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INTEGRATION BY PARTS FOR THE L^r HENSTOCK-KURZWEIL INTEGRAL

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Abstract. Musial and Sagher [4] described a Henstock-Kurzweil type integral that integrates L^r-derivatives. In this article, we develop a product rule for the L^r-derivative and then an integration by parts formula.

1. Introduction

Definition 1.1 ([4]). A real-valued function f defined on $[a,b]$ is said to be L^r Henstock-Kurzweil integrable ($f \in HK_r[a,b]$) if there exists a function $F \in L^r[a,b]$ so that for any $\varepsilon > 0$ there exists a gauge function $\delta(x) > 0$ so that whenever $\{(x_i,[c_i,d_i])\}$ is a δ-fine tagged partition of $[a,b]$ we have

$$\sum_{i=1}^{n} \Big(\frac{1}{d_i - c_i} (L) \int_{c_i}^{d_i} |F(y) - F(x_i) - f(x_i)(y - x_i)|^r \, dy \Big)^{1/r} < \varepsilon.$$

In the sequel, if an integral is not specified, it is a Lebesgue integral. It is shown in [4] that if f is HK_r-integrable on $[a,b]$, the following function is well-defined for all $x \in [a,b]$:

$$F(x) = (HK_r) \int_a^x f(t)\, dt \tag{1.1}$$

Here the function F is called the *indefinite HK_r integral of f*. Our aim is to establish an integration by parts formula for the HK_r integral. In a manner similar to L. Gordon [2] we state the following

Theorem 1.2. *Suppose that f is HK_r-integrable on $[a,b]$, and G is absolutely continuous on $[a,b]$ with $G' \in L^{r'}([a,b])$, where $1 \le r < \infty, r' = r/(r-1)$ if $r > 1$, and $r' = \infty$ if $r = 1$. Then fG is HK_r-integrable on $[a,b]$ and if F is the indefinite HK_r integral of f, then*

$$(HK_r) \int_a^b f(t)G(t)\, dt = F(b)G(b) - \int_a^b F(t)G'(t)\, dt.$$

We note that if $r = 1$ so that $r' = \infty$, the condition on G is that it is a Lipschitz function of order 1 on $[a,b]$.

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In the classical case where f is Henstock-Kurzweil integrable ($r = \infty, r' = 1$), Theorem 1.2 holds, but it is enough to assume that G is of bounded variation on $[a, b]$. In that case the integral on the right is the Riemann-Stieltjes integral $\int_a^b F dG$. See [3] for a proof of this statement.

To prove Theorem 1.2 we will need a product rule for the L^r-derivative. We will also utilize a characterization of the space of HK_r-integrable functions that involves generalized absolute continuity in L^r sense ($ACG_r([a, b])$).

2. Product rule for the L^r-derivative

Definition 2.1 ([1]). For $1 \le r < \infty$, a function $F \in L^r([a, b])$ is said to be L^r-differentiable at $x \in [a, b]$ if there exists $a \in \mathbb{R}$ such that

$$\int_{-h}^{h} |F(x+t) - F(x) - at|^r dt = o(h^{r+1}).$$

It is clear that if such a number a exists, then it is unique. We say that a is the *L^r-derivative* of F at x, and denote the value a by $F'_r(x)$.

Theorem 2.2. *For $1 \le r < \infty$, let $x \in \mathbb{R}$ and suppose $F \in L^r(I)$ where I is an interval having x in its interior, and suppose F is L^r-differentiable at x. Suppose also that $G \in L^\infty(I)$ and that G is L^r-differentiable at x. Then FG is L^r-differentiable at x and $(FG)'_r(x) = F'_r(x)G(x) + F(x)G'_r(x)$.*

Proof. Let $\varepsilon > 0$. We need to choose γ so that for $0 < h < \gamma$

$$\int_{-h}^{h} |F(x+t)G(x+t) - F(x)G(x) - H(x)t|^r dt < \varepsilon h^{r+1} \tag{2.1}$$

where $H(x) = F'_r(x)G(x)+F(x)G'_r(x)$. We add and subtract the terms $F(x)G(x+t)$ and $F'_r(x)G(x+t)t$ to the part of the integrand inside the absolute value signs. We also note that if a, b and c are non-negative numbers then

$$(a + b + c)^r \le C(a^r + b^r + c^r)$$

where C is a positive constant that depends on r.

Choose $\gamma_0 > 0$ and $N > 0$ so that $F \in L^r([x - \gamma_0, x + \gamma_0])$ and that

$$\operatorname{esssup}_{[x-\gamma_0,x+\gamma_0]} G < N.$$

We then have that if $0 < h < \gamma_0$ then the integral in (2.1) is less than or equal to

$$C \int_{-h}^{h} |G(x+t)|^r |F(x+t) - F(x) - F'_r(x)t|^r dt \tag{2.2}$$

$$+ C \int_{-h}^{h} |F(x)|^r |G(x+t) - G(x) - G'_r(x)t|^r dt \tag{2.3}$$

$$+ C \int_{-h}^{h} |F'_r(x)|^r |(G(x+t) - G(x))t|^r dt. \tag{2.4}$$

For (2.2), choose $\gamma_1 < \gamma_0$ so that if $0 < h < \gamma_1$ we have

$$\int_{-h}^{h} |F(x+t) - F(x) - F'_r(x)t|^r dt < \frac{\varepsilon h^{r+1}}{4CN^r}$$

so that

$$C \int_{-h}^{h} |G(x+t)|^r |F(x+t) - F(x) - F'_r(x)t|^r dt < \frac{\varepsilon h^{r+1}}{4}.$$

For (2.3), choose $\gamma_2 < \gamma_1$ so that if $0 < h < \gamma_2$ we have

$$\int_{-h}^{h} |G(x+t) - G(x) - G_r'(x)t|^r dt < \frac{\varepsilon h^{r+1}}{4C(|F(x)|^r + 1)}$$

so that

$$C\int_{-h}^{h} |F(x)|^r |G(x+t) - G(x) - G_r'(x)t|^r dt < \frac{\varepsilon h^{r+1}}{4}$$

For (2.4), we note that

$$\begin{aligned}
&C\int_{-h}^{h} |F_r'(x)|^r |(G(x+t) - G(x))t|^r dt \\
&= C|F_r'(x)|^r \int_{-h}^{h} |(G(x+t) - G(x) - G_r'(x)t + G_r'(x)t)t|^r dt \\
&\le C^2|F_r'(x)|^r h^r \Big(\int_{-h}^{h} |(G(x+t) - G(x) - G_r'(x)t)|^r dt \\
&\quad + \int_{-h}^{h} |G_r'(x)t|^r dt\Big) \\
&\le C^2|F_r'(x)|^r h^r \Big(\int_{-h}^{h} |(G(x+t) - G(x) - G_r'(x)t)|^r dt\Big) \\
&\quad + 2C^2|F_r'(x)|^r h^{2r+1}|G_r'(x)|^r.
\end{aligned}$$

Now we note that we can choose

$$0 < \gamma < \min\Big(1, \gamma_2, \big(\varepsilon/\big(8C^2(|G_r'(x)| + 1)(|F_r'(x)| + 1)\big)\big)^{1/r}\Big)$$

so that if $0 < h < \gamma$ we have

$$\Big(\int_{-h}^{h} |(G(x+t) - G(x) - G_r'(x)t)|^r dt\Big) < \frac{\varepsilon h^{r+1}}{4C^2(|F_r'(x)|^r + 1)}$$

We then have that if $0 < h < \gamma$, then

$$\begin{aligned}
&C^2|F_r'(x)|^r h^r \Big(\int_{-h}^{h} |(G(x+t) - G(x) - G_r'(x)t)|^r dt\Big) \\
&< (C^2|F_r'(x)|^r h^r)\Big(\frac{\varepsilon h^{r+1}}{4C^2(|F_r'(x)|^r + 1)}\Big) \\
&\le \frac{\varepsilon h^{2r+1}}{4} < \frac{\varepsilon h^{r+1}}{4}
\end{aligned}$$

and that

$$\begin{aligned}
&2C^2|F_r'(x)|^r h^{2r+1}|G_r'(x)|^r \\
&\le 2C^2|F_r'(x)|^r h^{r+1}|G_r'(x)|^r \Big(\frac{\varepsilon}{8C^2(|F_r'(x)| + 1)(|G_r'(x)| + 1)}\Big) \\
&\le \frac{\varepsilon h^{r+1}}{4}.
\end{aligned}$$

We can then conclude that (2.1) holds and the theorem is therefore proved. $\square$

In [4] we find sufficient conditions for HK_r-integrability. We will need the following definitions.

Definition 2.3 ([4])**.** We say that $F \in AC_r(E)$ if for all $\varepsilon > 0$ there exist $\eta > 0$ and a gauge function $\delta(x)$ defined on E so that if $\mathcal{P} = \{(x_i, [c_i, d_i])\}$ is a finite collection of non-overlapping δ-fine tagged intervals having tags in E and satisfying

$$\sum_{i=1}^{q}(d_i - c_i) < \eta$$

then

$$\sum_{i=1}^{q}\Big(\frac{1}{d_i - c_i}\int_{c_i}^{d_i}|F(y) - F(x_i)|^r dy\Big)^{1/r} < \varepsilon.$$

Definition 2.4 ([4])**.** We say that $F \in ACG_r(E)$ if E can be written

$$E = \cup_{i=1}^{\infty} E_i$$

and $F \in AC_r(E_i)$ for all i.

Lemma 2.5. *Suppose that F and G are in $ACG_r([a, b])$, and that $G \in L^\infty([a, b])$. Then $FG \in ACG_r([a, b])$.*

Proof. The function $F \in ACG_r([a, b])$ and so we can find a sequence of sets $\{A_n\}_{n=1}^{\infty}$ so that $[a, b] = \cup_{n=1}^{\infty} A_n$ and $F \in AC_r(A_n)$ for all n. Since G belongs to $ACG_r([a, b])$, we can also find a sequence of sets $\{B_m\}_{m=1}^{\infty}$ so that $[a, b] = \cup_{m=1}^{\infty} B_m$ and $G \in AC_r(B_m)$ for all m. We can then write

$$[a, b] = \cup_{n=1}^{\infty} \cup_{m=1}^{\infty} (A_n \cap B_m).$$

We will rewrite the sequence $\{A_n \cap B_m\}_{n,m\ge 1}$ as $\{E_k\}_{k\ge 1}$. We then have that both F and G are in $AC_r(E_k)$ for all $k \ge 1$. We will show that $FG \in ACG_r(E_k)$ for all k.

Let $N = 1 + \|G\|_\infty$ and fix k. For $j \ge 1$ let

$$U_j = \{x \in E_k : j - 1 \le |F(x)| < j\}$$

We then have

$$E_k = \cup_{j=1}^{\infty} U_j.$$

We will show that $FG \in AC_r(U_j)$ for all j.

Let $\varepsilon > 0$. There exist $\eta > 0$ and a gauge function $\delta(x)$ defined on U_j so that if $\mathcal{P} = \{x_i, [c_i, d_i]\}$ is a finite collection of non-overlapping δ-fine tagged intervals having tags in U_j and satisfying

$$\sum_{i=1}^{q}(d_i - c_i) < \eta$$

then

$$\sum_{i=1}^{q}\Big(\frac{1}{d_i - c_i}\int_{c_i}^{d_i}|F(y) - F(x_i)|^r dy\Big)^{1/r} < \frac{\varepsilon}{2N},$$
$$\sum_{i=1}^{q}\Big(\frac{1}{d_i - c_i}\int_{c_i}^{d_i}|G(y) - G(x_i)|^r dy\Big)^{1/r} < \frac{\varepsilon}{2j}.$$

Then for such $\mathcal{P}$,

$$\sum_{i=1}^{q}\Big(\frac{1}{d_i - c_i}\int_{c_i}^{d_i}|F(y)G(y) - F(x_i)G(x_i)|^r dy\Big)^{1/r}$$

$$
\begin{aligned}
&\le \sum_{i=1}^{q}\Big(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|F(y)G(y)-F(x_i)G(y)|^r\,dy\Big)^{1/r}\\
&\quad+\sum_{i=1}^{q}\Big(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|F(x_i)G(y)-F(x_i)G(x_i)|^r\,dy\Big)^{1/r}.\\
&\le N\Big(\sum_{i=1}^{q}\Big(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|F(y)-F(x_i)|^r\,dy\Big)^{1/r}\Big)\\
&\quad+|F(x_i)|\Big(\sum_{i=1}^{q}(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|G(y)-G(x_i)|^r\,dy)^{1/r}\Big)\\
&\le N\big(\frac{\varepsilon}{2N}\big)+j\big(\frac{\varepsilon}{2j}\big)=\varepsilon.
\end{aligned}
$$

Now we can conclude that for $\mathcal{P}$,

$$
\sum_{i=1}^{q}\Big(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|F(y)G(y)-F(x_i)G(x_i)|^r\,dy\Big)^{1/r}<\varepsilon
$$

and so that $FG\in ACG_r([a,b])$. $\square$

3. Linearity of $ACG_r(E)$

We now show that $ACG_r(E)$ is a linear space.

Theorem 3.1. *Suppose F and G are in $ACG_r(E)$. Then for any constants a and b we have that $aF+bG\in ACG_r(E)$.*

Proof. Write E as $\cup_{n=1}^{\infty}E_n$. We will show that $aF+bG\in AC_r(E_n)$ for every n.

First we show that $aF\in AC_r(E_n)$. Let $\varepsilon>0$ and choose $\eta>0$ and a gauge function $\delta(x)$ defined on E_n so that if $\mathcal{P}=\{x_i,[c_i,d_i]\}$ is a finite collection of non-overlapping δ-fine tagged intervals having tags in E and satisfying

$$
\sum_{i=1}^{q}(d_i-c_i)<\eta
$$

then

$$
\sum_{i=1}^{q}\Big(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|F(y)-F(x_i)|^r\,dy\Big)^{1/r}<\frac{\varepsilon}{|a|+1}.
$$

Then

$$
\begin{aligned}
&\sum_{i=1}^{q}\Big(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|aF(y)-aF(x_i)|^r\,dy\Big)^{1/r}\\
&=|a|\Big(\sum_{i=1}^{q}\Big(\frac{1}{d_i-c_i}\int_{c_i}^{d_i}|F(y)-F(x_i)|^r\,dy\Big)^{1/r}\Big)\\
&<|a|\big(\frac{\varepsilon}{|a|+1}\big)<\varepsilon.
\end{aligned}
$$

Now we show that $F+G\in ACG_r(E)$. Let $\varepsilon>0$ and choose $\eta>0$ and a gauge function $\delta(x)$ defined on E_n so that if $\mathcal{P}=\{x_i,[c_i,d_i]\}$ is a finite collection of non-overlapping δ-fine tagged intervals having tags in E and satisfying

$$\sum_{i=1}^{q}(d_i - c_i) < \eta\,,$$

then

$$\sum_{i=1}^{q}\Big(\frac{1}{d_i - c_i}\int_{c_i}^{d_i}|F(y) - F(x_i)|^r dy\Big)^{1/r} < \frac{\varepsilon}{2},$$
$$\sum_{i=1}^{q}\Big(\frac{1}{d_i - c_i}\int_{c_i}^{d_i}|G(y) - G(x_i)|^r dy\Big)^{1/r} < \frac{\varepsilon}{2}.$$

Then we have for this $\mathcal{P}$, using Minkowski's inequality,

$$\begin{aligned}
&\sum_{i=1}^{q}\Big(\frac{1}{d_i - c_i}\int_{c_i}^{d_i}|F(y) + G(y) - (F(x_i) + G(x_i))|^r dy\Big)^{1/r}\\
&\le \sum_{i=1}^{q}\Big(\frac{1}{d_i - c_i}\int_{c_i}^{d_i}|F(y) + F(x_i)|^r dy\Big)^{1/r}\\
&\quad + \sum_{i=1}^{q}\Big(\frac{1}{d_i - c_i}\int_{c_i}^{d_i}|G(y) - G(x_i)|^r dy\Big)^{1/r}\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

□

We will use the following characterization of HK_r-integrable functions.

Theorem 3.2 ([4])**.** *Let $1 \le r < \infty$. A function f is HK_r-integrable on $[a, b]$ if and only if there exists a function $F \in ACG_r([a, b])$ so that $F_r^{'} = f$ a.e.*

4. Integration by Parts

We are now ready to give the proof of Theorem 1.2.

Proof. Define

$$V(x) = f(x)G(x),$$
$$J(x) = F(x)G(x) - \int_a^x F(t)G'(t)\,dt.$$

We note that FG' is integrable by Hölder's inequality [5]. Our task is to show that J is the HK_r-integral of V. By Theorem 3.2, we see that it is sufficient to demonstrate that $J \in ACG_r([a, b])$ and that $J_r^{'} = V$ a.e.

We note that the function

$$\int_a^x F(t)G'(t)\,dt$$

is absolutely continuous on $[a, b]$ and therefore is in $ACG_r([a, b])$ [4]. Its derivative, and therefore its L^r-derivative, is equal to $F(x)G'(x)$ a.e. in $[a, b]$.

Using Theorem 2.2 we can see that FG has an L^r-derivative equal to $F_r'G + FG'$ a.e. in $[a, b]$. Using the linearity of the L^r-derivative, we have that $J_r^{'} = V$ a.e. Thus all that remains is to show that $J \in ACG_r([a, b])$. By Theorem 3.1 it is sufficient to show that $FG \in ACG_r([a, b])$.

The function $F \in ACG_r([a,b])$. Since $G \in AC([a,b])$, it is also in $ACG_r([a,b])$ and G is also in L^∞ so by Lemma 2.5, $FG \in ACG_r([a,b])$ and Theorem 1.2 is proved. $\square$

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