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SOLVABILITY OF MULTIPOINT DIFFERENTIAL OPERATORS OF FIRST ORDER

ZAMEDDIN I. ISMAILOV, PEMBE IPEK

In memory of acad. M. G. Gasymov (1939-2008)

Abstract. Based on methods of operator theory, we describe all boundedly solvable extensions of the minimal operator generated by a linear multipoint functional differential-operator expression of first order, in a direct sum of Hilbert space of vector-functions. We also study the structure of spectrum of these extensions.

1. Introduction

The general theory of extension of densely defined linear operator in Hilbert spaces was started by von Neumann with his important work [10] in 1929. Later in 1949 and 1952 Vishik in [15, 16] studied the boundedly (compact, regular and normal) invertible extensions of any unbounded linear densely defined operator in Hilbert spaces Generalization of these results to the nonlinear and complete additive Hausdorff topological spaces in abstract terms have been done by Kokebaev, Otelbaev and Synybekov in [7, 12]. By Dezin [2] another approach to the description of regular extensions for some classes of linear differential operators in Hilbert spaces of vector-functions at finite interval has been offered.

On other hand the role of the two point and multipoint theory of functional differential equations in our lives is indisputable. The general theory of delay differential equations is presented in many books (for example [4, 14]). Applications of this theory can be found in economy, biology, control theory, electrodinamics, chemistry, ecology, epidemiology, tumor growth, neural networks and etc. (see [3, 11]).

In addition note that oscillation and boundness properties of solutions of different classes of dynamic equations have been investigated in the book by Agarwal, Bohner and Li [1].

Let us remember that an operator $S : D(S) \subset H \to H$ on Hilbert spaces is called boundedly invertible, if S is one-to-one, $SD(S) = H$ and $S^{-1} \in L(H)$.

The main goal of this work is to describe all boundedly solvable extensions of the minimal operator generated by linear multipoint functional differential-operator expression for first order in the direct sum of Hilbert spaces of vector-functions at

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finite intervals and investigate the structure of spectrum of these extensions. In the second section all boundedly solvable extensions of the above mentioned minimal operator are described. Structure of spectrum of these extensions is studied in third section Finally, the obtained results are illustrated by applications.

2. Description of Boundedly Solvable Extensions

Throughout this work H_n is a separable Hilbert space, $\mathcal{H}_n = L^2(H_n, \Delta_n)$, $\Delta_n = (a_n, b_n) \subset \mathbb{R}$ for each $n \geq 1$ with property $-\infty < \inf_{n\geq 1} a_n < \sup_{n\geq 1} b_n < \infty$, $\inf_{n\geq 1} |\Delta_n| > 0$ and $\mathcal{H} = \oplus_{n=1}^{\infty} \mathcal{H}_n$.

We consider the linear multipoint functional differential-operator expression of first order in $\mathcal{H}$ of the form

$$l(u) = (l_n(u_n)),\ u = (u_n),$$

where:

1. $l_n(u_n) = u_n'(t) + A_n(t)u(\alpha_n(t))$, $n \geq 1$;
2. operator-function $A_n(\cdot) : [a_n, b_n] \to L(H_n)$, $n \geq 1$ is continuous on the uniformly operator topology and $\sup_{n\geq 1} \sup_{t\in\Delta_n} \|A_n(t)\| < \infty$;
3. For any $n \geq 1$, $\alpha_n : [a_n, b_n] \to [a_n, b_n]$ is invertible and $\alpha_n, (\alpha_n^{-1})' \in C[a_n, b_n]$, also $\sup_{n\geq 1}(\|(\alpha_n^{-1}(t))'\|_\infty)^{1/2} < \infty$

By standard way the minimal L_{n0} and maximal L_n operators corresponding to the differential expression $l_n(\cdot)$ in $\mathcal{H}_n$ can be defined for any $n \geq 1$. It is clear that for every $n \geq 1$ domains of minimal L_{n0} and maximal L_n operators in $\mathcal{H}_n$ are in the forms

$$D(L_{n0}) = \overset{o}{W}{}_2^1 (H_n, \Delta_n), \quad \text{and} \quad D(L_n) = W_2^1(H_n, \Delta_n),$$

respectively. For any scalar function $\varphi : [a_n, b_n] \to [a_n, b_n]$ and $n \geq 1$ now define an operator P_φ in $\mathcal{H}_n$ in the form

$$P_\varphi u_n(t) = u_n(\varphi(t)), \quad u_n \in \mathcal{H}_n\,.$$

If a function $\varphi \in C^1[a_n, b_n]$ and $\varphi'(t) > 0$ (< 0) for $t \in [a_n, b_n]$, then for any $u_n \in \mathcal{H}_n$,

$$\begin{aligned}
\|P_\varphi u_n\|_{\mathcal{H}_n}^2 &= \int_{a_n}^{b_n} \|u_n(\varphi(t))\|_{H_n}^2 \, dt \\
&= \int_{\varphi(a_n)}^{\varphi(b_n)} \|u_n(\varphi(x))\|_{H_n}^2 (\varphi^{-1})'(x) dx \\
&\leq \Big| \int_{\varphi(a_n)}^{\varphi(b_n)} \|u_n(x)\|_{H_n}^2 |(\varphi^{-1})'(x)| dx \Big| \\
&\leq \|(\varphi^{-1})'\|_\infty \int_{a_n}^{b_n} \|u_n(x)\|_{H_n}^2 dx \\
&= \|(\varphi^{-1})'\|_\infty \|u_n\|_{\mathcal{H}_n}^2
\end{aligned}$$

Consequently, for any strictly monotone function $\varphi \in C^1[a_n, b_n]$, the operator P_φ belongs to $L(\mathcal{H}_n)$ and $\|P_\varphi\| \leq \sqrt{\|(\varphi^{-1})'\|_\infty}$. In actually, the expression $l(\cdot)$ in $\mathcal{H}$ can be written in the form

$$l(u) = u'(t) + A_\alpha(t)u(t), \tag{2.1}$$

where $u = (u_n)$,

$$A(t) = \begin{pmatrix} A_1(t) & & & & \\ & A_2(t) & & 0 & \\ & & \ddots & & \\ & 0 & & A_n(t) & \\ & & & & \ddots \end{pmatrix}, \quad P_\alpha = \begin{pmatrix} P_{\alpha_1} & & & & \\ & P_{\alpha_2} & & 0 & \\ & & \ddots & & \\ & 0 & & P_{\alpha_n} & \\ & & & & \ddots \end{pmatrix}$$

and $A_\alpha(t) = A(t)P_\alpha,\ A_\alpha \in L(\mathcal{H})$.

The operators $L_0(M_0)$ and $L(M)$ be a minimal and maximal operators corresponding to (2.1) (with $m(\cdot) = \frac{d}{dt}$) in $\mathcal{H}$, respectively. Let us define

$$L_0 = M_0 + A_\alpha(t), \quad L = M + A_\alpha(t),$$
$$L_0 : D(L_0) \subset \mathcal{H} \to \mathcal{H}, \quad L : D(L) \subset \mathcal{H} \to \mathcal{H},$$
$$D(L_0) = \{(u_n) \in \mathcal{H} : u_n \in \overset{o}{W}{}_2^1 (H_n, \Delta_n),\ n \ge 1,\ \sum_{n=1}^\infty \|L_{n0}u_n\|^2_{\mathcal{H}_n} < \infty\},$$
$$D(L) = \{(u_n) \in \mathcal{H} : u_n \in W_2^1(H_n, \Delta_n),\ n \ge 1,\ \sum_{n=1}^\infty \|L_n u_n\|^2_{\mathcal{H}_n} < \infty\},$$

The main goal in this section is to describe all boundedly solvable extensions of the minimal operator L_0 in $\mathcal{H}$ in terms in the boundary values. Before that, we prove the validity the following assertion.

Lemma 2.1. *The kernel and image sets of L_0 in $\mathcal{H}$ satisfy* $\ker L_0 = \{0\}$ *and* $\overline{\operatorname{Im}(L_0)} \ne \mathcal{H}$.

Proof. Firstly, we prove that for any $n \ge 1$ $\ker L_{n0} = \{0\}$. Consider the boundary values problems

$$u_n'(t) + A_n(t)P_{\alpha_n} u_n(t) = 0,$$
$$u_n(a_n) = u_n(b_n) = 0, \quad n \ge 1$$

The general solution of these differential equations are

$$u_n(t) = \exp\Big(-\int_{a_n}^t A_n(s)P_{\alpha_n}\,ds\Big) f_n, \quad n \ge 1$$

Then from boundary value conditions we have $f_n = 0$ for $n \ge 1$. Hence $\ker L_{n0} = \{0\}$ for $n \ge 1$. Then $\ker L_0 = \{0\}$.

To show that $\overline{\operatorname{Im}(L_0)} \ne \mathcal{H}$ it is sufficient to show that $\overline{\operatorname{Im}(L_{n0})} \ne \mathcal{H}_n$ for some $n \ge 1$. So we consider the boundary value problem

$$L_{n0}^* u_n = -u_n'(t) + (A_n(t)P_{\alpha_n})^* u_n(t) = 0, \quad u_n \in \mathcal{H}_n$$

The solutions of this equation are

$$u_n(t) = \exp\Big(\int_{a_n}^t (A_n(s)P_{\alpha_n})^*\,ds\Big) g_n, \quad g_n \in H_n$$

Consequently, $H_n \subset \ker L_{n0}^*$, $n \ge 1$ This means that for any $n \ge 1$ $\overline{\operatorname{Im}(L_{n0})} \ne \mathcal{H}_n$. Then $\overline{\operatorname{Im}(L_0)} \ne \mathcal{H}$. $\square$

Theorem 2.2. *If $\widetilde{L}$ is any extension of L_0 in $\mathcal{H}$, then*

$$\widetilde{L} = \oplus_{n=1}^{\infty} \widetilde{L}_n,$$

where $\widetilde{L}_n$ is a extension of L_{n0}, $n \geq 1$, in $\mathcal{H}_n$.

Proof. Indeed, in this case for any $n \geq 1$ the linear manifold

$$\widetilde{M}_n := \{u_n \in D(L_n) : (u_n) \in D(\widetilde{L})\}$$

contains $D(L_{n0})$. That is,

$$D(L_{n0}) \subset \widetilde{M}_n \subset D(L_n), \quad n \geq 1$$

Then the operator

$$\widetilde{L}_n u_n = l_n(u_n), \quad u_n \in \widetilde{M}_n, \ n \geq 1$$

is an extensions of L_{n0}. Consequently, for any $n \geq 1$,

$$\widetilde{L}_n : D(\widetilde{L}_n) = \widetilde{M}_n \subset \mathcal{H}_n \to \mathcal{H}_n$$

From this, we obtain

$$\widetilde{L} = \oplus_{n=1}^{\infty} \widetilde{L}_n\,.$$

For the boundedly solvable extensions of L_0 and L_{n0}, $n \geq 1$, the following statement holds. □

Theorem 2.3. *For the boundedly solvability of any extension $\widetilde{L} = \oplus_{n=1}^{\infty} \widetilde{L}_n$ of the minimal operator L_0, the necessary and sufficient conditions are the boundedly solvability of the coordinate operators $\widetilde{L}_n$ of the minimal operator L_{n0}, $n \geq 1$ and $\sup_{n\geq 1} \|\widetilde{L}_n^{-1}\| < \infty$.*

Proof. It is clear that the extension $\widetilde{L}$ is one-to-one operator in $\mathcal{H}$ if and only if the all coordinate extensions $\widetilde{L}_n$, $n \geq 1$ of $\widetilde{L}$ are one-to-one operators in $\mathcal{H}_n$, $n \geq 1$. On the other hand for the boundedness of $\widetilde{L}^{-1} = \oplus_{n=1}^{\infty} \widetilde{L}_n^{-1}$ of $\mathcal{H}$ the necessary and sufficient condition is $\sup_{n\geq 1} \|\widetilde{L}_n^{-1}\| < \infty$ (see [5, 9]).

Now let $U_n(t,s)$, $t,s \in \Delta_n$, $n \geq 1$, be the family of evolution operators corresponding to the homogeneous differential equation

$$\frac{\partial U_n}{\partial t}(t,s)f + A_n(t)P_{\alpha_n} U_n(t,s)f = 0, \quad t,s \in \Delta_n$$

with the boundary condition

$$U_n(s,s)f = f, \ f \in H_n\,.$$

The operator $U_n(t,s)$, $t,s \in \Delta_n$, $n \geq 1$ is linear continuous boundedly invertible in H_n with the property (see [8])

$$U_n^{-1}(t,s) = U_n(s,t), \ s,t \in \Delta_n\,.$$

Lets us introduce the operator $U_n : \mathcal{H}_n \to \mathcal{H}_n$ as

$$U_n z_n(t) =: U_n(t,0) z_n(t), \quad t \in \Delta_n\,.$$

It is clear that if $\widetilde{L}_n$ is any extension of the minimal operator L_{n0}, that is, $L_{n0} \subset \widetilde{L}_n \subset L_n$, then

$$U_n^{-1} L_{n0} U_n = M_{n0}, \quad M_{n0} \subset U_n^{-1} \widetilde{L}_n U_n = \widetilde{M}_n \subset M_n, \quad U_n^{-1} L_n U_n = M_n$$

In addition, for any $n \geq 1$,

$$\begin{aligned}
\|U_n\| &= \Big\| \exp\Big(-\int_{a_n}^{t} A_n(s)P_{\alpha_n}\,ds\Big)\Big\| \\
&\leq \exp\Big(\int_{a_n}^{b_n} \|A_n(s)\|\|P_{\alpha_n}\|\,ds\Big) \\
&\leq \exp\Big(|\Delta_n|\|P_{\alpha_n}\| \sup_{t\in\Delta_n} \|A_n(t)\|\Big) < \infty
\end{aligned}$$

and

$$\begin{aligned}
\|U_n^{-1}\| &= \| \exp\Big(\int_{a_n}^{t} A_n(s)P_{\alpha_n}\,ds\Big)\| \\
&\leq \exp\Big(|\Delta_n|\|P_{\alpha_n}\| \sup_{t\in\Delta_n} \|A_n(t)\|\Big) < \infty\,.
\end{aligned}$$

□

Theorem 2.4. *Let $n \geq 1$. Each boundedly solvable extension $\widetilde{L}_n$ of the minimal operator L_{n0} in $\mathcal{H}_n$ is generated by the differential-operator expression $l_n(\cdot)$ and the boundary condition*

$$(K_n + E_n)u_n(a_n) = K_n U_n(a_n, b_n)u_n(b_n), \tag{2.2}$$

where $K_n \in L(H_n)$ and $E_n : H_n \to H_n$ is identity operator. The operator K_n is determined by the extension $\widetilde{L}_n$ uniquely, i.e. $\widetilde{L}_n = L_{K_n}$.

On the contrary, the restriction of the maximal operator L_n in $\mathcal{H}_n$ to the linear manifold of vector-functions satisfy the condition (2.2) for some bounded operator $K_n \in L(H_n)$, is a boundedly solvable extension of the minimal operator L_{n0}.

On other hand, for $n \geq 1$

$$\|\widetilde{L}_n^{-1}\| \leq |\Delta_n|^{1/2} \exp\Big(2|\Delta_n|\|P_{\alpha_n}\| \sup_{t\in\Delta_n} \|A_n(t)\|\Big)$$

Proof. For any $n \geq 1$, the description the all boundedly solvable extensions of $\widetilde{L}_n$ of the minimal operator L_{n0} in $\mathcal{H}_n$, $n \geq 1$ have been given in work [6]. From the relation $U_n^{-1}L_{K_n}U_n = M_{K_n}$ we obtain

$$L_{K_n}^{-1} = U_n M_{K_n}^{-1} U_n^{-1},$$

where $M_{K_n}u_n(t) = u_n'(t)$ with the boundary condition

$$(K_n + E_n)u_n(a_n) = K_n u_n(b_n), \quad n \geq 1\,.$$

Hence for $f_n \in L^2(H_n, \Delta_n)$,

$$\begin{aligned}
\|M_{K_n}^{-1} f_n\|_{\mathcal{H}_n}^2 &= \int_{a_n}^{b_n} \|K_n \int_{a_n}^{b_n} f_n(s)\,ds + \int_{a_n}^{t} f_n(s)\,ds\|^2\,dt \\
&\leq 2\int_{a_n}^{b_n} \|K_n \int_{a_n}^{b_n} f_n(s)\,ds\|^2\,dt + 2\int_{a_n}^{b_n} \|\int_{a_n}^{t} f_n(s)\,ds\|^2\,dt \\
&\leq 2\|K_n\|^2 \int_{a_n}^{b_n}\int_{a_n}^{b_n} \|f_n(s)\|^2\,ds|\Delta_n|\,dt \\
&\quad + 2\int_{a_n}^{b_n} \Big(\int_{a_n}^{b_n} \|f_n(s)\|^2\,ds\Big)\,dt|\Delta_n|
\end{aligned}$$

$$
\begin{aligned}
&= \big(2\|K_n\|^2|\Delta_n|^2 + 2|\Delta_n|^2\big)\|f_n\|^2_{\mathcal{H}_n} \\
&= 2|\Delta_n|^2(1 + \|K_n\|^2)\|f_n\|^2_{\mathcal{H}_n}\,.
\end{aligned}
$$

Hence,

$$
\|M^{-1}_{K_n}\| \le \sqrt{2}|\Delta_n|\big(1 + \|K_n\|^2\big)^{1/2}, \quad n \ge 1
$$

From this and the properties of evolution operators, the validity of inequality is clear. This completes proof of theorem. □

Theorem 2.5. *Let L_{K_n} be a boundedly solvable extension of L_{n0} in $\mathcal{H}_n$ and*

$$
M_{K_n} = U_n^{-1} L_{K_n} U_n, \quad \text{for } n \ge 1\,.
$$

To have

$$
\sup_{n\ge 1}\|L^{-1}_{K_n}\| < \infty,
$$

the necessary and sufficient condition is

$$
\sup_{n\ge 1}\|M^{-1}_{K_n}\| < \infty\,.
$$

Theorem 2.6. *Assumed that*

$$
\begin{gathered}
M_{K_n} : W_2^1(H_n,\Delta_n) \subset L^2(H_n,\Delta_n) \to L^2(H_n,\Delta_n), \\
M_{K_n} u_n(t) = u_n'(t), \\
(K_n + E_n)u_n(0) = K_n u_n(1)
\end{gathered}
$$

To have

$$
\sup_{n\ge 1}\|M^{-1}_{K_n}\| < \infty,
$$

the necessary and sufficient condition is $\sup_{n\ge 1}\|K_n\| < \infty$.

Proof. Indeed, from the proof of Theorem 2.4,

$$
\|M^{-1}_{K_n}\| \le \sqrt{2}|\Delta_n|\big(1 + \|K_n\|^2\big)^{1/2}, \quad n \ge 1
$$

From this, if $\sup_{n\ge 1}\|K_n\| < \infty$, then $\sup_{n\ge 1}\|M^{-1}_{K_n}\| < \infty$. On the contrary, assumed that $\sup_{n\ge 1}\|M^{-1}_{K_n}\| < \infty$. Then in the relation

$$
K_n \int_{a_n}^{b_n} f_n(t)\,dt = M^{-1}_{K_n} f_n(t) - \int_{a_n}^{t} f_n(t)\,dt,\ f_n \in \mathcal{H}_n, \quad n \ge 1
$$

choosing the functions $f_n(t) = f_n^*$, $t \in \Delta_n$, $f_n^* \in H_n$, $n \ge 1$, we have

$$
\|K_n f_n^*\|_{H_n}|\Delta_n| \le \|M^{-1}_{K_n} f_n^*\|_{H_n} + |\Delta_n|\|f_n^*\|_{H_n}, \quad n \ge 1\,.
$$

Then

$$
\|K_n\| \le \frac{1}{|\Delta_n|}\|M^{-1}_{K_n}\| + 1\,.
$$

Consequently, from the above relation and the condition on Δ_n, $n \ge 1$, we obtain

$$
\sup_{n\ge 1}\|K_n\| \le \Big(\inf_{n\ge 1}|\Delta_n|\Big)^{-1}\sup_{n\ge 1}\|M^{-1}_{K_n}\| + 1 < \infty\,.
$$

□

Now using Theorems 2.4–2.6, we formulate an assertion on the description of all boundedly solvable extensions of to in $\mathcal{H}$.

Theorem 2.7. *Each boundedly solvable extension $\widetilde{L}$ of the minimal operator L_0 in $\mathcal{H}$ is generated by differential-operator expression* (2.1) *and boundary conditions*

$$(K_n + E_n)u_n(a_n) = K_n U_n(a_n, b_n)u_n(b_n), \quad n \geq 1,$$

where $K_n \in L(H_n)$, $K = \oplus_{n=1}^{\infty} K_n \in L\big(\oplus_{n=1}^{\infty} H_n\big)$ and $E_n : H_n \to H_n$ is identity operator. The operator K is determined by the extension $\widetilde{L}$ uniquely, i.e. $\widetilde{L} = L_K$ and vice versa.

Remark 2.8. If in the (2.1), $\alpha_n(t) = t$ ($\alpha_n(t) < t$), $t \in [a_n, b_n]$ for any $n \geq 1$, then this problem corresponds to the problem of theory of multipoint ordinary (delay) differential operators in Hilbert spaces of vector-functions.

3. Structure of spectrum of boundedly solvable extensions

In this section the structure of spectrum of boundedly solvable extensions of minimal operator L_0 in $\mathcal{H}$ is investigated. First we consider the spectrum for the boundedly solvable extension L_K, $K = (K_n)$ of the minimal operator L_0 in $\mathcal{H}$; that is,

$$L_K u = \lambda u + f, \quad \lambda \in \mathbb{C}, \quad u = (u_n), \quad f = (f_n) \in \mathcal{H}$$

From it follows that

$$\oplus_{n=1}^{\infty}(L_{K_n} - \lambda E_n)(u_n) = (f_n)$$

The last relation is equivalent to the equations

$$(L_{K_n} - \lambda E_n)u_n = f_n, \ n \geq 1, \quad \lambda \in \mathbb{C}, \quad f_n \in \mathcal{H}_n$$

That is, for any $n \geq 1$, we have

$$U_n(M_{K_n} - \lambda E_n)U_n^{-1}u_n = f_n$$

Therefore,

$$\sigma_p(L_{K_n}) = \sigma_p(M_n), \quad \sigma_c(L_{K_n}) = \sigma_c(M_n), \quad \sigma_r(L_{K_n}) = \sigma_r(M_n) \tag{3.1}$$

Consequently, we consider the the spectrum parts of M_{K_n}; that is,

$$M_{K_n}u_n = \lambda u_n + f_n, \quad \lambda \in \mathbb{C}, \quad f_n \in \mathcal{H}_n, \quad n \geq 1$$

Then from this we obtain

$$\begin{gathered} u_n' = \lambda u_n + f_n, \\ (K_n + E_n)u_n(a_n) = K_n u_n(b_n), \quad n \geq 1 \end{gathered}$$

Since the general solution of the above differential equation in $\mathcal{H}_n$ has the form

$$u_n(t, \lambda) = \exp(\lambda(t - a_n))f_n^0 + \int_{a_n}^{t} \exp(\lambda(t - s))f_n(s)\, ds, \quad t \in \Delta_n,$$

$f_n^0 \in H_n$, $n \geq 1$, from the boundary condition it is obtained that

$$\Big(E_n + K_n\big(1 - \exp(\lambda|\Delta_n|)\big)\Big)f_n^0 = K_n \int_{a_n}^{b_n} \exp(\lambda(b_n - s))f_n(s)\, ds, \quad n \geq 1$$

It is easy to show that $\lambda_{n,m} = \frac{2m\pi i}{|\Delta_n|} \in \rho(M_{K_n})$, $m \in \mathbb{Z}$, $n \geq 1$. Then for $\lambda_{n,m} \neq 2m\pi i/|\Delta_n|$, $m \in \mathbb{Z}$, $n \geq 1$, we have

$$\Big(K_n - \frac{1}{exp(\lambda|\Delta_n|) - 1}E_n\Big)f_n^0$$

$$= \big(1 - \exp(\lambda|\Delta_n|)\big)^{-1} K_n \int_{a_n}^{b_n} \exp(\lambda(b_n - s)) f_n(s)\, ds,$$

$f_n^0 \in H_n$, $f_n \in \mathcal{H}_n$, $n \geq 1$. From this and relations (3.1) it follows the validity of following statement.

Theorem 3.1. *In order for λ to belong to $\sigma_p(L_{K_n})$ $(\sigma_c(L_{K_n}), \sigma_r(L_{K_n}))$, it is necessary and sufficient that*

$$\mu = \frac{1}{\exp(\lambda|\Delta_n|) - 1} \in \sigma_p(K_n)\ (\sigma_c(K_n), \quad \sigma_r(K_n)).$$

Then a structure of spectrum of boundedly solvable extension L_{K_n} can be formulated in the form

Theorem 3.2. *The point spectrum of the boundedly solvable extension L_{K_n} has the form*

$$\sigma_p(L_{K_n}) = \Big\{\lambda \in \mathbb{C} : \lambda = \frac{1}{|\Delta_n|}\big\{ \ln|\frac{\mu+1}{\mu}| + i\arg(\frac{\mu+1}{\mu}) + 2m\pi i\big\}, \\ \mu \in \sigma_p(K_n)\backslash\{0, -1\},\ m \in \mathbb{Z}\Big\}$$

Similarly propositions on the continuous $\sigma_c(L_{K_n})$ and residual $\sigma_r(L_{K_n})$ spectrums are true.

Lastly, using the results on the spectrum parts of direct sum of operators in the direct sum of Hilbert spaces [13] in we can proved the following theorem.

Theorem 3.3. *For the parts of spectrum of the boundedly solvable extension $L_K = \oplus_{n=1}^{\infty} L_{K_n}$, $K = (K_n)$ in Hilbert spaces $\mathcal{H} = \oplus_{n=1}^{\infty}\mathcal{H}_n$ the following statements are true*

$$\sigma_p(L_K) = \cup_{n=1}^{\infty}\sigma_p(L_{K_n}),$$

$$\sigma_c(L_K) = \Big\{\Big(\cup_{n=1}^{\infty}\sigma_p(L_{K_n})\Big)^c \cap \Big(\cup_{n=1}^{\infty}\sigma_r(L_{K_n})\Big)^c \cap \Big(\cup_{n=1}^{\infty}\sigma_c(L_{K_n})\Big)\Big\} \\ \cup \Big\{\lambda \in \cap_{n=1}^{\infty}\rho(L_{K_n}) : \sup_{n\geq 1}\|R_\lambda(L_{K_n})\| = \infty\Big\},$$

$$\sigma_r(L_K) = \Big(\cup_{n=1}^{\infty}\sigma_p(L_{K_n})\Big)^c \cap \Big(\cup_{n=1}^{\infty}\sigma_r(L_{K_n})\Big)$$

4. Applications

In this section, we present an application of above results.

Example 4.1. For any $n \geq 1$ let us $H_n = (\mathbb{C}, |\cdot|)$, $a_n = 0$, $b_n = 1$, $A_n(t) = c_n$, $c_n \in \mathbb{C}$, $\sup_{n\geq 1}|c_n| < \infty$, $\alpha_n(t) = \alpha_n t$, $0 < \alpha_n < 1$ with property $\sup_{n\geq 1}(\frac{1}{\alpha_n}) < \infty$. Consider the pantograph type delay differential expression

$$l(u) = u_n'(t) + c_n u_n(\alpha_n t)$$

in $\mathcal{H} = \oplus_{n=1}^{\infty}\mathcal{H}_n$, where $\mathcal{H}_n = L^2(0,1)$.

In this case by Theorem 2.7, all boundedly solvable extensions L_k of the minimal operator L_0 generated by $l(\cdot)$ in $\mathcal{H}$ are described by the differential expression $l(\cdot)$ and the boundary conditions

$$(k_n + 1)u_n(0) = k_n U_n(0,1)u_n(1),$$

where $k_n \in \mathbb{C}$, $n \geq 1$, $\sup_{n\geq 1}|k_n| < \infty$ and vice versa.

On other hand in the case $k_n \notin \{0,-1\}$, for any $n \geq 1$ the point, the continuous and residual spectrums of L_{k_n} in $\mathcal{H}_n = L^2(0,1)$ is of the form

$$\sigma_p(L_{k_n}) = \Big\{\lambda \in \mathbb{C} : \lambda = ln|\frac{k_n+1}{k_n}| + i\arg(\frac{k_n+1}{k_n}) + 2m\pi i,\ m \in \mathbb{Z}\Big\},$$
$$\sigma_c(L_{k_n}) = \sigma_r(L_{k_n}) = \emptyset\,.$$

Hence by Theorem 3.3 spectrum parts of $L_k = \oplus_{n=1}^{\infty} L_{k_n}$ has the form

$$\sigma_p(L_k) = \cup_{n=1}^{\infty} \cup_{m\in\mathbb{Z}} \Big\{ln|\frac{k_n+1}{k_n}| + i\arg(\frac{k_n+1}{k_n}) + 2m\pi i\Big\},$$
$$\sigma_c(L_k) = \Big\{\lambda \in \cap_{n=1}^{\infty}\rho(L_{k_n}) : \sup_{n\geq 1}\|R_\lambda(L_{k_n})\| = \infty\Big\},$$
$$\sigma_r(L_k) = \emptyset$$

Example 4.2. Let $H_n = (\mathbb{C}, |\cdot|)$, (a_n), a sequence of real numbers, $\sup_{n\geq 1}|a_n| < \infty$, (b_n), $b_n = a_n + 1$, $\Delta_n = (a_n, b_n)$, $\mathcal{H}_n = L^2(H_n, \Delta_n)$,

$$l_n(u_n) = u_n' + u_n(\alpha_n(t)), \quad \alpha_n(t) = (t-a_n)^2 + a_n - \frac{1}{2}, \quad t \in \Delta_n, \quad n \geq 1,$$

$\mathcal{H} = \oplus_{n=1}^{\infty}\mathcal{H}_n$, $l(\cdot) = \oplus_{n=1}^{\infty} l_n(\cdot)$. In this case for any $n \geq 1$ the function $\alpha_n(\cdot)$ is increase, invertible and $\alpha_n^{-1}(t) = a_n + \sqrt{t - a_n + \frac{1}{2}}$ and $(\alpha_n^{-1}(t))' = \frac{1}{2\sqrt{t-a_n+\frac{1}{2}}}$. Hence $\sup_{n\geq 1}\|(\alpha_n^{-1})'\|_\infty \leq \frac{1}{2}$. In this case all boundedly solvable extensions of minimal operator L_0 in $\mathcal{H}$ are described by $l(\cdot)$ and boundary conditions

$$(k_n+1)u_n(a_n) = k_n U_n(a_n, b_n) u_n(b_n), \quad n \geq 1,$$

where $\sup_{n\geq 1}|k_n| < \infty$. On the other hand for $k_n \notin \{0,-1\}$ spectrum of each boundedly solvable extension L_{k_n} has the form

$$\sigma_p(L_{k_n}) = \Big\{\lambda \in \mathbb{C} : \lambda = \ln|\frac{k_n+1}{k_n}| + i\arg(\frac{k_n+1}{k_n}) + 2m\pi i,\ m \in \mathbb{Z}\Big\},$$
$$\sigma_c(L_{k_n}) = \sigma_r(L_{k_n}) = \emptyset, \quad n \geq 1$$

Hence by Theorem 3.3, the spectrum parts of $L_k = \oplus_{n=1}^{\infty} L_{k_n}$ have the form

$$\sigma_p(L_k) = \cup_{n=1}^{\infty} \cup_{m\in\mathbb{Z}} \big\{ln|\frac{k_n+1}{k_n}| + i\arg(\frac{k_n+1}{k_n}) + 2m\pi i\big\},$$
$$\sigma_c(L_k) = \big\{\lambda \in \cap_{n=1}^{\infty}\rho(L_{k_n}) : \sup_{n\geq 1}\|R_\lambda(L_{k_n})\| = \infty\big\},$$
$$\sigma_r(L_k) = \emptyset$$

Remark 4.3. Similar to the problems in Example 4.1 and 4.2, we can investigate the case $\alpha_n(t) = b_n - t$, $a_n \leq t \leq b_n$, $n \geq 1$.

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Zameddin I. Ismailov
Karadeniz Technical University, Department of Mathematics, 61080, Trabzon, Turkey
E-mail address: zameddin.ismailov@gmail.com

Pembe Ipek
Karadeniz Technical University, Institute of Natural Sciences, 61080 Trabzon, Turkey
E-mail address: pembeipek@hotmail.com