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EXISTENCE AND UNIQUENESS FOR A DISLOCATION MODEL WITH SHORT-RANGE INTERACTIONS AND VARYING STRESS FIELD

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Abstract. In this article, we consider a coupled singular parabolic system, describing the dynamics of dislocation densities in a bounded domain. The model takes into consideration the short-range interactions between dislocations, which causes the singularity that appears under the form of dividing by a gradient term. We prove a long time existence and uniqueness under the assumption that the applied stresses on the domain is bounded in space and time. The proof relies on a comparison principle to avoid singularity, and on exponential gradient estimates for the long time existence.

1. Introduction

1.1. Setting of the problem and main result. In this article, we study the singular parabolic system

$$\begin{aligned} \kappa_t &= \varepsilon\kappa_{xx} + \frac{\rho_x\rho_{xx}}{\kappa_x} + \sigma\rho_x \quad \text{in } (0,1)\times(0,\infty) \\ \rho_t &= (1+\varepsilon)\rho_{xx} + \sigma\kappa_x \quad \text{in } (0,1)\times(0,\infty), \end{aligned} \tag{1.1}$$

with the initial conditions

$$\kappa(x,0) = \kappa^0(x), \quad \rho(x,0) = \rho^0(x), \quad x \in (0,1), \tag{1.2}$$

and the Dirichlet boundary conditions

$$\rho(0,t) = \rho(1,t) = \kappa(0,t) = 0, \quad \kappa(1,t) = 1, \quad t > 0. \tag{1.3}$$

In what follows, we adopt the notation

$$I_T = I \times (0,T) \quad \text{with } I = (0,1) \text{ and } T > 0.$$

System (1.1) is an approximate model of the one introduced by Groma, Csikor and Zaiser [5] to describe the dynamics of dislocation densities in a constrained channel submitted to an applied stress (here represented by the function $\sigma = \sigma(x,t)$). A dislocation is a defect, or irregularity within a crystal structure that can be observed by electron microscopy. The theory was originally developed by Vito Volterra in 1905. Dislocations are a non-stationary phenomena and their motion is

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the main explanation of the plastic deformation in metallic crystals (see [12, 6] and the references therein for a recent and mathematical presentation).

The approximation stated above is based on ε-Δ regularization of a spatially differentiated system of (1.1), which explains the presence of the different factors ε and $(1+\varepsilon)$ in the two equations. The stress function

$$\sigma : \overline{I}_\infty \longrightarrow \mathbb{R},$$

is assumed bounded and regular, namely:

$$\sigma \in C^2(\overline{I}_\infty) \quad \text{with } b := \|\sigma\|_{L^\infty(\overline{I}_\infty)} < +\infty, \tag{1.4}$$

while the initial data ρ^0 and κ^0 are assumed smooth over $\overline{I}$.

The goal of the present paper is to show the long time existence and uniqueness of a smooth solution of (1.1), (1.2) and (1.3). The same question has been raised and solved in [7] (see also [9] for a brief study focusing on the main ideas) but for a constant $\sigma \in \mathbb{R}$. The proof relied on a comparison principle on the gradient κ_x leading to the inequality:

$$\kappa_x > |\rho_x| \quad \text{on } \overline{I}_\infty, \tag{1.5}$$

that was used, first, to avoid the singularity in the first equation of (1.1), and second, to linearise it in order to obtain some *a priori* estimates ensuring the long time existence. We show how to adapt this comparison principle to cover the case (1.4), and how to obtain the *a priori* estimates. This will finally lead to our main result.

Theorem 1.1. *Let $\rho^0, \kappa^0 \in C^\infty(\overline{I})$ satisfy:*

$$\begin{gathered} \rho^0(0) = \rho^0(1) = 0, \\ \kappa^0(0) = 0, \quad k^0(1) = 1, \\ (1+\varepsilon)\rho^0_{xx} + \sigma(\cdot,0)\kappa^0_x = 0 \quad \text{on } \partial I, \\ (1+\varepsilon)\kappa^0_{xx} + \sigma(\cdot,0)\rho^0_x = 0 \quad \text{on } \partial I, \end{gathered} \tag{1.6}$$

and

$$\kappa^0_x > |\rho^0_x| \quad \text{in } \overline{I}. \tag{1.7}$$

Then, for every $0 < \alpha < 1$, there exists a unique solution:

$$\rho, \kappa \in C^{3+\alpha, \frac{3+\alpha}{2}}(\overline{I}_\infty) \cap C^\infty(\overline{I} \times (0,\infty)),$$

of system (1.1)*,* (1.2) *and* (1.3) *satisfying* (1.5)*.*

It is worth mentioning that the boundary condition (1.6) on ρ^0 and κ^0 is natural. In fact, it appears from the Hölder regularity of the solution as well as (1.1), (1.2) and (1.3). However, condition (1.5) is of physical origin as it represents the positivity of the dislocation densities (see for example [8] for the derivation).

Let us briefly state the strategy of our proof. The existence and uniqueness is made by using a fixed point argument after an artificial modification of (1.1) in order to avoid dividing by zero. We will first show the short time existence, proving in particular that

$$\kappa_x(x,t) \geq \sqrt{\gamma^2(t) + \rho_x^2(x,t)},$$

for a well chosen function $\gamma(t) = ce^{-ct}$, where $c > 0$ depends on σ and the initial data. Here we obtain one of the key estimates $|\frac{\rho_x}{\kappa_x}| \leq 1$ which, in addition to the boundedness of σ, somehow linearise the first equation of (1.1). Consequently, and due to some *a priori* estimates, we can prove the global time existence.

To our knowledge, systems of equations involving the singularity in $1/\kappa_x$ as in (1.1) has not been directly handled elsewhere in the literature. However, parabolic problems involving singular terms have been widely studied in various aspects (see for instance [1, 2, 3, 4, 10, 11]).

1.2. **Organization of this article.** This paper is organized as follows. In Section 2, we show a comparison principle associated to (1.1) that plays a crucial rule in the long time existence of the solution as well as the positivity of κ_x. In Section 3, we present a result of short time existence, uniqueness and regularity by using fixed point and bootstrap arguments. Finally, in Section 4, we provide some exponential bounds on the solution and we prove our main result: Theorem 1.1.

2. A comparison principle

In this section, and for simplification reasons, we take $I = (-1, 1)$ and we set

$$G_a(x) := \sqrt{x^2 + a^2}, \quad x, a \in \mathbb{R}.$$

We now state our comparison principle on the gradient of the solution.

Proposition 2.1 (Comparison principle). *Let (ρ, κ) be a regular solution of* (1.1) *on the compact $\overline{I}_T$ with $\kappa_x > 0$, and the initial data (ρ^0, κ^0) satisfying:*

$$\kappa_x^0 \geq G_{\gamma_0}(\rho_x^0), \quad \gamma_0 \in (0, 1). \tag{2.1}$$

Then there exists a positive function $\gamma : [0, T] \mapsto \mathbb{R}$ such that

$$\kappa_x \geq G_\gamma(\rho_x) \quad \text{on } \overline{I}_T. \tag{2.2}$$

Proof. We define $M := \kappa_x - G_\gamma(\rho_x)$, where the function $\gamma = \gamma(t)$, $t \in [0, T]$, is to be determined in a way that $M \geq 0$ on $\overline{I}_T$. The proof is divided into three steps.

Step 1. (Differential inequality satisfied by M**)** Assuming the regularity of ρ, κ and γ, we compute

$$\begin{aligned} M_t &= \kappa_{xt} - G'_\gamma(\rho_x)\rho_{xt} - \Gamma, \\ M_x &= \kappa_{xx} - G'_\gamma(\rho_x)\rho_{xx}, \\ M_{xx} &= \kappa_{xxx} - G''_\gamma(\rho_x)\rho_{xx}^2 - G'_\gamma(\rho_x)\rho_{xxx}, \end{aligned} \tag{2.3}$$

where

$$\Gamma = \frac{\gamma\gamma'}{\sqrt{\rho_x^2 + \gamma^2}}.$$

Differentiating (1.1) with respect to x, we easily obtain

$$\begin{aligned} \kappa_{xt} &= \varepsilon\kappa_{xxx} + \frac{\rho_{xx}^2}{\kappa_x} + \frac{\rho_x\rho_{xxx}}{\kappa_x} - \frac{\rho_x\rho_{xx}\kappa_{xx}}{\kappa_x^2} + \sigma_x\rho_x + \sigma\rho_{xx}, \\ \rho_{xt} &= (1 + \varepsilon)\rho_{xxx} + \sigma_x\kappa_x + \sigma\kappa_{xx}. \end{aligned} \tag{2.4}$$

Using (2.3) and (2.4), direct computations lead to

$$M_t = \varepsilon M_{xx} + AM_x + BM + C - \Gamma, \tag{2.5}$$

where

$$\begin{aligned} A &= -\frac{\rho_x \rho_{xx}}{\kappa_x^2} - \sigma G'(\rho_x), \\ B &= \frac{\rho_{xx}^2}{\kappa_x^2} - \sigma_x G'(\rho_x) - \frac{G'_\gamma(\rho_x)\rho_{xxx}}{\kappa_x}, \\ C &= G(\rho_x)\Big(\frac{\rho_{xx}^2}{\kappa_x^2} - \sigma_x G'(\rho_x)\Big) - \frac{\rho_x \rho_{xx}^2}{\kappa_x^2} G'(\rho_x) - \sigma (G'(\rho_x))^2 \rho_{xx} \\ &\quad + \varepsilon G''(\rho_x)\rho_{xx}^2 + \sigma_x \rho_x + \sigma \rho_{xx}. \end{aligned} \tag{2.6}$$

We now estimate the term C and we show, in particular, that we can eliminate σ_x. Thus we return back to a case similar to a constant σ. In fact, since $G(\rho_x)G'(\rho_x) = \rho_x$, we obtain

$$\begin{aligned} C &= \frac{\rho_{xx}^2}{\kappa_x^2} G(\rho_x) - \sigma_x \rho_x - \frac{\rho_x \rho_{xx}^2}{\kappa_x^2} G'(\rho_x) - \sigma (G'(\rho_x))^2 \rho_{xx} + \varepsilon G''(\rho_x)\rho_{xx}^2 \\ &\quad + \sigma_x \rho_x + \sigma \rho_{xx} \\ &= \overbrace{\frac{\rho_{xx}^2}{\kappa_x^2}\,(G(\rho_x) - \rho_x G'(\rho_x))}^{I} + \overbrace{\sigma \rho_{xx}\big(1 - (G'(\rho_x))^2\big)}^{II} + \overbrace{\varepsilon G''(\rho_x)\rho_{xx}^2}^{III}. \end{aligned}$$

Notice that $G(\rho_x) - \rho_x G'(\rho_x) = \frac{\gamma^2}{G(\rho_x)}$, thus $I \geq 0$. However, simple computations give:

$$II = \frac{\sigma \gamma^2 \rho_{xx}}{\rho_x^2 + \gamma^2} \quad \text{and} \quad III = \frac{\varepsilon \gamma^2 \rho_{xx}^2}{(\rho_x^2 + \gamma^2)^{3/2}}.$$

Using (1.4), and applying Young's inequality, we obtain:

$$\Big|\frac{\sigma \gamma^2 \rho_{xx}}{\rho_x^2 + \gamma^2}\Big| \leq \frac{\gamma |\rho_{xx}|}{(\rho_x^2 + \gamma^2)^{3/4}} \frac{b\gamma}{(\rho_x^2 + \gamma^2)^{1/4}} \leq \frac{\varepsilon \gamma^2 \rho_{xx}^2}{(\rho_x^2 + \gamma^2)^{3/2}} + \frac{b^2 \gamma^2}{4\varepsilon (\rho_x^2 + \gamma^2)^{1/2}};$$

therefore

$$II \geq -III - \frac{b^2 \gamma^2}{4\varepsilon (\rho_x^2 + \gamma^2)^{1/2}}.$$

Consequently,

$$C \geq -\frac{b^2 \gamma^2}{4\varepsilon (\rho_x^2 + \gamma^2)^{1/2}}.$$

This inequality, together with (2.5) and (2.6), lead to the differential inequality

$$M_t \geq \varepsilon M_{xx} + A M_x + B M - \frac{b^2 \gamma^2}{4\varepsilon (\rho_x^2 + \gamma^2)^{1/2}} - \Gamma.$$

The choice of γ: We want to choose γ such that the above differential inequality in M is homogeneous. In fact, choosing

$$\gamma(t) = \gamma_0 e^{-\frac{b^2}{4\varepsilon} t},$$

we arrive at

$$M_t \geq \varepsilon M_{xx} + A M_x + B M. \tag{2.7}$$

Step 2. (Boundary analysis) Since ρ and κ are constants on the boundary $\partial I \times [0,T]$, we obtain

$$\begin{gathered}\varepsilon\kappa_{xx} + \frac{\rho_x\rho_{xx}}{\kappa_x} + \sigma\rho_x = 0 \quad \text{on } \partial I \times [0,T]\\ (1+\varepsilon)\rho_{xx} + \sigma\kappa_x = 0 \quad \text{on } \partial I \times [0,T];\end{gathered}$$

therefore

$$M_x = \frac{\sigma}{1+\varepsilon} G'_\gamma(\rho_x) M \quad \text{on } \partial I \times [0,T]. \tag{2.8}$$

Here we would like to show that the minimum $m(t) = \min_{\overline{I}} M(\cdot,t)$ is not attained on the boundary, and then to use (2.7) to show its positivity. However, this is not possible since the above boundary condition (2.8) carries no information that violates the presence of the minimal point on ∂I. For this reason, we carefully multiply M by a suitable positive function having large values on the boundary. In particular, we consider $\overline{M}$ defined by:

$$\overline{M}(x,t) = \cosh(\theta x) M(x,t), \quad \theta \in \mathbb{R},$$

where θ is to be adjusted in a way that $\overline{M}(\cdot,t)$ attains its minimum in I. In fact, elementary computations show that

$$\overline{M}_x = \Theta\overline{M} \quad \text{on } \partial I \times [0,T], \tag{2.9}$$

where

$$\Theta = \big(\theta\tanh(\theta x) + \frac{\sigma}{1+\varepsilon} G'_\gamma(\rho_x)\big).$$

The boundedness of $\frac{\sigma}{1+\varepsilon} G'_\gamma(\rho_x)$ on $\overline{I}_T$ permits the existence of θ large enough so that $\Theta(1,t) > 0$ and $\Theta(-1,t) < 0$ for all $t \in [0,T]$. Hence, by (2.9), the function $\overline{M}(\cdot,t)$ can not have a positive minimum on ∂I.

Step 3. (Conclusion) We now write the partial differential inequality satisfied by $\overline{M}$:

$$\begin{aligned}\overline{M}_t \geq\ & \varepsilon\overline{M}_{xx} + \Big(-\frac{\rho_x\rho_{xx}}{\kappa_x^2} - \sigma G'(\rho_x) - 2\theta\varepsilon\tanh(\theta x)\Big)\overline{M}_x + \Big[\frac{\rho_{xx}^2}{\kappa_x^2}\\ & - \sigma_x G'(\rho_x) - \frac{G'_\gamma(\rho_x)\rho_{xxx}}{\kappa_x} - \theta\tanh(\theta x)\Big(-\frac{\rho_x\rho_{xx}}{\kappa_x^2} - \sigma G'(\rho_x)\Big)\\ & + \varepsilon\theta^2\left(2\tanh^2(\theta x) - 1\right)\Big]\overline{M}.\end{aligned} \tag{2.10}$$

Due to the regularity of $\overline{M}$, we may find a curve $t \mapsto (x(t),t)$ such that

$$\overline{m}(t) = \min_{x\in\overline{I}} \overline{M}(x,t) = \overline{M}(x(t),t) \quad t \in [0,T].$$

Without loss of generality, we assume that $\kappa_x^0 > \sqrt{(\rho_x^0)^2 + \gamma_0^2}$. In fact, it suffices to adjust γ_0 in (2.1). Therefore (see Step 2):

$$\overline{m}(0) > 0 \quad \text{and} \quad x(0) \in I.$$

Again, the regularity of $\overline{M}$ ensures that $x(t) \in I$ for all $0 \leq t \leq t_0 \leq T$. We claim that $t_0 = T$. Indeed, if not, we obtain $x(t_0) \in \partial I$. Let us show that this can not be true. Indeed, since $x(t) \in I$ for $0 \leq t < t_0$ we directly obtain

$$\overline{M}_x(x(t),t) = 0 \quad \text{and} \quad \overline{M}_{xx}(x(t),t) \geq 0.$$

Moreover, using (2.10), we obtain, for some constant $c \in \mathbb{R}$, the following ordinary differential inequality involving $\overline{m}$:

$$\overline{m}' \geq c\overline{m} \quad \text{for } 0 \leq t < t_0,$$

and therefore

$$\overline{m}(t) \geq \overline{m}(0)e^{ct} \quad \text{for } 0 \leq t < t_0. \tag{2.11}$$

Since $\overline{m}(0) > 0$, the above inequality gives $\overline{m}(t_0) > 0$. Consequently (see Step 2) $x(t_0) \in I$ and the claim is true.

Now as we obtain $x(t) \in I$ for all $t \in [0, T]$, the inequality (2.11) also holds true for all $t \in [0, T]$ with $\overline{m}(0) > 0$. Hence, we can infer that $\overline{M} \geq 0$ on I_T and therefore $M \geq 0$ on I_T. □

3. Short time existence and uniqueness

In this section we prove the short time existence and uniqueness for (1.1), (1.2) and (1.3). The main idea is to find a solution of a truncated system where we carefully truncate the gradients ρ_x and κ_x in order to make use of the fixed point theorem. After that, and due to the regularity of the obtained solution, we can eliminate the artificial modification and get back to our original system. Before stating the main result of this section, let us present some basic tools used in our analysis.

Basic tools: We first consider, for real values $a \geq 0$, the real valued function $\mathcal{I}_a$ defined by:

$$\mathcal{I}_a(x) = x\mathbb{1}_{\{|x| \leq a\}} + a\mathbb{1}_{\{x \geq a\}} - a\mathbb{1}_{\{x \leq -a\}},$$

where $\mathbb{1}_A$ is the indicator function of the set $A \subseteq \mathbb{R}$. It is easily seen that $\mathcal{I}_a$ is a truncation of the identity function which is bounded and Lipschitz on $\mathbb{R}$, and this property will be repeatedly used hereafter in this section.

We now define the spaces and some fundamental estimates we are use. We may sometimes use the differentiation notation:

$$D_z^k(u) = \frac{\partial^k u}{\partial z^k}.$$

For $p > 3$, we consider the parabolic Sobolev space:

$$Y = W_p^{2,1}(I_T) = \{u \in L^p(I_T);\, D_t^r D_x^s u \in L^p(I_T) \text{ for } 2r + s \leq 2\},$$

equipped with the norm

$$\|u\|_Y = \sum_{2r+s \leq 2} \|D_t^r D_x^s u\|_{L^p(I_T)}.$$

The value $p > 3$ is taken to emphasis some regularity properties on the solution. Indeed, it is well known (see [13] for the details) that if $p > 3$ then Y is continuously embedded in a parabolic Hölder space:

$$Y \hookrightarrow C^{1+\alpha, \frac{1+\alpha}{2}}(\overline{I}_T), \quad \alpha = 1 - \frac{3}{p}, \tag{3.1}$$

with the following fundamental estimate, valid for $u = 0$ on the parabolic boundary,

$$\partial^p(I_T) = (\partial I \times [0, T]) \cup (I \times \{0\})$$

that reads

$$\|u_x\|_{L^\infty(I_T)} \leq cT^{\frac{p-3}{2p}} \|u\|_Y, \tag{3.2}$$

where $c = c(p) > 0$ is a constant depending only on p.

Recall that [13, Section 1], for $l > 0$, the parabolic Hölder space $C^{l,l/2}(\overline{I}_T)$ is the Banach space of functions v that are continuous in $\overline{I}_T$, together with all derivatives of the form $D_t^r D_x^s v$ for $2r+s < l$, and have a finite norm $|v|_{I_T}^{(l)} = \langle v\rangle_{I_T}^{(l)} + \sum_{j=0}^{[l]} \langle v\rangle_{I_T}^{(j)}$, where

$$\langle v\rangle_{I_T}^{(0)} = |v|_{I_T}^{(0)} = \|v\|_{L^\infty(I_T)}, \quad \langle v\rangle_{I_T}^{(j)} = \sum_{2r+s=j} |D_t^r D_x^s v|_{I_T}^{(0)},$$
$$\langle v\rangle_{I_T}^{(l)} = \langle v\rangle_{x,I_T}^{(l)} + \langle v\rangle_{t,I_T}^{(l/2)}, \quad \langle v\rangle_{x,I_T}^{(l)} = \sum_{2r+s=[l]} \langle D_t^r D_x^s v\rangle_{x,I_T}^{(l-[l])},$$
$$\langle v\rangle_{t,I_T}^{(l/2)} = \sum_{0<l-2r-s<2} \langle D_t^r D_x^s v\rangle_{t,I_T}^{(\frac{l-2r-s}{2})},$$

with

$$\langle v\rangle_{x,I_T}^{(\alpha)} = \inf\{c;\ |v(x,t) - v(x',t)| \le c|x-x'|^\alpha,\ (x,t),(x',t) \in \overline{I}_T\}, \quad 0 < \alpha < 1,$$
$$\langle v\rangle_{t,I_T}^{(\alpha)} = \inf\{c;\ |v(x,t) - v(x,t')| \le c|t-t'|^\alpha,\ (x,t),(x,t') \in \overline{I}_T\}, \quad 0 < \alpha < 1.$$

Another very useful inequality in our study is a Sobolev estimate for parabolic equations (see [7, Lemma 2.3]). To state this estimate, we consider solutions $u \in Y$, $u = 0$ on $\partial^p(I_T)$, of

$$u_t = \varepsilon u_{xx} + f, \quad f \in L^p(I_T) \text{ called the source term.} \tag{3.3}$$

Then we have

$$\frac{\|u\|_{L^p(I_T)}}{T} + \frac{\|u_x\|_{L^p(I_T)}}{\sqrt{T}} + \|u_{xx}\|_{L^p(I_T)} + \|u_t\|_{L^p(I_T)} \le c\|f\|_{L^p(I_T)}, \tag{3.4}$$

where $c = c(\varepsilon, p) > 0$ is a constant depending only on p and ε. Now, we may state the main proposition of this section.

Proposition 3.1 (Short time existence and uniqueness)**.** *Let $p > 3$ and let*

$$\rho^0, \kappa^0 \in C^\infty(\overline{I}),$$

be two functions such that $\rho^0(0) = \rho^0(1) = \kappa^0(0) = 0$ and $\kappa^0(1) = 1$. Suppose furthermore that

$$\begin{gathered} \kappa_x^0 \ge \gamma_0 \quad \text{on} \quad I, \\ \|D_x^s \rho^0, D_x^s \kappa^0\|_{L^\infty(I)} \le M_0, \quad s = 1, 2, \end{gathered} \tag{3.5}$$

with $\gamma_0 > 0$ and $M_0 > 0$. Then there exists a unique solution $(\rho, \kappa) \in Y^2$ of (1.1), (1.2) and (1.3) where

$$T = T(M_0, \gamma_0, \varepsilon, b, p), \quad 0 < T < 1. \tag{3.6}$$

Moreover, this solution satisfies

$$\begin{gathered} \kappa_x \ge \frac{\gamma_0}{2} \quad \text{on } \overline{I}_T, \\ |\rho_x| \le 2M_0 \quad \text{on } \overline{I}_T. \end{gathered} \tag{3.7}$$

Proof. Throughout the proof, and in various estimates we suppose that $0 < T < 1$. This is in no way a problem since we have to choose T small enough to construct our solution. The proof uses a fixed point argument on a closed subspace of Y.

Looking at (3.7), we can artificially modify (1.1) using suitable truncations. To simplify our computations, we set $\mathcal{I} := \mathcal{I}_{2M_0}$, and consider

$$\begin{gathered} \kappa_t = \varepsilon\kappa_{xx} + \frac{\rho_{xx}\mathcal{I}(\rho_x)}{(\gamma_0/2) + (\kappa_x - \gamma_0/2)^+} + \sigma\rho_x \quad \text{on } I_T \\ \rho_t = (1+\varepsilon)\rho_{xx} + \sigma\kappa_x \quad \text{on } I_T, \end{gathered} \tag{3.8}$$

with the same initial and boundary conditions (1.2) and (1.3). It is worth noticing that when (3.7) is satisfied then (3.8) coincides with (1.1). On the other hand, condition (3.7) also suggests that we consider functions $u \in Y$ of bounded gradients, i.e. $\|u_x\|_{L^p(I_T)} \le \lambda$ where $\lambda > 0$ is a fixed sufficiently large constant. For this reason, define the spaces Y_λ^ρ and Y_λ^κ as follows:

$$\begin{gathered} Y_\lambda^\rho = \{u \in Y;\ \|u_x\|_{L^p(I_T)} \le \lambda,\ u = \rho^0 \ \text{ on } \ \partial^p(I_T)\}, \\ Y_\lambda^\kappa = \{u \in Y;\ \|u_x\|_{L^p(I_T)} \le \lambda,\ u = \kappa^0 \ \text{ on } \ \partial^p(I_T)\}. \end{gathered}$$

Define the application: $\Psi : Y_\lambda^\rho \times Y_\lambda^\kappa \longmapsto Y_\lambda^\rho \times Y_\lambda^\kappa$ by

$$(\hat\rho, \hat\kappa) \longmapsto \Psi(\hat\rho, \hat\kappa) = (\rho, \kappa)$$

where (ρ, κ) is the solution of

$$\begin{gathered} \kappa_t = \varepsilon\kappa_{xx} + \frac{\rho_{xx}\mathcal{I}(\hat\rho_x)}{(\gamma_0/2) + (\hat\kappa_x - \gamma_0/2)^+} + \sigma\hat\rho_x \quad \text{on } I_T \\ \rho_t = (1+\varepsilon)\rho_{xx} + \sigma\hat\kappa_x \quad \text{on } I_T. \end{gathered} \tag{3.9}$$

The application Ψ is well defined: The existence and uniqueness of $(\rho, \kappa) \in Y_\lambda^\rho \times Y_\lambda^\kappa$ solution of (3.9) is obtained in two steps. In a first step, while having the initial and boundary conditions (1.2) and (1.3), we find a solution $\rho \in Y$ of the second equation of (3.9), then we plug it into the first equation to get a solution $\kappa \in Y$. Here, the existence and uniqueness of both solutions are guaranteed by [13, Theorem 9.1]. It is worth mentioning that [13, Theorem 9.1] requires a compatibility condition of order 0 on the initial and boundary data. Those conditions are satisfied by our boundary assumptions on ρ^0 and κ^0 (see Proposition 3.1).

In a second step, we use (3.4), basically on the functions:

$$\bar\rho = \rho - \rho^0 \quad \text{and} \quad \bar\kappa = \kappa - \kappa^0,$$

together with (3.5), to gain the L^p bounds on ρ_x and κ_x if we choose T small enough. The above steps ensure that the application Ψ is well defined at least for sufficiently small time.

The application Ψ is a contraction:
We now show that Ψ is a contraction. In fact, let $\Psi(\hat\rho, \hat\kappa) = (\rho, \kappa)$ and $\Psi(\hat\rho', \hat\kappa') = (\rho', \kappa')$. The couple $(\rho - \rho', \kappa - \kappa')$ is the solution of the system

$$\begin{gathered} (\kappa - \kappa')_t = \varepsilon(\kappa - \kappa')_{xx} + \mathcal{F}_1 \quad \text{on } I_T \\ (\rho - \rho')_t = (1+\varepsilon)(\rho - \rho')_{xx} + \mathcal{F}_2 \quad \text{on } I_T \end{gathered} \tag{3.10}$$

where

$$\begin{gathered} \mathcal{F}_1 = \frac{\rho_{xx}\mathcal{I}(\hat\rho_x)}{(\gamma_0/2) + (\hat\kappa_x - \gamma_0/2)^+} - \frac{\rho'_{xx}\mathcal{I}(\hat\rho'_x)}{(\gamma_0/2) + (\hat\kappa'_x - \gamma_0/2)^+} + \sigma(\hat\rho_x - \hat\rho'_x), \\ \mathcal{F}_2 = \sigma(\hat\kappa_x - \hat\kappa'_x), \end{gathered}$$

with

$$(\rho - \rho', \kappa - \kappa') = (0, 0) \quad \text{on } \partial^p(I_T).$$

In the remaining part of the proof, the variable $c > 0$ is a generic constant that depends on all constants in Proposition 3.1 but independent of T. Using (3.4) and (3.10), we deduce that

$$\begin{aligned} \|\kappa - \kappa'\|_Y &\le c\|\mathcal{F}_1\|_{L^p(I_T)}, \\ \|\rho - \rho'\|_Y &\le c\|\mathcal{F}_2\|_{L^p(I_T)}. \end{aligned} \tag{3.11}$$

Estimate of $\|\rho - \rho'\|_Y$. We note that if $v = \hat\kappa - \hat\kappa'$, then v satisfies

$$\begin{gathered} v_t = v_{xx} + f \quad \text{on } I_T, \\ v = 0 \quad \text{on } \partial^p(I_T), \end{gathered}$$

with $f = (\hat\kappa - \hat\kappa')_t - (\hat\kappa - \hat\kappa')_{xx}$. Then, by (3.4), we obtain

$$\|v_x\|_{L^p(I_T)} \le c\sqrt{T}\big\|(\hat\kappa - \hat\kappa')_t - (\hat\kappa - \hat\kappa')_{xx}\big\|_{L^p(I_T)} \le c\sqrt{T}\|\hat\kappa - \hat\kappa'\|_Y;$$

therefore,

$$\|\mathcal{F}_2\|_{L^p(I_T)} \le cb\sqrt{T}\|\hat\kappa - \hat\kappa'\|_Y. \tag{3.12}$$

Using the second equation of (3.11) and (3.12), we finally obtain

$$\|\rho - \rho'\|_Y \le cb\sqrt{T}\|\hat\kappa - \hat\kappa'\|_Y. \tag{3.13}$$

Estimate of $\|\kappa - \kappa'\|_Y$. We write

$$\mathcal{F}_1 = \mathcal{A}_1 + \mathcal{A}_2 + \mathcal{A}_3 + \mathcal{A}_4,$$

with

$$\begin{gathered} \mathcal{A}_1 = \frac{\mathcal{I}(\hat\rho_x)}{(\gamma_0/2) + (\hat\kappa_x - \gamma_0/2)^+}\left(\rho_{xx} - \rho'_{xx}\right), \\ \mathcal{A}_2 = \frac{\rho'_{xx}(\mathcal{I}(\hat\rho_x) - \mathcal{I}(\hat\rho'_x))}{(\gamma_0/2) + (\hat\kappa_x - \gamma_0/2)^+}, \\ \mathcal{A}_3 = \rho'_{xx}\mathcal{I}(\hat\rho'_x)\Big(\frac{1}{(\gamma_0/2) + (\hat\kappa_x - \gamma_0/2)^+} - \frac{1}{(\gamma_0/2) + (\hat\kappa'_x - \gamma_0/2)^+}\Big), \\ \mathcal{A}_4 = \sigma(\hat\rho_x - \hat\rho'_x). \end{gathered} \tag{3.14}$$

We estimate the L^p norms of $\mathcal{A}_i, i = 1, 2, 3, 4$. First remark that the coefficient of $(\rho_{xx} - \rho'_{xx})$ in $\mathcal{A}_1$ is bounded, hence by (3.13) we deduce that

$$\|\mathcal{A}_1\|_{L^p(I_T)} \le cb\sqrt{T}\|\hat\kappa - \hat\kappa'\|_Y. \tag{3.15}$$

For the term $\mathcal{A}_2$, we proceed as follows. We apply the L^∞ control of the spatial derivative (3.2) to the function $\hat\rho - \hat\rho'$, we obtain

$$\|(\hat\rho - \hat\rho')_x\|_{L^\infty(I_T)} \le cT^{\frac{p-3}{2p}}\|\hat\rho - \hat\rho'\|_Y. \tag{3.16}$$

For the term ρ'_{xx} in $\mathcal{A}_2$, we first remark that if we let $\bar\rho' = \rho' - \rho^0$, this function satisfies (see the second equation of (3.9)):

$$\begin{gathered} \bar\rho'_t = (1+\varepsilon)\bar\rho'_{xx} + (1+\varepsilon)\rho^0_{xx} - \sigma\hat\kappa'_x \quad \text{on } I_T, \\ \bar\rho' = 0 \quad \text{on } \partial^p I_T, \end{gathered}$$

and hence, by (3.4) and (3.5), we deduce that

$$\|\rho'_{xx}\|_{L^p(I_T)} \le c(M_0 + \lambda). \tag{3.17}$$

Knowing that $\mathcal{I}$ is Lipschitz, we use (3.16) and (3.17) to obtain

$$\|\mathcal{A}_2\|_{L^p(I_T)} \le c\frac{(M_0 + \lambda)}{\gamma_0}T^{\frac{p-3}{2p}}\|\hat\rho - \hat\rho'\|_Y. \tag{3.18}$$

For $\mathcal{A}_3$, we use the same arguments as for $\mathcal{A}_2$, and we are lead to:

$$\|\mathcal{A}_3\|_{L^p(I_T)} \leq c\frac{M_0(M_0+\lambda)}{\gamma_0^2}T^{\frac{p-3}{2p}}\|\hat{\kappa}-\hat{\kappa}'\|_Y. \tag{3.19}$$

Here we have used that the function $x \to \frac{1}{(\gamma_0/2)+(x-\gamma_0/2)^+}$ is Lipschitz continuous over $\mathbb{R}$. Finally,

$$\|\mathcal{A}_4\|_{L^p(I_T)} \leq cb\sqrt{T}\|\hat{\rho}-\hat{\rho}'\|_Y. \tag{3.20}$$

Using (3.15), (3.18), (3.19) and (3.20), we finally obtain

$$\|\mathcal{F}_1\|_{L^p(I_T)} \leq cT^{\frac{p-3}{2p}}\left(\|\hat{\kappa}-\hat{\kappa}'\|_Y + \|\hat{\rho}-\hat{\rho}'\|_Y\right), \tag{3.21}$$

hence, by (3.21), and the first equation of (3.11), we arrive at

$$\|\kappa-\kappa'\|_Y \leq cT^{\frac{p-3}{2p}}\left(\|\hat{\kappa}-\hat{\kappa}'\|_Y + \|\hat{\rho}-\hat{\rho}'\|_Y\right). \tag{3.22}$$

Conclusion: Equations (3.13) and (3.22) show that, for $T > 0$ sufficiently small, the application Ψ is a contraction, and eventually it has a unique fixed point (ρ,κ) solution of system (3.8), (1.2) and (1.3). Lastly, to get rid of the artificial truncations ($\mathcal{I}$ and the positive part in the denominator in (3.8)), and to show (3.7), we use the embedding (3.1) and the initial conditions (3.5). This is again with the possibility of reducing the time T. □

Lemma 3.2. *If κ is the solution obtained in Proposition 3.1, then*

$$\sigma\kappa_x \in C^{\alpha,\alpha/2}(\bar{I}_T).$$

Proof. The embedding (3.1) infers that $\kappa_x \in C^{\alpha,\alpha/2}(\bar{I}_T)$ and hence $\sigma\kappa_x \in C(\bar{I}_T)$. We then compute, for $x, x' \in [0,1]$ and $t, t' \in [0,T]$, $0 < T < 1$:

$$\begin{aligned}
&|\sigma(x,t)\kappa_x(x,t) - \sigma(x',t)\kappa_x(x',t)| \\
&\leq b\langle\kappa_x\rangle_{x,I_T}^{(\alpha)}|x-x'|^\alpha + \|\kappa_x\|_{L^\infty(I_T)}\|\sigma_x\|_{L^\infty(I_T)}|x-x'| \\
&\leq c|x-x'|^\alpha
\end{aligned}$$

and

$$\begin{aligned}
&|\sigma(x,t)\kappa_x(x,t) - \sigma(x,t')\kappa_x(x,t')| \\
&\leq b\langle\kappa_x\rangle_{t,I_T}^{(\alpha/2)}|t-t'|^{\alpha/2} + \|\kappa_x\|_{L^\infty(I_T)}\|\sigma_t\|_{L^\infty(I_T)}|t-t'| \\
&\leq c|t-t'|^{\alpha/2}.
\end{aligned}$$

Therefore, $\sigma\kappa_x \in C^{\alpha,\alpha/2}(\bar{I}_T)$. □

Remark 3.3. Lemma 3.2 suggests that we may have a better regularity for the solution ρ obtained in Proposition 3.1, which in turn, can also lead to a better regularity on κ due to the coupling in (1.1). This is better illustrated by the next proposition.

Proposition 3.4 (Regularity by a bootstrap argument)**.** *Under the same hypothesis of Proposition 3.1 and if, in addition, the functions ρ^0, κ^0 satisfy the condition*

$$\begin{aligned}
(1+\varepsilon)\rho_{xx}^0 + \sigma(\cdot,0)\kappa_x^0 = 0 \quad \text{on } \partial I, \\
(1+\varepsilon)\kappa_{xx}^0 + \sigma(\cdot,0)\rho_x^0 = 0 \quad \text{on } \partial I,
\end{aligned} \tag{3.23}$$

then the solution (ρ,κ) obtained in Proposition 3.1 satisfies:

$$\rho, \kappa \in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar{I}_T) \cap C^\infty(\bar{I}\times(0,T)). \tag{3.24}$$

Remark 3.5 (Comments on condition (3.23))**.** The assumption (3.23) on the initial data, together with the constant boundary values, define a compatibility condition of order 1. In other words, we obtain

$$\begin{gathered} 0=\kappa_t(x,0)=\varepsilon\kappa_{xx}(x,0)+\frac{\rho_x(x,0)\rho_{xx}(x,0)}{\kappa_x(x,0)}+\sigma(x,0)\rho_x(x,0)\quad \text{for } x\in\partial I,\\ 0=\rho_t(x,0)=(1+\varepsilon)\rho_{xx}(x,0)+\sigma(x,0)\kappa_x(x,0),\quad \text{for } x\in\partial I. \end{gathered}$$

This boundary condition let us conclude that (see [13, Theorem 5.2]), if the source terms (see equation (3.3)) in (1.1) is of class $C^{\beta,\beta/2}(\overline{I}_T)$ for $0<\beta<2$, then the solution (ρ,κ) will be of class $C^{2+\beta,\frac{2+\beta}{2}}(\overline{I}_T)$.

Proof of Proposition 3.4. The proof follows the idea of Remark 3.3. In fact, Lemma 3.2 show that the source term $\sigma\kappa_x$ of the second equation of (1.1) is of class $C^{\alpha,\alpha/2}(\overline{I}_T)$. Remark 3.5 with $0<\beta=\alpha<2$ and the compatibility conditions (3.23), ensures that $\rho\in C^{\alpha+2,\frac{\alpha+2}{2}}(\overline{I}_T)$, which, by its turn (thanks to similar computations as those done in Lemma 3.2), shows that the the source term $\frac{\rho_x\rho_{xx}}{\kappa_x}+\sigma\rho_x$ of the first equation of (1.1) is of class $C^{\alpha,\alpha/2}(\overline{I}_T)$. Finally, we also deduce that $\kappa\in C^{\alpha+2,\frac{\alpha+2}{2}}(\overline{I}_T)$ and then $\kappa_x\in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{I}_T)$.

We now repeat exactly the same ideas but with this new regularity on κ_x, taking into consideration that still $0<\beta=\alpha+1<2$, and hence Remark 3.5 is still applicable. In fact, since $\kappa_x\in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{I}_T)$ then, by similar arguments as in Lemma 3.2, we obtain $\sigma\kappa_x\in C^{1+\alpha,\frac{1+\alpha}{2}}(\overline{I}_T)$ and the whole process may be repeated. Therefore, we are finally lead to $\rho,\kappa\in C^{3+\alpha,\frac{3+\alpha}{2}}(\overline{I}_T)$. Note that this is the maximum Hölder regularity that could be obtained up to the boundary since the compatibility condition (3.23) is not sufficient when $\beta>2$. Indeed, higher order compatibility assumptions are required if we to achieve more regularity on the boundary.

To get the interior C^∞ regularity, we need to carefully overcome the problem of compatibility at $t=0$, $x\in\partial I$. Indeed, let $0<\delta<T$, and define any test function $\phi_\delta\in C^\infty[0,T]$ by

$$\phi_\delta(t)=\begin{cases}0 & \text{if } 0\le t\le\frac{\delta}{3},\\ \phi_\delta(t)\in(0,1) & \text{if } \frac{\delta}{3}\le t\le\frac{2\delta}{3},\\ 1 & \text{if } \frac{2\delta}{3}\le t\le T.\end{cases} \tag{3.25}$$

By introducing

$$\bar\rho=\rho\phi_\delta\quad\text{and}\quad\bar\kappa=\kappa\phi_\delta, \tag{3.26}$$

we can easily see that $(\bar\rho,\bar\kappa)$ satisfy a parabolic system where higher order compatibility conditions on the initial data are satisfied. Hence, the parabolic Hölder regularity can be infinitely applied for $\bar\rho$ and $\bar\kappa$. Accordingly

$$(\bar\rho,\bar\kappa)\in C^\infty(\overline{I}_T).$$

From (3.25) and (3.26) we deduce that

$$(\rho,\kappa)=(\bar\rho,\bar\kappa)\quad\text{in } [\frac{2\delta}{3},T],\quad\forall\, 0<\delta<T.$$

Thus $\rho,\kappa\in C^\infty(\bar I\times(0,T))$ and the proof follows. ☐

Remark 3.6. It is worth noticing that the regularity (3.24) of the solution (ρ,κ) (obtained by Proposition 3.1) is sufficient to make use of the comparison principle (Proposition 2.1 in Section 2).

4. Exponential estimates and long time existence

In this section, we give some exponential bounds of the solution given by Proposition 3.1, and having the regularity shown by Proposition 3.4. In this section, the generic constants $c > 0$ and $c(T) > 0$ may vary from line to line.

Proposition 4.1 (Exponential bound in time for ρ_x and κ_x)**.** *Let*

$$\rho, \kappa \in C^{3+\alpha, \frac{3+\alpha}{2}}(\overline{I}_\infty) \cap C^\infty(\bar{I} \times (0,\infty)),$$

be a solution of (1.1)*,* (1.2) *and* (1.3)*, with* $\rho^0(0) = \rho^0(1) = \kappa^0(0) = 0$ *and* $\kappa^0(1) = 1$*. Suppose furthermore that* (1.5) *is satisfied. Then there exists a constant* $c = c(\rho^0, \kappa^0, \varepsilon, b, p) > 0$ *such that*

$$\begin{aligned} \|\rho_x(\cdot,t)\|_{L^\infty(I)} &\le ce^{ct} \quad \textit{for all } t \ge 0, \\ \|\kappa_x(\cdot,t)\|_{L^\infty(I)} &\le ce^{ct} \quad \textit{for all } t \ge 0. \end{aligned} \tag{4.1}$$

Proof. The idea behind our exponential estimates is based upon linearising system (1.1). Indeed, condition (1.5) implies the boundedness of the coefficient $\frac{\rho_x}{\kappa_x}$ of the term ρ_{xx}, whereas (1.4) implies the boundedness of the other coefficients. We begin by estimating $\|\kappa_x\|_{L^p(I_T)}$. By applying (3.4) to the function $\kappa - \kappa'$ where κ' satisfies

$$\begin{aligned} \kappa'_t &= \kappa'_{xx} \quad \text{on } I_T \\ \kappa' &= \kappa \quad \text{on } \partial^p I_T, \end{aligned} \tag{4.2}$$

we obtain

$$\|\kappa_x\|_{L^p(I_T)} \le c(T)\left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + 1\right) + cb\sqrt{T}\|\rho\|_{W_p^{2,1}(I_T)}. \tag{4.3}$$

Here we have used the fact that $|\frac{\rho_x}{\kappa_x}| \le 1$, and the solvability of parabolic equations in Sobolev spaces (see for instance [13, Theorem 9.1]). The same estimate now applied to $\rho - \rho'$ (where ρ' satisfies an inequality similar to (4.2)) gives

$$\|\rho\|_{W_p^{2,1}(I_T)} \le c(T)\|\rho^0\|_{W_p^{2-2/p}(I)} + cb\|\kappa_x\|_{L^p(I_T)}. \tag{4.4}$$

Combining (4.3) and (4.4), we conclude that for

$$T^* = \frac{1}{2c^4b^4},$$

we have

$$\|\kappa_x\|_{L^p(I_{T^*})} \le c\Big(\|\kappa^0\|_{W_p^{2-2/p}(I)} + \|\rho^0\|_{W_p^{2-2/p}(I)} + 1\Big), \quad c = c(T^*) > 0.$$

Having the special coupling of system (1.1), together with the above estimate, we can deduce that

$$\|\rho\|_{W_p^{2,1}(I_{T^*})}, \|\kappa\|_{W_p^{2,1}(I_{T^*})} \le c\left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + \|\rho^0\|_{W_p^{2-2/p}(I)} + 1\right),$$

where $c = c(T^*) > 0$; hence, by (3.1), we obtain

$$|\rho|_{I_{T^*}}^{(1+\alpha)}, |\kappa|_{I_{T^*}}^{(1+\alpha)} \le c\left(\|\kappa^0\|_{W_p^{2-2/p}(I)} + \|\rho^0\|_{W_p^{2-2/p}(I)} + 1\right), \quad c = c(T^*) > 0.$$

The trace theorem for parabolic Sobolev spaces [13, Lemma 3.4] give

$$\|u(\cdot, t+T^*)\|_{W_p^{2-2/p}(I)} \le c\|u\|_{W_p^{2,1}(I\times(t,t+T^*))}, \quad c = c(T^*) > 0;$$

therefore, we can iterate backwards in time to finally obtain our result. ∎

Proposition 4.2. *Under the same hypothesis of Proposition 4.1, there exists a constant* $c = c(\rho^0, \kappa^0, \varepsilon, b, p) > 0$ *such that*

$$\begin{aligned} \|\rho_{xx}(\cdot,t)\|_{L^\infty(I)} &\le ce^{ct} \quad \text{for all } t \ge 0, \\ \|\kappa_{xx}(\cdot,t)\|_{L^\infty(I)} &\le ce^{ct} \quad \text{for all } t \ge 0. \end{aligned} \tag{4.5}$$

The proof of the above porposition can be found in [7, Propositions 5 and 6].

Proposition 4.3. *Under the same hypothesis of Proposition 4.1, there exists a constant* $c = c(\rho^0, \kappa^0, \varepsilon, b, p) > 0$ *such that*

$$\|\kappa_x(\cdot,t)\|_{L^\infty(I)} \ge ce^{-ct} \quad \text{for all } 0 \le t \le T. \tag{4.6}$$

The proof of the above proposition Follows immediately from (1.7) and (2.2). Now we are ready to show the main result of this paper, namely Theorem 1.1.

Proof of Theorem 1.1. Define the set $\mathcal{B}$ by

$$\begin{gathered} \mathcal{B} = Big\{T > 0;\ \exists!\ \text{ solution } (\rho,\kappa) \in C^{3+\alpha,\frac{3+\alpha}{2}}(\bar{I}_T) \text{ of} \\ (1.1), (1.2) \text{ and } (1.3), \text{ satisfying } (1.5) \end{gathered}$$

This set is non empty by the short time existence result (Theorem 3.1). Set

$$T_\infty = \sup \mathcal{B}.$$

We claim that $T_\infty = \infty$. Assume, by contradiction that $T_\infty < \infty$. In this case, let $\delta > 0$ be an arbitrary small positive constant, and apply the short time existence result (Theorem 3.1) with $T_0 = T_\infty - \delta$. Indeed, by the exponential bounds (4.1), (4.5) and (4.6), we deduce that the time of existence T given by (3.6) is in fact independent of δ. Hence, choosing δ small enough, we obtain $T_0 + T \in \mathcal{B}$ with $T_0 + T > T_\infty$ and hence a contradiction. □

References

[1] J. A. Carrillo, J. L. Vázquez; *Fine asymptotics for fast diffusion equations*, Comm. Partial Differential Equations, **28** (2003), 1023-1056.

[2] C. Y. Chan, H. G. Kaper; *Quenching for semilinear singular parabolic problems*, SIAM J. Math. Anal., **20** (1989), 558-566.

[3] E. Chasseigne, J. L. Vázquez; *Theory of extended solutions for fast-diffusion equations in optimal class of data*, Arch. Ration. Mech. Anal., **164** (2002), 133–187.

[4] E. DiBenedetto; *Degenerate parabolic equations*, Universitext, Springer-Verlag, New York, 1993.

[5] I. Groma, F. F. Czikor, M. Zaiser; *Spatial correlations and higher-order gradient terms in a continuum description of dislocation dynamics*, Acta Mater, **51** (2003), 1271-1281.

[6] J. R. Hirth, L. Lothe; *Theory of dislocations*, second edition, Kreiger publishing company, Florida 32950, 1982.

[7] H. Ibrahim, M. Jazar, R. Monneau; *Dynamics of dislocation densities in a bounded channel. Part I: smooth solutions to a singular coupled parabolic system*, Commun. Pure Appl. Anal., **9** (2009), no. 3, 703-719.

[8] H. Ibrahim, M. Jazar and R. Monneau; *Dynamics of dislocation densities in a bounded channel. Part II: existence of weak solutions to a singular Hamilton-Jacobi/parabolic strongly coupled system*, Comm. Partial Differential Equations, **34** (2009), no. 8, 889-917.

[9] H. Ibrahim, M. Jazar, R. Monneau; *Global existence of solutions to a singular parabolic /Hamilton-Jacobi coupled system with Dirichlet conditions*, C. R. Math. Acad. Sci. Paris, **346** (2008), 945-950.

[10] A. Maugeri; *A boundary value problem for a class of singular parabolic equations*, Boll. Un. Mat. Ital. B (5), **17** (1980), 325–339.

[11] F. Merle, H. Zaag; *Optimal estimates for blowup rate and behaviour for nonlinear heat equations*, Comm. Pure Appl. Math., **51**(1998), 139-196.

[12] F. R. N. Nabarro; *Theory of crystal dislocations*, Oxford, Clarendon Press, 1969.

[13] O. A. Ladyzenskaja, V. A. Solonnikov, N. N. Uralceva; *Linear and quasilinear equations of parabolic type*, Translated from the Russian by S. Smith, Translations of Mathematical Monographs, Vol. 23, American Mathematical Society, Providence, RI, 1967.

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