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MULTIPLICITY OF SOLUTIONS TO THE SUM OF POLYHARMONIC EQUATIONS WITH CRITICAL SOBOLEV EXPONENTS

WEI LIU, GAO JIA, LU-QIAN GUO

Abstract. In this article, we prove multiplicity of solutions for the sum of polyharmonic equation with critical Sobolev exponent. The proof is based upon the methods of weakly lower semi-continuous of the functionals and the Mountain Pass Lemma without (PS) conditions.

1. Introduction

In this article, we discuss the multiplicity of solutions for the sum polyharmonic equation

$$\begin{gathered} \sum_{i=0}^{k}(-\Delta)^{i}u=\lambda|u|^{q-2}u+|u|^{N-2}u+\mu f(x),\quad \text{in } \Omega,\\ u\in H_0^k(\Omega), \end{gathered} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain, k is positive integer, q is a real number with $2 < q < N$, $N = 2n/(n-2k)$ is the critical Sobolev exponent in the embedding $H_0^k(\Omega) \hookrightarrow L^N(\Omega)$, λ, μ are both positive real parameters and $f(x)$ is continuous with not identical to 0 in Ω. Our main result is the following theorem.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain, $n > 2k$ and $f(x)$ be continuous and not identical to 0 in Ω. Then there exist $\lambda_0 > 0$ and $\mu_0 > 0$, such that for any $\lambda > \lambda_0$ and $0 < \mu < \mu_0$, problem* (1.1) *admits at least two distinct weak solutions u_1 with positive energy and u_2 with negative energy.*

Remark 1.2. For the highest order term $(-\Delta)^k u$ of problem (1.1), we need to discuss that k is odd or even. In fact, no matter k is odd or even, we obtain the similar result of Theorem 1.1. For the sake of simplicity, in the following discussion, we let k be an even, that is $k = 2m$ and m is positive integer.

Higher-order elliptic boundary problems have abundant applications in physics and engineering [16] and have also been studied in many areas of mathematics, including conformal geometry [12], some geometry invariants [5] and non-linear elasticity [13].

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The existence of the solutions of the Brezis-Nirenberg problem [9] for the higher-order equations has been studied in many papers [1, 3, 7, 11, 14, 18]. Grunau [15] considered the existence of positive solution for semilinear polyharmonic Dirichlet problem with critical Sobolev exponent

$$\begin{gathered} (-\Delta)^k u = \lambda u + |u|^{s-2} u \quad \text{in } B, \\ D^\alpha u = 0, |\alpha| \le k-1 \quad \text{on } \partial B, \end{gathered} \tag{1.2}$$

where $k \in \mathbb{N}$, B is the unit ball centered at the origin, $\lambda \in \mathbb{R}$, $n > 2k$, $s = 2n/(n-2k)$ is the critical Sobolev exponent. He proved the existence of a positive radial solution for: $\lambda \in (0, \lambda_1)$, if $n \ge 4k$; $\lambda \in (\overline{\lambda}, \lambda_1)$ for some $\overline{\lambda} = \overline{\lambda}(n,k) \in (0, \lambda_1)$, if $2k+1 \le n \le 4k-1$, where λ_1 is the first eigenvalue of $(-\Delta)^k$ with homogeneous Dirichlet boundary conditions.

Recently, Benalili and Tahri [6] considered the multiplicity of solutions considered for the equation

$$\Delta^2 u - \nabla^i (a\rho^{-\sigma} \Delta_i u) + b\rho^{-\mu} u = \lambda |u|^{q-2} u + f(x)|u|^{s-2} u \tag{1.3}$$

where the function $a(x)$ and $b(x)$ are smooth on M and $1 < q < 2$. $s = \frac{2n}{n-4}$ is the critical Sobolev exponent. They proved that when $0 < \sigma < 2$ and $0 < \mu < 4$, there is $\lambda_* > 0$ such that if $\lambda \in (0, \lambda_*)$, the equation (1.3) possesses at least two distinct nontrivial solutions in the distribution sense.

The multiplicity of solutions for higher-order equations can be founded in [4] and the references therein.

Here, our motivation comes from the recent papers [6, 15]. We consider the situation of the multiplicity of the higher-order equation with critical Sobolev exponent when $k \ge 1$ and $q > 2$.

The paper is organized as follows. In Section 2, we will introduce the Sobolev spaces and the embedding theorem which is applicable to problem (1.1). In Section 3, since a lack of compactness, we use analytic techniques and variational arguments to overcome the difficulty and establish some basic lemmas. In Section 4, we give the proof of two distinct weak solutions of Theorem 1.1. Our methods are mainly based on the weakly lower semi-continuous of the functional and the Mountain Pass Lemma without (PS) condition.

2. Preliminaries

Suppose $\Omega \subset \mathbb{R}^n$ is a bounded smooth open domain. We let $H_0^{2m}(\Omega)$ be the Sobolev space which is the completion of the space $C_0^\infty(\Omega)$ with respect to the norm

$$\|u\|_{H_{2m}} = (\|\Delta^m u\|_2^2 + \|\nabla \Delta^{m-1} u\|_2^2 + \dots + \|\nabla u\|_2^2 + \|u\|_2^2)^{1/2}. \tag{2.1}$$

It is well known that a weak solution of the equation (1.1) is a critical point of the following functional

$$\begin{aligned} I_{\lambda,\mu}(u) = \frac{1}{2} \int_\Omega &((\Delta^m u)^2 + |\nabla \Delta^{m-1} u|^2 + \dots + (\Delta u)^2 + |\nabla u|^2 + u^2) \\ &- \frac{1}{q} \lambda \int_\Omega |u|^q - \frac{1}{N} \int_\Omega |u|^N - \mu \int_\Omega f(x) u. \end{aligned} \tag{2.2}$$

Under the above assumptions, it is easy to know that $I_{\lambda,\mu}(u) \in C^1(H_0^{2m}(\Omega), \mathbb{R})$ and with the Gâteaux derivative

$$\begin{aligned} \langle \nabla I_{\lambda,\mu}(u), v\rangle &= \int_\Omega (\Delta^m u \Delta^m v) + (\nabla\Delta^{m-1}u \cdot \nabla\Delta^{m-1}v) + \dots + \nabla u \cdot \nabla v + uv) \\ &\quad - \lambda \int_\Omega |u|^{q-2}uv - \int_\Omega |u|^{N-2}uv - \mu \int_\Omega f(x)v \end{aligned} \tag{2.3}$$

for every $v \in H_0^{2m}(\Omega)$ (see [15]).

Lemma 2.1 (Mountain Pass Theorem [2]). *Let E be a real Banach space and let $I(u) \in C^1(E, \mathbb{R})$. Suppose $I(0) = 0$ and*

(I1) *there is a constant $\rho > 0$ such that $I|_{\partial B_\rho(0)} > 0$,*
(I2) *there is an $e \in E \backslash \overline{B_\rho(0)}$ such that $I(e) \le 0$.*

Set

$$C = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I(\gamma(t)) > 0 \tag{2.4}$$

where Γ denotes the class of paths joining 0 to e. Conclusion: there is a sequence $\{u_k\}$ in E, such that

$$I(u_k) \to C \quad \text{and} \quad \nabla I(u_k) \to 0 \quad \text{in a dual space } E'.$$

Lemma 2.2 (Sobolev-Rellich-Knodrakov Theorem [19]). *Assume that $\Omega \subset \mathbb{R}^n$ is a bounded domain with Lipschitz boundary, k is positive integer and $1 \le p < \infty$. Then the following hold:*

- *if $n > kp$, then $W^{k,p}(\Omega) \hookrightarrow L^s(\Omega)$, for $1 \le s \le p^* = \frac{np}{n-kp}$;*
- *the embedding is compact, for $s < \frac{np}{n-kp}$.*

3. Basic Lemmas

To complete the proof of Theorem 1.1, the following lemmas are our main tools.

Lemma 3.1. *For each fixed $\lambda > 0$, there exist $\delta > 0$, $\mu_0 > 0$ and $\eta > 0$, such that for all $u \in H_0^{2m}(\Omega)$ with $\|u\|_{H_{2m}} = \delta$ and any $0 < \mu < \mu_0$, it holds $I_{\lambda,\mu}(u) > \eta > 0$.*

Proof. From (2.1), (2.2) and the Hölder inequality, we deduce that

$$\begin{aligned} I_{\lambda,\mu}(u) &= \frac{1}{2}\int_\Omega ((\Delta^m u)^2 + (\nabla\Delta^{m-1}u)^2 + \dots + (\Delta u)^2 + |\nabla u|^2 + u^2) \\ &\quad - \frac{\lambda}{q}\int_\Omega |u|^q - \frac{1}{N}\int_\Omega |u|^N - \mu\int_\Omega f(x)u \\ &\ge \frac{1}{2}\|u\|_{H_{2m}}^2 - \frac{\lambda}{q}|\Omega|^{1-\frac{q}{N}}\|u\|_N^q - \frac{1}{N}\|u\|_N^N - \mu \max_{x\in\Omega} f(x)|\Omega|^{1-\frac{1}{N}}\|u\|_N. \end{aligned} \tag{3.1}$$

By (3.1) and Lemma 2.2, we infer that

$$\begin{aligned} I_{\lambda,\mu}(u) &\ge \frac{1}{2}\|u\|_{H_{2m}}^2 - \frac{\lambda}{q}(C)^q|\Omega|^{1-\frac{q}{N}}\|u\|_{H_{2m}}^q - \frac{1}{N}(C)^N\|u\|_{H_{2m}}^N \\ &\quad - \mu\max_{x\in\Omega} f(x)|\Omega|^{1-\frac{1}{N}}C\|u\|_{H_{2m}} \\ &= \Big(\big(\frac{1}{2} - \lambda C_1\|u\|_{H_{2m}}^{q-2} - C_2\|u\|_{H_{2m}}^{N-2}\big)\cdot\|u\|_{H_{2m}} - \mu C_3\Big)\|u\|_{H_{2m}}, \end{aligned}$$

with some positive constants C_1, C_2, C_3 and $2 < q < N$.

Thus for any $\lambda > 0$, there exist $\delta = \delta(\lambda) > 0$, sufficiently small $\mu_0 = \mu_0(\delta) > 0$, and $\eta_0 = \eta_0(\mu_0) > 0$, such that for all $u \in H_0^{2m}(\Omega)$ with $\|u\|_{H_{2m}} = \delta$ and for any $0 < \mu < \mu_0$, it holds $I_{\lambda,\mu}(u) > \eta_0$ $\square$

Lemma 3.2. *Suppose $f(x)$ is continuous and not identical to 0 in Ω. For any $\mu_0 > 0$, there exist $\lambda_0 > 0$ and $v_0 \in H_0^{2m}(\Omega)$, such that for any $\lambda \geq \lambda_0$, we have*

$$0 < \sup_{t \geq 0} I_{\lambda,\mu}(tv_0) < \frac{2m}{n}(C^*)^{-n/(2m)}, \tag{3.2}$$

where C^* is the best Sobolev constant of $H_0^{2m}(\Omega) \hookrightarrow L^N(\Omega)$, $N = 2n/(n-4m)$.

Proof. By the conditions of $f(x)$, we can choose $v_0 \in H_0^{2m}(\Omega)$, such that

$$\int_\Omega f(x)v_0 > 0 \quad \text{and} \quad \int_\Omega |v_0|^N = 1.$$

Thus from (2.2), we obtain

$$I_{\lambda,\mu}(tv_0) = \frac{t^2}{2}\|v_0\|_{H_{2m}}^2 - t^q\frac{\lambda}{q}\int_\Omega |v_0|^q - \frac{1}{N}t^N - t\mu\int_\Omega f(x)v_0. \tag{3.3}$$

For any $\lambda, \mu > 0$, we have

$$\lim_{t\to+\infty} I_{\lambda,\mu}(tv_0) = -\infty. \tag{3.4}$$

Using Lemma 3.1 and (3.4), there exists $t_{\lambda,\mu} > 0$, such that

$$I_{\lambda,\mu}(t_{\lambda,\mu}v_0) = \sup_{t\geq 0} I_{\lambda,\mu}(tv_0) > 0. \tag{3.5}$$

By (3.3) and (3.5), one gets

$$\frac{1}{2}t_{\lambda,\mu}^2\|v_0\|_{H_{2m}}^2 - (\frac{\lambda}{q}t_{\lambda,\mu}^q\|v_0\|_q^q + \frac{1}{N}t_{\lambda,\mu}^N) - t_{\lambda,\mu}\mu\int_\Omega f(x)v_0 > 0. \tag{3.6}$$

That is,

$$t_{\lambda,\mu}^{q-1}\left(\frac{\lambda}{q}\|v_0\|_q^q + \frac{1}{N}t_{\lambda,\mu}^{N-q}\right) < \frac{1}{2}\|v_0\|_{H_{2m}}^2 - \mu\int_\Omega f(x)v_0.$$

By simple analysis, we obtain

$$\lim_{\lambda\to+\infty}(\frac{\lambda}{q}\|v_0\|_q^q + \frac{1}{N}t_{\lambda,\mu}^{N-q}) = +\infty,$$

and

$$\lim_{\lambda\to+\infty} t_{\lambda,\mu} = 0. \tag{3.7}$$

From (3.3) and (3.7), we obtain

$$\lim_{\lambda\to+\infty} \lambda t_{\lambda,\mu}^{q-1} \leq 0. \tag{3.8}$$

Taking into account of (3.5), (3.7) and (3.8), we obtain

$$\lim_{\lambda\to+\infty}\sup_{t\geq 0} I_{\lambda,\mu}(t_{\lambda,\mu}v_0) = 0. \tag{3.9}$$

Then there exist λ_0 such that for any $\lambda > \lambda_0$, we have

$$0 < \sup_{t\geq 0} I_{\lambda,\mu}(tv_0) < \frac{2m}{n}(C^*)^{-n/(2m)}.$$

The proof is complete. $\square$

Lemma 3.3. *For any $\lambda > 0$, there exists sufficiently small $\mu_0 > 0$, such that for any $0 < \mu < \mu_0$, the $I_{\lambda,\mu}(u)$ satisfies the $(PS)_{C_{\lambda,\mu}}$-condition for all $C_{\lambda,\mu}$ in the interval*

$$0 < C_{\lambda,\mu} < \frac{2m}{n}(C^*)^{-n/(2m)}. \tag{3.10}$$

Proof. First we prove that each $(PS)_{C_{\lambda,\mu}}$ sequence is bounded in $H_0^{2m}(\Omega)$. Let $\{u_k\} \subset H_0^{2m}(\Omega)$ be a $(PS)_{C_{\lambda,\mu}}$ sequence for $I_{\lambda,\mu}(u)$, defined by (2.2), i.e.,

$$I_{\lambda,\mu}(u_k) \to C_{\lambda,\mu}, \quad \text{and} \quad \nabla I_{\lambda,\mu}(u_k) \to 0, \quad \text{in } H_0^{2m}(\Omega)', \text{ as } k \to \infty.$$

That is,

$$\begin{aligned} I_{\lambda,\mu}(u_k) &= \frac{1}{2}\int_\Omega ((\Delta^m u_k)^2 + (\nabla\Delta^{m-1}u_k)^2 + \dots + (\Delta u_k)^2 + |\nabla u_k|^2 + u_k^2) \\ &\quad - \frac{\lambda}{q}\int_\Omega |u_k|^q - \frac{1}{N}\int_\Omega |u_k|^N - \mu\int_\Omega f(x)u_k \\ &= C_{\lambda,\mu} + o(1) \end{aligned} \tag{3.11}$$

and

$$\begin{aligned} \langle \nabla I_{\lambda,\mu}(u_k), u_k\rangle &= \int_\Omega ((\Delta^m u_k)^2 + (\nabla\Delta^{m-1}u_k)^2 + \dots + (\Delta u_k)^2 + |\nabla u_k|^2 + u_k^2) \\ &\quad - \lambda\int_\Omega |u_k|^q - \int_\Omega |u_k|^N - \mu\int_\Omega f(x)u_k \\ &= o(1)\|u_k\|_{H_{2m}}, \end{aligned} \tag{3.12}$$

as $k \to \infty$. By (3.11), (3.12), the Hölder inequality and Lemma 2.2, we obtain

$$\begin{aligned} &I_{\lambda,\mu}(u_k) - \frac{1}{q}\langle \nabla I_{\lambda,\mu}(u_k), u_k\rangle \\ &= (\frac{1}{2} - \frac{1}{q})\|u_k\|_{H_{2m}}^2 + (\frac{1}{q} - \frac{1}{N})\int_\Omega |u_k|^N - (1 - \frac{1}{q})\mu\int_\Omega f(x)u_k \\ &\ge (\frac{1}{2} - \frac{1}{q})\|u_k\|_{H_{2m}}^2 - (1 - \frac{1}{q})\mu\int_\Omega f(x)u_k \\ &\ge (\frac{1}{2} - \frac{1}{q})\|u_k\|_{H_{2m}}^2 - (1 - \frac{1}{q})\mu\max_{x\in\Omega} f(x)|\Omega|^{1-\frac{1}{N}}(C^*)\|u_k\|_{H_{2m}}. \end{aligned} \tag{3.13}$$

It follows from (3.11), (3.12) and (3.13) that

$$\begin{aligned} &o(1) + C_{\lambda,\mu} + o(1)\|u_k\|_{H_{2m}} \\ &\ge (\frac{1}{2} - \frac{1}{q})\|u_k\|_{H_{2m}}^2 - (1 - \frac{1}{q})\mu\max_{x\in\Omega} f(x)|\Omega|^{1-\frac{1}{N}}(C^*)\|u_k\|_{H_{2m}}, \end{aligned}$$

i.e.

$$\begin{aligned} &(\frac{1}{2} - \frac{1}{q})\|u_k\|_{H_{2m}}^2 \\ &\le o(1) + C_{\lambda,\mu} + \big[(1 - \frac{1}{q})\mu\max_{x\in\Omega} f(x)|\Omega|^{1-\frac{1}{N}}(C^*) + o(1)\big]\|u_k\|_{H_{2m}}, \end{aligned} \tag{3.14}$$

where $q > 2$. Hence, for each λ , $\mu > 0$, fixed $C_{\lambda,\mu} \in \mathbb{R}$, we conclude that the sequence $\{u_k\}$ is bounded in $H_0^{2m}(\Omega)$.

Now, we show that the $(PS)_{C_{\lambda,\mu}}$ sequence contains a strongly convergent subsequence.

Since the sequence $\{u_k\}$ is bounded in $H_0^{2m}(\Omega)$ and the well-known Sobolev's embedding, there exists a subsequence, still denoted by $\{u_k\}$, and $u \in H_0^{2m}(\Omega)$, such that

$$
\begin{gathered}
u_k \rightharpoonup u \quad \text{weakly in } H_0^{2m}(\Omega), \\
u_k \to u \quad \text{strongly in } L^i(\Omega), \text{ for } 1 < i < N = \frac{2n}{n-4m}, \\
\nabla u_k \to \nabla u \quad \text{strongly in } L^2(\Omega), \\
\Delta u_k \to \Delta u \quad \text{strongly in } L^2(\Omega), \\
\dots \\
\nabla \Delta^{m-1} u_k \to \nabla \Delta^{m-1} u, \quad \text{strongly in } L^2(\Omega), \\
u_k \to u \quad \text{a.e. in } \Omega.
\end{gathered}
$$

Thus

$$
\begin{gathered}
\int_\Omega u_k^2 \to \int_\Omega u^2, \\
\int_\Omega |\nabla u_k|^2 \to \int_\Omega |\nabla u|^2, \\
\dots \\
\int_\Omega |\nabla \Delta^{m-1} u_k|^2 \to \int_\Omega |\nabla \Delta^{m-1} u|^2, \\
\int_\Omega |u_k|^q \to \int_\Omega |u|^q, \ q < N, \\
\int_\Omega f(x) u_k \to \int_\Omega f(x) u,
\end{gathered}
$$

as $k \to \infty$. By Brezis-Lieb Lemma [8], we have

$$
\begin{gathered}
\|\Delta^m u_k\|_2^2 - \|\Delta^m u\|_2^2 = \|\Delta^m (u_k - u)\|_2^2 + o(1), \\
\int_\Omega (|u_k|^N - |u|^N) = \int_\Omega |u_k - u|^N + o(1).
\end{gathered}
$$

Now, by doing some calculations, we obtain

$$
I_{\lambda,\mu}(u_k) - I_{\lambda,\mu}(u) = \frac{1}{2}\|\Delta^m (u_k - u)\|_2^2 - \frac{1}{N}\int_\Omega |u_k - u|^N + o(1). \tag{3.15}
$$

By (3.12) and $\{u_k\}$ being bounded, we have

$$
\begin{aligned}
o(1) &= \langle \nabla I_{\lambda,\mu}(u_k), u_k - u \rangle \\
&= \int_\Omega (\Delta^m (u_k - u))^2 - \int_\Omega (|u_k|^N - |u|^N) + o(1).
\end{aligned} \tag{3.16}
$$

From (3.15) and (3.16), we have

$$
I_{\lambda,\mu}(u_k) - I_{\lambda,\mu}(u) = (\frac{1}{2} - \frac{1}{N})\|\Delta^m (u_k - u)\|_2^2 + o(1). \tag{3.17}
$$

On the other hand, the Vitali convergence theorem see [17, chap. III.2] yields

$$
\int_\Omega |u_k|^N - \int_\Omega (|u_k|^{N-2} |u_k - u|^2) \to \int_\Omega |u|^N, \quad \text{as } k \to \infty. \tag{3.18}
$$

From (3.15) and (3.18), one gets

$$I_{\lambda,\mu}(u_k) - I_{\lambda,\mu}(u) = \frac{1}{2}\|\Delta^m(u_k-u)\|_2^2 - \frac{1}{N}\int_\Omega |u_k|^{N-2}|u_k-u|^2 + o(1). \tag{3.19}$$

Using (3.19), Lemma 2.2 and Hölder inequality, we obtain

$$I_{\lambda,\mu}(u_k) - I_{\lambda,\mu}(u) \ge \Big(\frac{1}{2} - \frac{1}{N}\|u_k\|_N^{N-2}(C^*)^2\Big)\|\Delta^m(u_k-u)\|_2^2 + o(1). \tag{3.20}$$

Taking account of (3.17) and (3.20), we obtain

$$(\frac{1}{2} - \frac{1}{N})\|\Delta^m(u_k-u)\|_2^2 \ge \Big(\frac{1}{2} - \frac{1}{N}\|u_k\|_N^{N-2}(C^*)^2\Big)\|\Delta^m(u_k-u)\|_2^2 + o(1),$$

i.e.

$$\Big(1 - \|u_k\|_N^{N-2}(C^*)^2\Big)\,\|\Delta^m(u_k-u)\|_2^2 \le o(1).$$

Hence

$$\lim_{k\to+\infty}\sup \|u_k\|_N < (C^*)^{-\frac{2}{N-2}}, \tag{3.21}$$

which implies

$$\|\Delta^m(u_k-u)\|_2^2 = o(1), \quad k\to\infty.$$

Thus $u_k \to u$ strongly in $H_0^{2m}(\Omega)$.

Now, we verify (3.21). Using (3.11), (3.12) and that the sequence $\{u_k\}$ is bounded in $H_0^{2m}(\Omega)$, we have

$$(\frac{1}{2} - \frac{1}{N})\|u_k\|_N^N + (\frac{1}{2} - \frac{1}{q})\lambda\|u_k\|_q^q - \frac{\mu}{2}\int_\Omega f u_k = C_{\lambda,\mu} + o(1). \tag{3.22}$$

By (3.14), for any $\varepsilon > 0$ and μ is sufficiently small, we have

$$\frac{\mu}{2}\Big|\int_\Omega f u_k\Big| < \varepsilon.$$

Thus for any $\lambda > 0$ and μ is sufficiently small, we obtain

$$(\frac{1}{2} - \frac{1}{N})\|u_k\|_N^N \le C_{\lambda,\mu}. \tag{3.23}$$

By the assumption $0 < C_{\lambda,\mu} < \frac{2m}{n}(C^*)^{-n/(2m)}$, we have thus (3.21). $\square$

Lemma 3.4. *For all $\lambda > 0$ and $\mu > 0$, the function $I_{\lambda,\mu}(u)$ is weak lower semi-continuous on the set*

$$\{u \in H_0^{2m}(\Omega) : \|u\|_{H_{2m}} \le r_0\},$$

where $r_0 = \big(\frac{N}{2^{2N-2}(C^*)^N}\big)^{1/(N-2)}$.

Proof. Let $\{u_k\}$ be a sequence in $H_0^{2m}(\Omega)$, and $0 < r < \big(\frac{N}{2^{2N-3}(C^*)^N}\big)^{1/(N-2)}$, such that

$$u_k \rightharpoonup u, \quad \text{in } H_0^{2m}(\Omega) \text{ and } \|u_k\|_{H_{2m}} \le r.$$

Then we have $\|u\|_{H_{2m}} \le r$. Up to a subsequence, we obtain

$$u_k \to u, \quad \text{strongly in } L^p(\Omega), \text{ for all } p < N,$$
$$u_k \to u \quad \text{a.e. in } \Omega.$$

Thus

$$\int_\Omega |u_k|^q \to \int_\Omega |u|^q, \quad 2 < q < N, \tag{3.24}$$

$$\int_\Omega f(x)u_k \to \int_\Omega f(x)u. \tag{3.25}$$

By the Brezis-Lieb Lemma [8], we have

$$\|\Delta^i u_k\|_2^2 - \|\Delta^i u\|_2^2 = \|\Delta^i (u_k - u)\|_2^2 + o(1), \quad i = 0, 1, 2, \dots, m, \tag{3.26}$$

$$\|\nabla\Delta^i u_k\|_2^2 - \|\nabla\Delta^i u\|_2^2 = \|\nabla\Delta^i (u_k - u)\|_2^2 + o(1), \quad i = 0, 1, 2, \dots, m-1, \tag{3.27}$$

$$\int_\Omega (|u_k|^N - |u|^N) = \int_\Omega |u_k - u|^N + o(1). \tag{3.28}$$

From (3.26) and (3.27), we obtain

$$\|u_k\|_{H_{2m}}^2 - \|u\|_{H_{2m}}^2 = \|u_k - u\|_{H_{2m}}^2 + o(1). \tag{3.29}$$

Using (3.28) and Lemma 2.2, we have

$$\begin{aligned} \int_\Omega |u_k - u|^N &\le (C^*)^N \|u_k - u\|_{H_{2m}}^N \\ &\le (C^*)^N \|u_k - u\|_{H_{2m}}^2 2^{N-2} (\|u_k\|_{H_{2m}} + \|u\|_{H_{2m}})^{N-2} \\ &\le 2^{2N-4} (C^*)^N r^{N-2} \|u_k - u\|_{H_{2m}}^2. \end{aligned} \tag{3.30}$$

To sum up (3.24), (3.25), (3.29) and (3.30), we obtain

$$\begin{aligned} I_{\lambda,\mu}(u_k) - I_{\lambda,\mu}(u) &= \frac{1}{2}\|u_k - u\|_{H_{2m}}^2 - \frac{1}{N}\int_\Omega |u_k - u|^N + o(1) \\ &\ge \Big(\frac{1}{2} - 2^{2N-4}\frac{1}{N}(C^*)^N r^{N-2}\Big)\|u_k - u\|_{H_{2m}}^2 + o(1). \end{aligned}$$

Taking $r = r_0 = \big(\frac{N}{2^{2N-2}(C^*)^N}\big)^{1/(N-2)}$ in above equation,

$$I_{\lambda,\mu}(u_k) - I_{\lambda,\mu}(u) \ge \frac{1}{4}\|u_k - u\|_{H_{2m}}^2 + o(1).$$

If $\|u_k - u\|_{H_{2m}} \to 0$, as $k \to \infty$, by the (3.15) and (3.16), we have

$$\liminf_{k\to\infty} I_{\lambda,\mu}(u_k) = I_{\lambda,\mu}(u).$$

If $\|u_k - u\|_{H_{2m}} \to 0$, as $k \to \infty$, thus

$$\liminf_{k\to\infty} I_{\lambda,\mu}(u_k) \ge I_{\lambda,\mu}(u).$$

In brief, we obtain

$$\liminf_{k\to\infty} I_{\lambda,\mu}(u_k) \ge I_{\lambda,\mu}(u).$$

This completes the proof. □

4. Proof of main results

Proposition 4.1. *Suppose that $f(x)$ is continuous with not identical to 0 in Ω and $\lambda > 0$. For $\mu_0 > 0$ small enough such that for any $0 < \mu < \mu_0$, then (1.1) has a solution with negative energy.*

Proof. Since $f(x)$ is continuous and not identical to 0 in Ω, then there exists $\phi \in H_0^{2m}(\Omega)$, such that $\int_\Omega f(x)\phi > 0$. For any $t > 0$, we have

$$I_{\lambda,\mu}(t\phi) = \frac{1}{2}t^2\|\phi\|_{H_{2m}}^2 - \frac{1}{N}t^N\|\phi\|_N^N - \lambda\frac{1}{q}t^q\|\phi\|_q^q - \mu t\int_\Omega f(x)\phi. \tag{4.1}$$

Hence, there exists $t_0(\lambda,\mu) > 0$, such that $0 < t \le t_0(\lambda,\mu)$ and

$$I_{\lambda,\mu}(t\phi) < 0.$$

By Lemma 3.4, there exist $r_0 > 0$ and $v \in H_0^{2m}(\Omega)$ with $\|v\|_{H_{2m}} \le r_0$, such that

$$I_{\lambda,u}(v) = \inf_{\|u\|_{H_{2m}} \le r_0} I_{\lambda,\mu}(u) < 0. \tag{4.2}$$

Thus v is a weak solution of (1.1) with negative energy. □

Proposition 4.2. *Suppose $f(x)$ is continuous and not identical to 0 in Ω. If $\lambda > 0$ is sufficiently large and $\mu > 0$ is enough small, then* (1.1) *has a weak solution with positive energy.*

Proof. Lemma 3.1 implies that $I_{\lambda,\mu}(u)$ satisfies the condition (I1) in Lemma 2.1. On the other hand, from (4.1), we obtain

$$\lim_{t \to +\infty} I_{\lambda,\mu}(t\phi) = -\infty.$$

There exists a constant $T > 0$, taking $e = T\phi$ with $\|e\|_{H_{2m}} > \delta$ such that

$$I_{\lambda,\mu}(e) < 0,$$

where $\delta > 0$ is the constant in Lemma 3.1. Thus the condition (I2) of Lemma 2.1 holds. Denote

$$C_{\lambda,\mu} = \inf_{\gamma \in \Gamma} \sup_{t \in [0,1]} I_{\lambda,\mu}(\gamma(t)), \tag{4.3}$$

where

$$\Gamma = \{\gamma \in C([0,1], H_0^{2m}(\Omega)) : \gamma(0) = 0, \quad \gamma(1) = e\}.$$

From Lemma 3.2 and (4.3), it follows that

$$0 < C_{\lambda,\mu} < \frac{2m}{n}(C^*)^{-n/(2m)}.$$

Applying Lemma 2.1, there exists a sequence $\{u_k\} \subset H_0^{2m}(\Omega)$, such that

$$I_{\lambda,\mu}(u_k) \to C_{\lambda,\mu}, \quad \text{and} \quad \nabla I_{\lambda,\mu}(u_k) \to 0, \quad \text{as } k \to \infty.$$

By Lemma 3.3, there exists a subsequence of $\{u_k\}$ which strongly converges to u in $H_0^{2m}(\Omega)$. Thus $I_{\lambda,\mu}(u)$ has a critical point u with $I_{\lambda,\mu}(u) = C_{\lambda,\mu} > 0$. Hence we obtain a weak solution of equation (1.1) with positive energy. □

Proof of Theorem 1.1. From Propositions 4.1 and 4.2, problem (1.1) has two distinctic solutions u_1, u_2 with $I_{\lambda,\mu}(u_1) < 0 < I_{\lambda,\mu}(u_2)$. □

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Wei Liu
College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China
E-mail address: liuweimaths@hotmail.com

Gao Jia (corresponding author)
College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China
E-mail address: gaojia89@163.com

Lu-Quian Guo
College of Science, University of Shanghai for Science and Technology, Shanghai 200093, China
E-mail address: xiaoshudian.2008@163.com