(x)-2}z\Big)\cdot(y-z)\ge\begin{cases}2^{2-p^+}|y-z|^{p(x)}, & \text{if } p(x)\ge 2,\\ (p^- -1)\frac{|y-z|^2}{(|y|+|z|)^{2-p(x)}}, & \text{if } 1<p(x)<2.\end{cases}$$

we compute the two parts in (3.19):

$$\begin{aligned} I^{(1)} &\le \frac{1}{2^{2-p^+}}\int_0^T\int_{\{x\in\Omega;p(x)\ge 2\}}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n\\ &\quad - |\nabla u|^{p(x)-2}\nabla u\Big)\cdot(\nabla u_n-\nabla u)\,dx\,dt\\ &\le 2^{p^+-2}E(n)\to 0. \end{aligned}\tag{3.20}$$

Using $p(x)$-Hölder's inequality, for $I^{(2)}$, by (3.7) and (3.18), we have

$$\begin{aligned} I^{(2)} &= \int_0^T\int_{\{x\in\Omega;1<p(x)<2\}}\frac{|\nabla u_n-\nabla u|^{p(x)}}{(|\nabla u_n|+|\nabla u|)^{\frac{p(x)}{2}(2-p(x))}}\Big(|\nabla u_n|\\ &\quad + |\nabla u|\Big)^{\frac{p(x)}{2}(2-p(x))}\,dx\,dt\\ &\le 2\Big|\frac{|\nabla u_n-\nabla u|^{p(x)}}{(|\nabla u_n|+|\nabla u|)^{\frac{p(x)}{2}(2-p(x))}}\Big|_{L^{\frac{2}{p(x)}}(Q_T)}\Big|\Big(|\nabla u_n|\\ &\quad + |\nabla u|\Big)^{\frac{p(x)}{2}(2-p(x))}\Big|_{L^{\frac{2}{2-p(x)}}(Q_T)}\\ &\le 2\max\Big\{\Big(\iint_{Q_T}\frac{|\nabla u_n-\nabla u|^2}{(|\nabla u_n|+|\nabla u|)^{2-p(x)}}\,dx\,dt\Big)^{\frac{p^{\pm}}{2}}\Big\}\\ &\quad\times\max\Big\{\Big(\iint_{Q_T}(|\nabla u_n|+|\nabla u|)^{p(x)}\,dx\,dt\Big)^{\frac{2-p^{\pm}}{2}}\Big\}\\ &\le C\max\Big\{\big(\frac{1}{p^- -1}\big)^{\frac{p^{\pm}}{2}}(E(n))^{\frac{p^{\pm}}{2}}\Big\} \end{aligned}$$

$$\times \max\Big\{\Big(\iint_{Q_T}\Big(|\nabla u_n|^{p(x)}+|\nabla u|^{p(x)}\Big)\,dx\,dt\Big)^{\frac{2-p^{\pm}}{2}}\Big\}\to 0. \tag{3.21}$$

Combining (3.19), (3.20) and (3.21), we arrive at

$$\iint_{Q_T}|\nabla u_n-\nabla u|^{p(x)}\,dx\,dt\to 0, \tag{3.22}$$

which implies

$$|\nabla u_n-\nabla u|_{L^{p(x)}(Q_T)}\to 0; \tag{3.23}$$

that is, $u_n\to u$ strongly in the solution space V (simultaneously, $u_n\to u$, strongly in $L^{p^-}(0,T;W_0^{1,p(x)}(\Omega))$).

Step 6: Passing to the limit. It follows from (3.23), the property of Nemytskii operator ([10, 20]) and generalized Lebesgue dominated convergence theorem that

$$\begin{gathered}|\nabla u_n|^{p(x)-2}\nabla u_n\to|\nabla u|^{p(x)-2}\nabla u,\ \text{strongly in}\ \Big(L^{p'(x)}(Q_T)\Big)^N,\\ \min\big\{|\nabla u_n|^{p(x)},n\big\}\to|\nabla u|^{p(x)},\ \text{strongly in}\,L^1(Q_T).\end{gathered}$$

For every $v\in V$,

$$\begin{aligned}&\big|\langle-\operatorname{div}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\Big),v\rangle_{V^*,V}\big|\\&=\big|\iint_{Q_T}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\Big)\cdot\nabla v\,dx\,dt\big|\\&\le 2\big||\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\big|_{L^{p'(x)}(Q_T)}|\nabla v|_{L^{p(x)}(Q_T)}.\end{aligned}$$

It follows that

$$\begin{aligned}&\|-\operatorname{div}\Big(|\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u\Big)\|_{V^*}\\&\le 2||\nabla u_n|^{p(x)-2}\nabla u_n-|\nabla u|^{p(x)-2}\nabla u|_{L^{p'(x)}(Q_T)}\to 0.\end{aligned}$$

Therefore, for the principal term in the approximate equation (3.2), we have

$$-\operatorname{div}(|\nabla u_n|^{p(x)-2}\nabla u_n)\to-\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u),\quad\text{strongly in } V^*.$$

As a consequence, one has $u_{nt}\to u_t$, strongly in $V^*+L^1(Q_T)$.

On the other hand, as stated in Step 3, $V^*+L^1(Q_T)\hookrightarrow L^1(0,T;H^{-s}(\Omega))$ for s sufficiently large. Therefore, from (3.8) and (3.12), we deduce (according to $W^{1,1}(0,T;H^{-s}(\Omega))\hookrightarrow C([0,T];H^{-s}(\Omega))$ in [8]) that $u_n\to u$, strongly in $C([0,T];H^{-s}(\Omega))$, from which $u_n(x,0)=u_0(x)$ makes a perfect sense.

Finally, since $u_n(x,0)\to u(x,0)$, strongly in $H^{-s}(\Omega)$, it follows that $u(x,0)=u_0(x)$. This proves that $u\in V\cap L^\infty(Q_T)$ is a weak solution to Problem (1.1). $\square$

4. Appendix

Proof of Lemma 2.1. Note that

$$\varphi'(s)=\begin{cases}\lambda e^{\lambda s}, & s\ge 0,\\ \lambda e^{-\lambda s}, & s\le 0.\end{cases}$$

(1) Obviously, $|\varphi(s)|=e^{\lambda|s|}-1\ge\lambda|s|$. Remember that $\lambda\ge\frac12+\frac{b}{a}$. If $s\ge 0$, then

$$a\lambda e^{\lambda s}-b(e^{\lambda s}-1)\ge(a\lambda-b)e^{\lambda s}\ge\frac{a}{2}e^{\lambda s}.$$

If $s \le 0$, then

$$a\lambda e^{-\lambda s} - b(e^{-\lambda s} - 1) \ge (a\lambda - b)e^{-\lambda s} \ge \frac{a}{2}e^{-\lambda s}.$$

(2) The inequality $\lambda e^{\lambda s} \le \lambda M[e^{\lambda \frac{s}{p^-}} - 1]^{p^-}$ is equivalent to $[\frac{\exp(\lambda \frac{s}{p^-})}{\exp(\lambda \frac{s}{p^-} - 1)}]^{p^-} \le M$, which, for $s \ge d$, is guaranteed by

$$\lim_{s\to+\infty} \frac{\exp(\lambda \frac{s}{p^-})}{\exp(\lambda \frac{s}{p^-} - 1)} = 1.$$

Likewise, the inequality $e^{\lambda s} - 1 \le M\left[e^{\lambda \frac{s}{p^-}} - 1\right]^{p^-}$ for $s \ge d$ is ensured by the limit

$$\lim_{s\to+\infty} \frac{\exp(\lambda s)}{\exp(\lambda \frac{s}{p^-} - 1)^{p^-}} = 1.$$

(3) We prove the case $1 < p^- < 2$ only; the proof of the case $p^- \ge 2$ is entirely similar. The desired inequalities follow easily from the following limits:

$$\lim_{s\to+\infty} \frac{\frac{1}{\lambda}(e^{\lambda s} - 1) - s}{(e^{\lambda \frac{s}{p^-}} - 1)^{p^-}} = \frac{1}{\lambda}; \quad \lim_{s\to+\infty} \frac{\frac{1}{\lambda}(e^{\lambda s} - 1) - s}{(e^{\lambda \frac{s}{p^-}} - 1)^{2}} = 2\lambda.$$

□

Proof of Lemma 2.2. Since $\pi \in C^1$ with $\pi(0) = 0$ and π, π' are bounded, it follows that $\pi(u) \in V \cap L^\infty(Q_T)$. The left-hand side of (2.6) exists. By Lemma 3.2 in [3] or [15], it follows from $u \in V$ with $u_t \in V^* + L^1(Q_T)$ that $u \in C([0,T]; L^1(\Omega))$ and hence $\Pi(u) \in C([0,T]; L^1(\Omega))$. So, the right-hand side of (2.6) does exist. For the decomposition of the time derivative $u_t = \alpha^{(1)} + \alpha^{(2)} \in V^* + L^1(Q_T)$, noting the embedding relationship

$$L^{p^+}(0,T; W_0^{1,p(x)}(\Omega)) \hookrightarrow V \hookrightarrow L^{p^-}(0,T; W_0^{1,p(x)}(\Omega)),$$

by standard mollification method in [18], there exist $u_n \in C^\infty([0,T]; W_0^{1,p(x)}(\Omega))$, $u_{nt} = \alpha_n^{(1)} + \alpha_n^{(2)}$, $\alpha_n^{(1)} \in C^\infty([0,T]; W^{-1,p'(x)}(\Omega))$, $\alpha_n^{(2)} \in C^\infty([0,T]; L^1(\Omega))$ such that $u_n \to u$, strongly in V; $\alpha_n^{(1)} \to \alpha^{(1)}$, strongly in V^*; $\alpha_n^{(2)} \to \alpha^{(2)}$, strongly in $L^1(0,T; L^1(\Omega))$. Because $\Pi(u_n) \in C^1([0,T]; L^1(\Omega))$ and $\pi(u_n) \in V \cap L^\infty(Q_T)$, we have

$$\begin{aligned} \Pi(u_n(T)) - \Pi(u_n(0)) &= \int_0^T [\Pi(u_n)]_t dt \\ &= \langle \alpha_n^{(1)}, \pi(u_n)\rangle_{V^*,V} + \iint_{Q_T} \alpha_n^{(2)} \pi(u_n)\, dx\, dt. \end{aligned} \tag{4.1}$$

Since $u_n \to u$, strongly in $C([0,T]; L^1(\Omega))$, we have $u_n \to u$, a.e. in Q_T. (If necessary, by a further subsequence to be denoted by the same u_n.) Furthermore, the sequence $\pi(u_n) \to \pi(u)$, a.e. in Q_T and remains bounded; hence $\pi(u_n) \to \pi(u)$, weakly* in $L^\infty(Q_T)$. Combing with $\alpha_n^{(2)} \to \alpha^{(2)}$, strongly in $L^1(Q_T)$, one has $\iint_{Q_T} \alpha_n^{(2)} \pi(u_n)\, dx\, dt \to \iint_{Q_T} \alpha^{(2)} \pi(u)\, dx\, dt$. Moreover, from $u_n \to u$, strongly in V and the properties of π, one has $\pi(u_n) \to \pi(u)$, strongly in V. Together with $\alpha_n^{(1)} \to \alpha^{(1)}$, strongly in V^*, it yields $\langle \alpha_n^{(1)}, \pi(u_n)\rangle_{V^*,V} \to \langle \alpha^{(1)}, \pi(u)\rangle_{V^*,V}$. Finally, $\Pi(u_n) \to \Pi(u)$ in $C([0,T]; L^1(\Omega)) \hookrightarrow L^1(Q_T)$. Meanwhile $\Pi(u_n(T)) \to \Pi(u(T))$ and $\Pi(u_n(0)) \to \Pi(u(0))$, strongly in $L^1(Q_T)$. Consequently, $\int_\Omega \Pi(u_n(T))dx -$

$\int_\Omega \Pi(u_n(0))dx \to \int_\Omega \Pi(u(T))dx - \int_\Omega \Pi(u(0))dx$. Hence (2.6) follows from (4.1) by passing to the limit as $n \to \infty$. □

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