Electronic Journal of Differential Equations, Vol. 2015 (2015), No. 45, pp. 1–11.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

THREE SOLUTIONS FOR A FOURTH-ORDER BOUNDARY-VALUE PROBLEM

GHASEM A. AFROUZI, SAEID SHOKOOH

Abstract. Using two three-critical points theorems, we prove the existence of at least three weak solutions for one-dimensional fourth-order equations. Some particular cases and two concrete examples are then presented.

1. Introduction

In this note, we consider the fourth-order boundary-value problem

$$\begin{gathered} u''''h(x,u') - u'' = [\lambda f(x,u) + g(u)]h(x,u'), \quad \text{in } (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{gathered} \tag{1.1}$$

where λ is a positive parameter, $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function, $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|g(t_1) - g(t_2)| \le L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$, with $g(0) = 0$, and $h : [0,1] \times \mathbb{R} \to [0,+\infty)$ is a bounded and continuous function with $m := \inf_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t) > 0$.

Due to the importance of fourth-order two-point boundary value problems in describing a large class of elastic deflection, many researchers have studied the existence and multiplicity of solutions for such a problem, we refer the reader to [1, 2, 3, 6, 11] and references therein. For example, authors in [2], using Ricceri's Variational Principle [10, Theorem 1], established the existence three weak solutions for the problem

$$\begin{gathered} u'''' + \alpha u'' + \beta u = \lambda f(x,u) + \mu g(x,u), \quad \text{in } (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{gathered}$$

where α, β are real constants, $f, g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and $\lambda, \mu > 0$.

In this article, employing two three-critical points theorems which we recall in the next section (Theorems 2.1 and 2.2), we establish the existence three weak solutions for (1.1). A special case of Theorem 3.1 is the following theorem.

2000 *Mathematics Subject Classification.* 34B15, 34B18, 58E05.
Key words and phrases. Dirichlet boundary condition; variational method; critical point.

Submitted November 14, 2014. Published February 17, 2015.

Theorem 1.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that*

$$2^{12} \int_0^2 f(x)\,dx < \int_0^{3\sqrt{3}} f(x)\,dx,$$

$$\limsup_{|\xi|\to+\infty} \frac{\int_0^\xi f(x)\,dx}{\xi^2} \le 0.$$

Then, for each

$$\lambda \in \Big] \frac{2^{13}(\pi^4+\pi^2+1)}{\pi^4 \int_0^{3\sqrt{3}} f(x)\,dx}, \frac{2(\pi^4+\pi^2+1)}{\pi^4 \int_0^2 f(x)\,dx} \Big[,$$

the problem

$$\begin{gathered} u'''' - u'' + u = f(u), \quad \text{in } (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1) \end{gathered}$$

admits at least three weak solutions.

The following result is a consequence of Theorem 3.6.

Theorem 1.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a non-negative continuous function such that*

$$2^{11} \int_0^2 f(x)\,dx < \int_0^3 f(x)\,dx,$$

$$\int_0^{2^{10}} f(x)\,dx < 2^7 \int_0^3 f(x)\,dx,$$

Then, for each

$$\lambda \in \Big] \frac{2^{13}(\pi^4+\pi^2+1)}{\pi^4 \int_0^3 f(x)\,dx}, \frac{(\pi^4+\pi^2+1)}{\pi^4} \min\Big\{ \frac{2}{\int_0^2 f(x)\,dx}, \frac{2^{20}}{\int_0^{1024} f(x)\,dx} \Big\} \Big[,$$

the problem

$$\begin{gathered} u'''' - u'' - u = f(u), \quad \text{in } (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1) \end{gathered}$$

admits at least three weak solutions.

2. Preliminaries

We now state two critical point theorems established by Bonanno and coauthors [4, 5] which are the main tools for the proofs of our results. The first result has been obtained in [5] and it is a more precise version of Theorem 3.2 of [4]. The second one has been established in [4]. In the first one the coercivity of the functional $\Phi - \lambda\Psi$ is required, in the second one a suitable sign hypothesis is assumed.

Theorem 2.1 ([5, Theorem 2.6])**.** *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a sequentially weakly lower semicontinuous, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on X^*, $\Psi : X \to \mathbb{R}$ be a sequentially weakly upper semicontinuous, continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*

$$\Phi(0) = \Psi(0) = 0\,.$$

Assume that there exist $r > 0$ and $\bar{x} \in X$, with $r < \Phi(\bar{x})$ such that

(i) $\sup_{\Phi(x)\le r}\Psi(x) < r\Psi(\bar{x})/\Phi(\bar{x})$,
(ii) *for each* λ *in*
$$\Lambda_r := \Big]\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(x)}\Big[,$$
the functional $\Phi - \lambda\Psi$ *is coercive.*

Then, for each $\lambda \in \Lambda_r$ *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points in* X.

Theorem 2.2 ([4, Theorem 3.2]). *Let* X *be a reflexive real Banach space;* $\Phi : X \to \mathbb{R}$ *be a convex, coercive and continuously Gâteaux differentiable functional whose Gâteaux derivative admits a continuous inverse on* X^*, $\Psi : X \to \mathbb{R}$ *be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact, such that*
$$\inf_X \Phi = \Phi(0) = \Psi(0) = 0\,.$$
Assume that there exist two positive constants $r_1, r_2 > 0$ *and* $\bar{x} \in X$, *with* $2r_1 < \Phi(\bar{x}) < r_2/2$, *such that*

(j) $\sup_{\Phi(x)\le r_1}\Psi(x)/r_1 < (2/3)\Psi(\bar{x})/\Phi(\bar{x})$,
(jj) $\sup_{\Phi(x)\le r_2}\Psi(x)/r_2 < (1/3)\Psi(\bar{x})/\Phi(\bar{x})$,
(jjj) *for each* λ *in*
$$\Lambda^*_{r_1,r_2} := \Big]\frac{3}{2}\frac{\Phi(\bar{x})}{\Psi(\bar{x})}, \min\Big\{\frac{r_1}{\sup_{\Phi(x)\le r_1}\Psi(x)}, \frac{r_2}{2\sup_{\Phi(x)\le r_2}\Psi(x)}\Big\}\Big[$$
and for every $x_1, x_2 \in X$, *which are local minima for the functional* $\Phi - \lambda\Psi$, *and such that* $\Psi(x_1) \ge 0$ *and* $\Psi(x_2) \ge 0$, *one has* $\inf_{t\in[0,1]}\Psi(tx_1 + (1-t)x_2) \ge 0$.

Then, for each $\lambda \in \Lambda^*_{r_1,r_2}$ *the functional* $\Phi - \lambda\Psi$ *has at least three distinct critical points which lie in* $\Phi^{-1}(]-\infty, r_2[)$.

Let us introduce some notation which will be used later. Define
$$H^1_0([0,1]) := \big\{u \in L^2([0,1]) : u' \in L^2([0,1]),\ u(0) = u(1) = 0\big\},$$
$$H^2([0,1]) := \big\{u \in L^2([0,1]) : u', u'' \in L^2([0,1])\big\}.$$
Take $X = H^2([0,1]) \cap H^1_0([0,1])$ endowed with the usual norm
$$\|u\| := \Big(\int_0^1 |u''(x)|^2\,dx\Big)^{1/2}.$$
We recall the following Poincaré type inequalities (see, for instance, [8, Lemma 2.3]):
$$\|u'\|^2_{L^2([0,1])} \le \frac{1}{\pi^2}\|u\|^2, \tag{2.1}$$
$$\|u\|^2_{L^2([0,1])} \le \frac{1}{\pi^4}\|u\|^2 \tag{2.2}$$
for all $u \in X$. For the norm in $C^1([0,1])$,
$$\|u\|_\infty := \max\Big\{\max_{x\in[0,1]}|u(x)|, \max_{x\in[0,1]}|u'(x)|\Big\},$$
we have the following relation.

Proposition 2.3. *Let $u \in X$. Then*

$$\|u\|_\infty \le \frac{1}{2\pi}\|u\|. \tag{2.3}$$

Proof. Taking (2.1) into account, the conclusion follows from the well-known inequality $\|u\|_\infty \le \frac{1}{2}\|u'\|_{L^2([0,1])}$. □

For an excellent overview of the most significant mathematical methods employed in this paper we refer to [7, 9].

Let $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function with the Lipschitz constant $L > 0$, i.e.,

$$|g(t_1) - g(t_2)| \le L|t_1 - t_2|$$

for every $t_1, t_2 \in \mathbb{R}$, and $g(0) = 0$, $h : [0,1] \times \mathbb{R} \to [0,+\infty)$ is a bounded and continuous function with $m := \inf_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t) > 0$, and $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ be an L^1-Carathéodory function.

We recall that $f : [0,1] \times \mathbb{R} \to \mathbb{R}$ is an L^1-Carathéodory function if

(a) the mapping $x \mapsto f(x,\xi)$ is measurable for every $\xi \in \mathbb{R}$;
(b) the mapping $\xi \mapsto f(x,\xi)$ is continuous for almost every $x \in [0,1]$;
(c) for every $\rho > 0$ there exists a function $l_\rho \in L^1([0,1])$ such that

$$\sup_{|\xi|\le\rho} |f(x,\xi)| \le l_\rho(x)$$

for almost every $x \in [0,1]$.

Corresponding to f, g and h we introduce the functions $F : [0,1] \times \mathbb{R} \to \mathbb{R}$, $G : \mathbb{R} \to \mathbb{R}$ and $H : [0,1] \times \mathbb{R} \to [0,+\infty)$, respectively, as follows

$$F(x,t) := \int_0^t f(x,\xi)\,d\xi, \quad G(t) := -\int_0^t g(\xi)\,d\xi,$$
$$H(x,t) := \int_0^t \Big(\int_0^\tau \frac{1}{h(x,\delta)}d\delta\Big)d\tau$$

for all $x \in [0,1]$ and $t \in \mathbb{R}$.

In the following, we let $M := \sup_{(x,t)\in[0,1]\times\mathbb{R}} h(x,t)$ and suppose that the Lipschitz constant L of the function g satisfies $0 < L < \pi^4$.

We say that a function $u \in X$ is a *weak solution* of (1.1) if

$$\begin{aligned}&\int_0^1 u''(x)v''(x)\,dx + \int_0^1 \Big(\int_0^{u'(x)} \frac{1}{h(x,\tau)}d\tau\Big)v'(x)\,dx\\ &- \lambda\int_0^1 f(x,u(x))v(x)\,dx - \int_0^1 g(u(x))v(x)\,dx = 0\end{aligned}$$

holds for all $v \in X$.

3. Main results

Put

$$A := \frac{\pi^4 - L}{2\pi^4}, \quad B := \frac{\pi^2 + m(\pi^4 + L)}{2m\pi^4},$$

and suppose that $B \le 4A\pi^2$. We formulate our main results as follows.

Theorem 3.1. *Assume that there exist two positive constants c, d, satisfying $c < 32d/(3\sqrt{3}\pi)$, such that*

(A1) *$F(x,t) \geq 0$ for all $(x,t) \in ([0,3/8] \cup [5/8,1]) \times [0,d]$;*

(A2)
$$\frac{\int_0^1 \max_{|t|\leq c} F(x,t)\,dx}{c^2} < \frac{27}{4096}\frac{\int_{3/8}^{5/8} F(x,d)\,dx}{d^2};$$

(A3)
$$\limsup_{|\xi|\to+\infty} \frac{\sup_{x\in[0,1]} F(x,\xi)}{\xi^2} \leq \frac{\pi^4 A}{B}\frac{\int_0^1 \max_{|t|\leq c} F(x,t)\,dx}{c^2}.$$

Then, for every λ in
$$\Lambda := \Big]\frac{4096Bd^2}{27\int_{3/8}^{5/8} F(x,d)\,dx}, \frac{Bc^2}{\int_0^1 \max_{|t|\leq c} F(x,t)\,dx}\Big[,$$
problem (1.1) *has at least three distinct weak solutions.*

Proof. Fix λ as in the conclusion. Our aim is to apply Theorem 2.1 to our problem. To this end, for every $u \in X$, we introduce the functionals $\Phi, \Psi : X \to \mathbb{R}$ by setting
$$\Phi(u) := \frac{1}{2}\|u\|^2 + \int_0^1 H(x,u'(x))\,dx + \int_0^1 G(u(x))\,dx,$$
$$\Psi(u) := \int_0^1 F(x,u(x))\,dx,$$
and putting
$$I_\lambda(u) := \Phi(u) - \lambda\Psi(u) \quad \forall\, u \in X.$$
Note that the weak solutions of (1.1) are exactly the critical points of I_λ. The functionals Φ, Ψ satisfy the regularity assumptions of Theorem 2.1. Indeed, by standard arguments, we have that Φ is Gâteaux differentiable and sequentially weakly lower semicontinuous and its Gâteaux derivative is the functional $\Phi'(u) \in X^*$, given by
$$\Phi'(u)(v) = \int_0^1 u''(x)v''(x)\,dx + \int_0^1 \Big(\int_0^{u'(x)} \frac{1}{h(x,\tau)}d\tau\Big)v'(x)\,dx - \int_0^1 g(u(x))v(x)\,dx$$
for any $v \in X$. Furthermore, the differential $\Phi' : X \to X^*$ is a Lipschitzian operator. Indeed, taking (2.1) and (2.2) into account, for any $u, v \in X$, there holds
$$\begin{aligned}
\|\Phi'(u) - \Phi'(v)\|_{X^*} &= \sup_{\|w\|\leq 1} |(\Phi'(u) - \Phi'(v), w)| \\
&\leq \ sup_{\|w\|\leq 1} \int_0^1 |u''(x) - v''(x)||w''(x)|\,dx \\
&\quad + \sup_{\|w\|\leq 1} \int_0^1 \Big|\int_{u'(x)}^{v'(x)} \frac{1}{h(x,\tau)}d\tau\Big|\,|w'(x)|\,dx \\
&\quad + \sup_{\|w\|\leq 1} \int_0^1 |g(u(x)) - g(v(x))||w(x)|\,dx \\
&\leq \|u-v\| + \frac{1}{m}\sup_{\|w\|\leq 1} \|u'-v'\|_{L^2(0,1)}\|w'\|_{L^2(0,1)} \\
&\quad + L\sup_{\|w\|\leq 1} \|u-v\|_{L^2(0,1)}\|w\|_{L^2(0,1)} \\
&\leq \Big(1 + \frac{1}{m\pi^2} + \frac{L}{\pi^4}\Big)\|u-v\| = 2B\|u-v\|.
\end{aligned}$$

Recalling that g is Lipschitz continuous and h is bounded away from zero, the claim is true. In particular, we derive that Φ is continuously differentiable. Also, for any $u, v \in X$, we have

$$\begin{aligned}(\Phi'(u) - \Phi'(v), u - v) &= \|u - v\|^2 + \int_0^1 \Big(\int_{u'(x)}^{v'(x)} \frac{1}{h(x,\tau)} d\tau \Big)(u'(x) - v'(x))\, dx \\ &\quad - \int_0^1 (g(u(x)) - g(v(x)))(u(x) - v(x))\, dx \\ &\geq \|u - v\|^2 + \frac{1}{M}\|u' - v'\|^2_{L^2(0,1)} - L\|u - v\|^2_{L^2(0,1)} \\ &\geq \|u - v\|^2 - \frac{L}{\pi^4}\|u - v\|^2 = 2A\|u - v\|^2.\end{aligned}$$

By the assumption $L < \pi^4$, it turns out that Φ' is a strongly monotone operator. So, by applying Minty-Browder theorem [12, Theorem 26.A], $\Phi' : X \to X^*$ admits a Lipschitz continuous inverse. On the other hand, the fact that X is compactly embedded into $C^0([0,1])$ implies that the functional Ψ is well defined, continuously Gâteaux differentiable and with compact derivative, whose Gâteaux derivative is given by

$$\Psi'(u)(v) = \int_0^1 f(x, u(x))v(x)\, dx$$

for any $v \in X$.

Since g is Lipschitz continuous and satisfies $g(0) = 0$, while h is bounded away from zero, the inequalities (2.1) and (2.2) yield for any $u \in X$ the estimate

$$A\|u\|^2 \leq \Phi(u) \leq B\|u\|^2. \tag{3.1}$$

We will verify (i) and (ii) of Theorem 2.1. Put $r = Bc^2$. Taking (2.3) into account, for every $u \in X$ such that $\Phi(u) \leq r$, one has $\max_{x\in[0,1]} |u(x)| \leq c$. Consequently,

$$\sup_{\Phi(u)\leq r} \Psi(u) \leq \int_0^1 \max_{|t|\leq c} F(x,t)\, dx;$$

that is,

$$\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} \leq \frac{\int_0^1 \max_{|t|\leq c} F(x,t)\, dx}{Bc^2}\,.$$

Hence,

$$\frac{\sup_{\Phi(u)\leq r} \Psi(u)}{r} < \frac{1}{\lambda}\,. \tag{3.2}$$

Put

$$w(x) = \begin{cases} -\frac{64d}{9}(x^2 - \frac{3}{4}x), & x \in [0, \frac{3}{8}[, \\ d, & x \in [\frac{3}{8}, \frac{5}{8}], \\ -\frac{64d}{9}(x^2 - \frac{5}{4}x + \frac{1}{4}), & x \in]\frac{5}{8}, 1]. \end{cases}$$

It is easy to verify that $w \in X$ and, in particular,

$$\|w\|^2 = \frac{4096}{27} d^2.$$

So, taking (3.1) into account, we deduce

$$\frac{4096}{27} A d^2 \leq \Phi(w) \leq \frac{4096}{27} B d^2.$$

Hence, from $c < \frac{32}{3\sqrt{3}\pi} d$ and $B \le 4A\pi^2$, we obtain $r < \Phi(w)$.

Since $0 \le w(x) \le d$ for each $x \in [0,1]$, assumption (A1) ensures that

$$\int_0^{3/8} F(x, w(x))\, dx + \int_{5/8}^1 F(x, w(x))\, dx \ge 0,$$

and so

$$\Psi(w) \ge \int_{3/8}^{5/8} F(x, d)\, dx.$$

Therefore, we obtain

$$\frac{\Psi(w)}{\Phi(w)} \ge \frac{27}{4096} \frac{\int_{3/8}^{5/8} F(x,d)\, dx}{Bd^2} > \frac{1}{\lambda}. \tag{3.3}$$

Therefore, from (3.2) and (3.3), condition (i) of Theorem 2.1 is fulfilled.

Now, to prove the coercivity of the functional I_λ. By (A3), we have

$$\limsup_{|\xi| \to +\infty} \frac{\sup_{x \in [0,1]} F(x, \xi)}{\xi^2} < \frac{\pi^4 A}{\lambda}.$$

So, we can fix $\varepsilon > 0$ satisfying

$$\limsup_{|\xi| \to +\infty} \frac{\sup_{x \in [0,1]} F(x, \xi)}{\xi^2} < \varepsilon < \frac{\pi^4 A}{\lambda}.$$

Then, there exists a positive constant θ such that

$$F(x,t) \le \varepsilon |t|^2 + \theta \quad \forall x \in [0,1],\ \forall t \in \mathbb{R}.$$

Taking into account (2.2) and (3.1), we have

$$I_\lambda(u) = \Phi(u) - \lambda \Psi(u) \ge A\|u\|^2 - \lambda\varepsilon \|u\|^2_{L^2[0,1]} - \lambda\theta \ge \big(A - \frac{\lambda\varepsilon}{\pi^4}\big)\|u\|^2 - \lambda\theta$$

for all $u \in X$. So, the functional I_λ is coercive. Now, the conclusion of Theorem 2.1 can be used. It follows that for every

$$\lambda \in \Lambda \subseteq \Big] \frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(u) \le r} \Psi(u)} \Big[,$$

the functional I_λ has at least three distinct critical points in X, which are the weak solutions of the problem (1.1). This completes the proof. $\square$

Now, we present a consequence of Theorem 3.1.

Corollary 3.2. *Let $\alpha \in L^1([0,1])$ be a non-negative and non-zero function and let $\gamma : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $\alpha_0 := \int_{3/8}^{5/8} \alpha(x) dx$, $\|\alpha\|_1 := \int_0^1 \alpha(x)\, dx$ and $\Gamma(t) = \int_0^t \gamma(\xi) d\xi$ for all $t \in \mathbb{R}$, and assume that there exist two positive constants c, d, with $c < \frac{32}{3\sqrt{3}\pi} d$, such that*

- (A1') $\Gamma(t) \ge 0$ *for all* $t \in [0, d]$;
- (A2')
$$\frac{\max_{|t| \le c} \Gamma(t)}{c^2} < \frac{27}{4096} \frac{\alpha_0}{\|\alpha\|_1} \frac{\Gamma(d)}{d^2};$$
- (A3') $\limsup_{|\xi| \to +\infty} \Gamma(\xi)/\xi^2 \le 0$.

Then, for every

$$\lambda \in \Big]\frac{4096}{27}\frac{Bd^2}{\alpha_0\Gamma(d)}, \frac{Bc^2}{\|\alpha\|_1 \max_{|t|\le c}\Gamma(t)}\Big[,$$

the problem

$$\begin{gathered} u''''h(x,u') - u'' = [\lambda\alpha(x)\gamma(u) + g(u)]h(x,u'), \quad \text{in } (0,1),\\ u(0) = u(1) = 0 = u''(0) = u''(1) \end{gathered} \tag{3.4}$$

has at least three weak solutions.

The proof of the above corollary follows from Theorem 3.1 by choosing $f(x,t) := \alpha(x)\gamma(t)$ for all $(x,t) \in [0,1]\times\mathbb{R}$.

Remark 3.3. Clearly, if γ is non-negative then assumption (A1') is satisfied and (A2') becomes

$$\frac{\Gamma(c)}{c^2} < \frac{27}{4096}\frac{\alpha_0}{\|\alpha\|_1}\frac{\Gamma(d)}{d^2}.$$

Remark 3.4. Theorem 1.1 in the introduction is an immediate consequence of Corollary 3.2, on choosing $g(u) = -u$, $h \equiv 1$, $c = 2$ and $d = 3\sqrt{3}$.

The following lemma will be crucial in our arguments.

Lemma 3.5. *Assume that $f(x,t) \ge 0$ for all $(x,t) \in [0,1]\times\mathbb{R}$. If u is a weak solution of (1.1), then $u(x) \ge 0$ for all $x \in [0,1]$.*

Proof. Arguing by contradiction, if we assume that u is negative at a point of $[0,1]$, the set

$$\Omega := \{x \in [0,1] : u(x) < 0\},$$

is non-empty and open. Let us consider $\bar{v} := \min\{u,0\}$, one has, $\bar{v} \in X$. So, taking into account that u is a weak solution and by choosing $v = \bar{v}$, from our assumptions, one has

$$\begin{aligned} 0 &\ge \lambda\int_\Omega f(x,u(x))u(x)\,dx\\ &= \int_\Omega |u''(x)|^2\,dx + \int_\Omega\Big(\int_0^{u'(x)}\frac{1}{h(x,\tau)}d\tau\Big)u'(x)\,dx - \int_\Omega g(u(x))u(x)\,dx\\ &\ge \frac{\pi^4 - L}{\pi^4}\|u\|^2_{H^2(\Omega)\cap H^1_0(\Omega)}. \end{aligned}$$

Therefore, $\|u\|_{H^2(\Omega)\cap H^1_0(\Omega)} = 0$ which is absurd. Hence, the conclusion is achieved. □

Our other main result is as follows.

Theorem 3.6. *Assume that there exist three positive constants c_1, c_2, d, satisfying $\frac{3\sqrt{3}\pi}{16\sqrt{2}}c_1 < d < \frac{3\sqrt{3}}{64\sqrt{2}}c_2$, such that*

(B1) *$f(x,t) \ge 0$ for all $(x,t) \in [0,\,1]\times[0,c_2]$;*

(B2)

$$\frac{\int_0^1 F(x,c_1)\,dx}{c_1^2} < \frac{9}{2048}\frac{\int_{3/8}^{5/8} F(x,d)\,dx}{d^2};$$

(B3)
$$\frac{\int_0^1 F(x,c_2)\,dx}{c_2^2} < \frac{9}{4096}\frac{\int_{3/8}^{5/8} F(x,d)\,dx}{d^2}.$$

Let
$$\Lambda' := \Big]\frac{2048}{9}\frac{Bd^2}{\int_{3/8}^{5/8} F(x,d)\,dx}, B\,\min\Big\{\frac{c_1^2}{\int_0^1 F(x,c_1)\,dx}, \frac{c_2^2}{2\int_0^1 F(x,c_2)\,dx}\Big\}\Big[.$$

Then, for every $\lambda \in \Lambda'$ the problem (1.1) *has at least three weak solutions u_i, $i = 1, 2, 3$, such that $0 < \|u_i\|_\infty \le c_2$.*

Proof. Without loss of generality, we can assume $f(x,t) \ge 0$ for all $(x,t) \in [0,1]\times\mathbb{R}$. Fix λ as in the conclusion and take X, Φ and Ψ as in the proof of Theorem 3.1. Put w as in Theorem 3.1, $r_1 = Bc_1^2$ and $r_2 = Bc_2^2$. Therefore, one has $2r_1 < \Phi(w) < \frac{r_2}{2}$ and we have
$$\begin{aligned}
\frac{1}{r_1}\sup_{\Phi(u)<r_1}\Psi(u) &\le \frac{1}{Bc_1^2}\int_0^1 F(x,c_1)\,dx < \frac{1}{\lambda}\\
&< \frac{9}{2048}\frac{\int_{3/8}^{5/8} F(x,d)\,dx}{Bd^2}\\
&\le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)},
\end{aligned}$$
and
$$\begin{aligned}
\frac{2}{r_2}\sup_{\Phi(u)<r_2}\Psi(u) &\le \frac{2}{Bc_2^2}\int_0^1 F(x,c_2)\,dx < \frac{1}{\lambda}\\
&< \frac{9}{2048}\frac{\int_{3/8}^{5/8} F(x,d)\,dx}{Bd^2}\\
&\le \frac{2}{3}\frac{\Psi(w)}{\Phi(w)}.
\end{aligned}$$
So, conditions (j) and (jj) of Theorem 2.2 are satisfied. Finally, let u_1 and u_2 be two local minima for $\Phi - \lambda\Psi$. Then, u_1 and u_2 are critical points for $\Phi - \lambda\Psi$, and so, they are weak solutions for the problem (1.1). Hence, owing to Lemma 3.5, we obtain $u_1(x) \ge 0$ and $u_2(x) \ge 0$ for all $x \in [0,1]$. So, one has $\Psi(su_1 + (1-s)u_2) \ge 0$ for all $s \in [0,1]$. From Theorem 2.2 the functional $\Phi - \lambda\Psi$ has at least three distinct critical points which are weak solutions of (1.1). This complete the proof. □

Now, we present a consequence of Theorem 3.6.

Corollary 3.7. *Let $\alpha \in L^1([0,1])$ be such that $\alpha(x) \ge 0$ a.e. $x \in [0,1]$, $\alpha \not\equiv 0$, and let $\gamma : \mathbb{R} \to \mathbb{R}$ be a continuous function. Put $\alpha_0 := \int_{3/8}^{5/8}\alpha(x)dx$, $\|\alpha\|_1 := \int_0^1 \alpha(x)\,dx$ and $\Gamma(t) = \int_0^t \gamma(\xi)d\xi$ for all $t \in \mathbb{R}$, and assume that there exist three positive constants c_1, c_2, d, with $\frac{3\sqrt{3}\pi}{16\sqrt{2}}c_1 < d < \frac{3\sqrt{3}}{64\sqrt{2}}c_2$, such that*

(B1') *$\gamma(t) \ge 0$ for all $t \in [0, c_2]$;*

(B2')
$$\frac{\Gamma(c_1)}{c_1^2} < \frac{9}{2048}\frac{\alpha_0}{\|\alpha\|_1}\frac{\Gamma(d)}{d^2};$$

(B3')
$$\frac{\Gamma(c_2)}{c_2^2} < \frac{9}{4096}\frac{\alpha_0}{\|\alpha\|_1}\frac{\Gamma(d)}{d^2}.$$

Then, for every
$$\lambda \in \Big]\frac{2048}{9}\frac{Bd^2}{\alpha_0\Gamma(d)}, B\min\Big\{\frac{c_1^2}{\|\alpha\|_1\Gamma(c_1)}, \frac{c_2^2}{2\|\alpha\|_1\Gamma(c_2)}\Big\}\Big[,$$
the problem (3.4) *has at least three weak solutions* u_i*,* $i = 1, 2, 3$*, such that* $0 < \|u_i\|_\infty \le c_2$*.*

The proof of the above corollary follows from Theorem 3.6 by choosing $f(x,t) := \alpha(x)\gamma(t)$ for all $(x,t) \in [0,1] \times \mathbb{R}$.

Remark 3.8. Theorem 1.2 in the introduction is an immediate consequence of Corollary 3.7, on choosing $g(u) = u$, $h \equiv 1$, $c_1 = 2$, $c_2 = 2^{10}$, and $d = 3$.

Finally, we present the following examples to illustrate our results.

Example 3.9. Consider the following problem
$$\begin{gathered} u'''' - u''(2 + x + \cos u') + u = \lambda f(u), \quad \text{in } (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{gathered} \tag{3.5}$$
where $f : \mathbb{R} \to \mathbb{R}$ is defined by
$$f(t) = \begin{cases} 2^{-10} & \text{if } |t| \le 1, \\ 2^{-10}\, t^4 & \text{if } 1 < |t| \le 32, \\ 2^{20}\, t^{-2} & \text{if } |t| > 32. \end{cases}$$
Here, $g(t) = -t$ and $h(x,t) = (2 + x + \cos t)^{-1}$ for all $x \in [0,1]$ and $t \in \mathbb{R}$. It is easy to verify that (A2') and (A3') are satisfied with $c = 1$ and $d = 32$. From Corollary 3.2, for each parameter
$$\lambda \in \Big]\frac{48(\pi^4 + 4\pi^2 + 1)}{\pi^4}, \frac{512(\pi^4 + 4\pi^2 + 1)}{\pi^4}\Big[,$$
problem (3.5) admits at least three weak solutions.

Example 3.10. Consider the problem
$$\begin{gathered} u'''' - u''(3 + \sin u') - 2u = \lambda f(u), \quad \text{in } (0,1), \\ u(0) = u(1) = 0 = u''(0) = u''(1), \end{gathered} \tag{3.6}$$
where $f : \mathbb{R} \to \mathbb{R}$ is defined by
$$f(t) = \begin{cases} 2^{-20} & \text{if } |t| \le 2^{-5}, \\ t^4 & \text{if } 2^{-5} < |t| \le 1, \\ t^{-2} & \text{if } |t| > 1. \end{cases}$$
Here, $g(t) = 2t$ and $h(x,t) = (3 + \sin t)^{-1}$ for all $x \in [0,1]$ and $t \in \mathbb{R}$. It is easy to verify that (B2') and (B3') are satisfied with $c_1 = 2^{-5}$, $d = 1$ and $c_2 = 2^{10}$. From Corollary 3.7, for each parameter
$$\lambda \in \Big]\frac{2276(\pi^4 + 4\pi^2 + 2)}{\pi^4}, \frac{2^{14}(\pi^4 + 4\pi^2 + 2)}{\pi^4}\Big[,$$
problem (3.6) admits at least three weak solutions u_i, $i = 1, 2, 3$, such that $0 < \|u_i\|_\infty \le 1024$.

References

[1] G. Afrouzi, A. Hadjian, V. D. Rădulescu; Variational approach to fourth-order impulsive differential equations with two control parameters, *Results in Mathematics*, **65** (2014), 371-384.

[2] G. A. Afrouzi, S. Heidarkhani, D. O'Regan; Existence of three solutions for a doubly eigenvalue fourth-order boundary value problem, *Taiwanese J. Math.*, **15** (2011), 201-210.

[3] G. Afrouzi, M. Mirzapour, V. D. Rădulescu; Nonlocal fourth-order Kirchhoff systems with variable growth: low and high energy solutions, *Collectanea Mathematica*, in press (DOI: 10.1007/s13348-014-0131-x).

[4] G. Bonanno, P. Candito; Non-differentiable functionals and applications to elliptic problems with discontinuous nonlinearities, *J. Differential Equations,* **244** (2008), 3031-3059.

[5] G. Bonanno, S. A. Marano; On the structure of the critical set of non-differentiable functionals with a weak compactness condition, *Appl. Anal.,* **89** (2010), 1-10.

[6] G. Chai; Existence of positive solutions for fourth-order boundary value problem with variable parameters, *Math. Comput. Modelling*, **66** (2007), 870-880.

[7] P. G. Ciarlet; Linear and Nonlinear Functional Analysis with Applications, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, 2013.

[8] L. A. Peletier, W. C. Troy, R. C. A. M. Van der Vorst; Stationary solutions of a fourth order nonlinear diffusion equation, (Russian) Translated from the English by V. V. Kurt. *Differentsialnye Uravneniya* **31** (1995), 327-337. English translation in *Differential Equations* **31** (1995), 301-314.

[9] V. D. Rădulescu; Qualitative Analysis of Nonlinear Elliptic Partial Differential Equations: Monotonicity, Analytic, and Variational Methods, *Contemporary Mathematics and Its Applications*, Vol. 6, Hindawi Publ. Corp., 2008.

[10] B. Ricceri; A three critical points theorem revisited, *Nonlinear Anal.*, **70** (2009), 3084-3089.

[11] V. Shanthi, N. Ramanujam; A numerical method for boundary value problems for singularly perturbed fourth-order ordinary differential equations, *Appl. Math. Comput.*, **129** (2002). 269-294.

[12] E. Zeidler; *Nonlinear Functional Analysis and its Applications*, vol. II/B and III, Berlin-Heidelberg-New York, 1990.

Ghasem A. Afrouzi

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

E-mail address: afrouzi@umz.ac.ir

Saeid Shokooh

Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar, Iran

E-mail address: saeid.shokooh@stu.umz.ac.ir