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NONEXISTENCE OF NON-TRIVIAL GLOBAL WEAK SOLUTIONS FOR HIGHER-ORDER NONLINEAR SCHRÖDINGER EQUATIONS

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Abstract. We study the initial-value problem for the higher-order nonlinear Schrödinger equation

$$i\partial_t u - (-\Delta)^m u = \lambda |u|^p,$$

subject to the initial data

$$u(x,0) = f(x),$$

where $u = u(x,t) \in \mathbb{C}$ is a complex-valued function, $(x,t) \in \mathbb{R}^N \times [0,+\infty)$, $p > 1$, $m \geq 1$, $\lambda \in \mathbb{C}\backslash\{0\}$, and $f(x)$ is a given complex-valued function. We prove nonexistence of a nontrivial global weak solution. Furthermore, we prove that the L^2-norm of the local in time L^2-solution blows up at a finite time.

1. Introduction

We consider the higher-order nonlinear Schrödinger equation

$$i\partial_t u - (-\Delta)^m u = \lambda |u|^p, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.1}$$

supplemented with the initial data

$$u(x,0) = f(x), \quad x \in \mathbb{R}^N, \tag{1.2}$$

where $u = u(x,t)$ is a complex-valued unknown function of (x,t), $\lambda = \lambda_1 + i\lambda_2 \in \mathbb{C}\backslash\{0\}$, $\lambda_i \in \mathbb{R}\,(i = 1,2)$, and $f = f(x) = f_1(x) + i f_2(x) \in \mathbb{C}$, $f_i(x) \in L^1(\mathbb{R})\,(i = 1,2)$ are real-valued given functions.

Let us first recall some previous results on nonlinear Schrödinger equations (NLS). Since there is a large amount of papers for NLS, we mention the ones related to our result. Many authors have studied NLS with a gauge invariant power type nonlinearity

$$i\partial_t u + \Delta u = \lambda_0 |u|^{p-1} u, \quad x \in \mathbb{R}^N,\, t \in \mathbb{R}, \tag{1.3}$$

where $\lambda_0 \in \mathbb{R}$, $p > 1$. In the case of $1 < p < 1 + \frac{4}{N}$, Tsutsumi [16] proved global existence of L^2-solution for an integral equation associated to (1.3) with the initial condition $u(x,0) = u_0(x) \in L^2$ without any size restriction (For other type of solutions see e.g. [6] etc). It is also well known that when $N \geq 2$, $p \geq 1 + \frac{4}{N}$ and $\lambda_0 < 0$, there are solutions of (1.3) that blow up in finite time for certain initial

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data (see, e.g. [11]). Ikeda and Wakasugi [7] studied the nonlinear Schrödinger equation with nongauge invariant power nonlinearity

$$i\partial_t u + \frac{1}{2}\Delta u = \lambda |u|^p,\ x \in \mathbb{R}^N,\ t > 0, \tag{1.4}$$

subject to the initial data $u(x,0) = \varepsilon f(x)$, where $f \in L^2$, $\varepsilon > 0$, $1 < p < 1 + \frac{2}{N}$ and $\lambda \in \mathbb{C}\backslash\{0\}$. They proved the nonexistence of a non-trivial global weak solution for the equation (1.4) with some initial data but without any size and coefficient restriction, which implies that "small data global existence" does not hold for (1.4). Furthermore, they also proved that the L^2-norm of a time local L^2-solution with a suitable initial data blows up in a finite time.

We will prove, using Banach fixed point theorem and Strichartz estimates (see [2, 16, 6, 14, 15]), a local existenceresult for problem (1.1)–(1.2) for any initial data $f \in L^2(\mathbb{R}^N)$. Moreover, by the test function method (see [3, 4, 8, 9, 10, 18] and the references therein), we will show nonexistence of non-trivial global weak solutions for problem (1.1)–(1.2). Next we prove that the L^2-norm of the time local L^2-solution blows up at a finite time.

2. Local existence

In this section, we prove the local existence and uniqueness of the L^2-solution to the problem (1.1)–(1.2). It is well known that $(-\Delta)^m$ is a self-adjoint operator in $L^2(\mathbb{R}^N)$ for every $m \geq 1$, and it generates a strongly continuous semigroup $S(t)$ on $L^2(\mathbb{R}^N)$ for $t > 0$. Using the semigroup theory (see, e.g. [17]), we can write problem (1.1)–(1.2) in the following equivalent integral equation

$$u(t) = S(t)f - i\int_0^t S(t-s)|u(s)|^p\,ds, \quad t \geq 0. \tag{2.1}$$

For $S(t) = \exp(it(-\Delta)^m)$, we have the following results.

Lemma 2.1. *Let ρ and r be positive numbers such that $\frac{1}{\rho} + \frac{1}{r} = 1$ and $2 \leq \rho \leq \infty$. For any $t > 0$, $S(t)$ is a bounded operator from L^r to L^ρ. Moreover, it satisfies the important estimate*

$$\|S(t)v\|_{L^\rho(\mathbb{R}^N)} \leq Ct^{-\frac{N}{2m}(\frac{1}{r}-\frac{1}{\rho})}\|v\|_{L^r(\mathbb{R}^N)}, \quad v \in L^r(\mathbb{R}^N),\ t > 0, \tag{2.2}$$

and for any $t > 0$, the map $t \mapsto S(t)$ is strongly continuous. For $\rho = 2$, $S(t)$ is unitary and strongly continuous for $t > 0$.

Definition 2.2. The triple (r, ρ, q) is called σ-admissible triple if $\frac{1}{r} = \sigma(\frac{1}{q} - \frac{1}{\rho})$, where $1 < r \leq q \leq \infty$ and $\sigma > 0$.

Now, we give the following Strichartz estimate.

Lemma 2.3. *Let $(r, \rho, 2)$ be $\frac{N}{2m}$-admissible. Then*

$$\|S(\cdot)v\|_{L^r((0,T);L^\rho(\mathbb{R}^N))} \leq C\|v\|_{L^2(\mathbb{R}^N)}, \tag{2.3}$$

where $C = C(N,p)$.

For the proof of Lemma 2.1 see, e.g [6]. For Lemma 2.3, see Strichartz [15] and Ginibre and Velo [5].

Let $1 < \rho,\ r < \infty$ and $a, b > 0$. We set

$$E := \Big\{v(t) \in L^\infty((0,T);L^2(\mathbb{R}^N)) \cap L^r((0,T);L^\rho(\mathbb{R}^N));$$

$$\|v(t)\|_{L^2(\mathbb{R}^N)} \le a,\ \|v\|_{L^r((0,T);L^\rho(\mathbb{R}^N))} \le b\Big\};$$

E is a closed subset in $L^r((0,T),L^\rho(\mathbb{R}^N))$.

Theorem 2.4. *Let $1<p<1+\frac{4m}{N}$, $\lambda\in\mathbb{C}\backslash\{0\}$ and $f\in L^2(\mathbb{R}^N)$. Then there exists a positive time $T=T(\|f\|_{L^2})>0$ and a unique solution $u\in C([0,T);L^2(\mathbb{R}^N))\cap L^r([0,T);L^\rho(\mathbb{R}^N))$ of the integral equation (2.1), where ρ and r are defined by $\rho=p+1$ and $\frac{2m}{r}=\frac{N}{2}-\frac{N}{\rho}$.*

Proof. We define the Banach space

$$\begin{aligned} E_T := \Big\{ & v(t)\in L^\infty(I_T;L^2(\mathbb{R}^N))\cap L^r(I_T;L^\rho(\mathbb{R}^N)); \\ & \|v(t)\|_{L^\infty(I_T;L^2(\mathbb{R}^N))} \le \|f\|_{L^2(\mathbb{R}^N)}, \quad \|v\|_{L^r(I_T;L^\rho(\mathbb{R}^N))} \le 2\delta\|f\|_{L^2(\mathbb{R}^N)}\Big\}, \end{aligned}$$

where $I_T:=(0,T)$ and δ is the constant appearing in (2.3), with $\rho=p+1$, $r=\frac{4m(p+1)}{N(p-1)}$ and T is a small positive constant to be determined later. Now, for every $u\in E_T$, we define

$$\Psi(u) := S(t)f(x) - \lambda i\int_0^t S(t-s)|u|^p\,ds.$$

As usual, we prove the existence of local solutions using the Banach fixed point theorem.

- Ψ is defined from E_T to E_T: Let $u\in E_T$. Setting

$$\tilde{u}(t) := \begin{cases} u(t), & \text{if } t\in I_T,\\ 0, & \text{otherwise.}\end{cases}$$

Now, we have

$$\begin{aligned} &\|\Psi(u)\|_{L^r(I_T;L^\rho(\mathbb{R}^N))} \\ &\le \delta\|f\|_{L^2(\mathbb{R}^N)} + C\Big\|\int_0^t (t-s)^{-\frac{N}{2m}(\frac{p}{\rho}-\frac{1}{\rho})}\|u(s)\|^p_{L^\rho(\mathbb{R}^N)}\,ds\Big\|_{L^r(I_T)} \\ &\le \delta\|f\|_{L^2(\mathbb{R}^N)} + C\Big\|\int_{-\infty}^{+\infty} |t-s|^{-\frac{N}{2m}(\frac{p}{\rho}-\frac{1}{\rho})}\|\tilde{u}(s)\|^p_{L^\rho(\mathbb{R}^N)}\,ds\Big\|_{L^r(\mathbb{R})}. \end{aligned}$$

By the generalized Young inequality [13], we have

$$\begin{aligned} \|\Psi(u)\|_{L^r(I_T;L^\rho(\mathbb{R}^N))} &\le \delta\|f\|_{L^2(\mathbb{R}^N)} + C\|\tilde{u}\|^p_{L^{\rho_1}(\mathbb{R};L^\rho(\mathbb{R}^N))} \\ &\le \delta\|f\|_{L^2(\mathbb{R}^N)} + C\|u\|^p_{L^{\rho_1}(I_T;L^\rho(\mathbb{R}^N))}, \end{aligned}\tag{2.4}$$

where $\rho_1=\frac{4mp(p+1)}{N+4m-(N-4m)p}$, and note that $1<\rho_1<r$ for $1<p<1+\frac{4m}{N}$. By Hölder's inequality we have, with $\frac{1}{\rho_1}=\frac{1}{\rho_2}+\frac{1}{r}$,

$$\begin{aligned} \|u\|_{L^{\rho_1}(I_T;L^\rho(\mathbb{R}^N))} &\le \Big(\int_0^t ds\Big)^{1/\rho_2}\|u\|_{L^r(I_T,L^\rho(\mathbb{R}^N))} \\ &\le CT^{1/\rho_2}\|u\|_{L^r(I_T,L^\rho(\mathbb{R}^N))}, \end{aligned}\tag{2.5}$$

where $\rho_2=\frac{4mp}{N+4m-Np}$ and $C=C(N,p)$. Next, (2.4) and (2.5) give us

$$\begin{aligned} \|\Psi(u)\|_{L^r(I;L^\rho(\mathbb{R}^N))} &\le \delta\|f\|_{L^2(\mathbb{R}^N)} + C_1T^{p/\rho_2}\|u\|^p_{L^r(I;L^\rho(\mathbb{R}^N))} \\ &\le \delta\|f\|_{L^2(\mathbb{R}^N)} + 2C_1T^{p/\rho_2}(2\delta)^{p-1}\|f\|^{p-1}_{L^2(\mathbb{R}^N)}\delta\|f\|_{L^2(\mathbb{R}^N)}, \end{aligned}$$

where $C_1 = C(N,p,\lambda)$. Now, if we choose T small enough such that

$$2C_1 T^{p/\rho_2}(2\delta)^{p-1}\|f\|^{p-1}_{L^2(\mathbb{R}^N)} \le 1,$$

we conclude that $\|\Psi(u)\|_{L^r(I;L^\rho(\mathbb{R}^N))} \le \delta\|f\|_{L^2(\mathbb{R}^N)}$, and then $\Psi(u) \in E_T$.

• Ψ is a contracting map. For u, $v \in E_T$, repeating the same calculations as above, we obtain

$$\begin{aligned}
&\|\Psi(u)-\Psi(v)\|_{L^r(I;L^\rho(\mathbb{R}^N))}\\
&\le C\Big\|\int_0^t (t-s)^{-\frac{N}{2m}(\frac{p}{\rho}-\frac{1}{\rho})}\||u|^p-|v|^p\|_{L^{\rho/p}(\mathbb{R}^N)}\,ds\Big\|_{L^r(I_T)}\\
&\le C\Big\|\int_0^t (t-s)^{-\frac{N}{2m}(\frac{p}{\rho}-\frac{1}{\rho})}\Big(\|u\|^{p-1}_{L^\rho(\mathbb{R}^N)}+\|v\|^{p-1}_{L^\rho(\mathbb{R}^N)}\Big)\|u(s)-v(s)\|_{L^\rho(\mathbb{R}^N)}\,ds\Big\|_{L^r(I_T)}\\
&\le C\Big(\|u\|^{p-1}_{L^{\rho_1}(I_T,L^\rho(\mathbb{R}^N))}+\|v\|^{p-1}_{L^{\rho_1}(I_T,L^\rho(\mathbb{R}^N))}\Big)\|u-v\|_{L^{\rho_1}(I_T,L^\rho(\mathbb{R}^N))}\\
&\le C_2T^{p/\rho_2}\Big(\|u\|^{p-1}_{L^r(I_T,L^\rho(\mathbb{R}^N))}+\|v\|^{p-1}_{L^r(I_T,L^\rho(\mathbb{R}^N))}\Big)\|u-v\|_{L^r(I_T,L^\rho(\mathbb{R}^N))}\\
&\le C_2T^{p/\rho_2}2(2\delta\|f\|_{L^2(\mathbb{R}^N)})^{p-1}\|u-v\|_{L^r(I_T,L^\rho(\mathbb{R}^N))}.
\end{aligned}$$

If we choose T so small such that

$$C_2T^{p/\rho_2}2(2\delta\|f\|_{L^2(\mathbb{R}^N)})^{p-1} \le \frac{1}{2},$$

then we have

$$\|\Psi(u)-\Psi(v)\|_{L^r(I;L^\rho(\mathbb{R}^N))} \le \frac{1}{2}\|u-v\|_{L^r(I_T,L^\rho(\mathbb{R}^N))}.$$

By the Banach fixed point theorem, there exists a solution $u \in L^\infty(I_T;L^2(\mathbb{R}^N)) \cap L^r(I_T;L^\rho(\mathbb{R}^N))$ to problem (1.1)–(1.2) on $[0,T]$.

As usual, the solution can be extended to a maximal time of existence $T_{\max} > 0$.

• Uniqueness of solution: We show that the solution of (1.1)–(1.2) is unique. Let u and v be two solutions in E_T for some $T>0$, we set

$$t_1 = \sup\{t \in [0,T_{\max} : u(t)=v(t)\}\,.$$

If $t_1 = T_{\max}$, then $u(t)=v(t)$ on $[0,T_{\max}]$, which is the desired result. If $t_1 < T_{\max}$, repeating the same calculations as before, and by the assumption on t_1, we have

$$\begin{aligned}
&\|u-v\|_{L^r((0,t_2);L^\rho(\mathbb{R}^N))}\\
&= \|u-v\|_{L^r((t_1,t_2);L^\rho(\mathbb{R}^N))}\\
&\le C\Big\|\int_{t_1}^{t_2}(t_2-t_1)^{-\frac{N}{2m}(\frac{p}{\rho}-\frac{1}{\rho})}\||u|^p-|v|^p\|_{L^{\rho/p}(\mathbb{R}^N)}\,ds\Big\|_{L^r(I_T)}\\
&\le C_2(t_2-t_1)^{p/\rho_2}\Big(\|u\|^{p-1}_{L^r((t_1,t_2),L^\rho(\mathbb{R}^N))}+\|v\|^{p-1}_{L^r((t_1,t_2),L^\rho(\mathbb{R}^N))}\Big)\\
&\quad\times\|u-v\|_{L^r((t_1,t_2),L^\rho(\mathbb{R}^N))}\\
&\le C_2(t_2-t_1)^{p/\rho_2}2\big(2\delta\|f\|_{L^2(\mathbb{R}^N)}\big)^{p-1}\|u-v\|_{L^r((t_1,t_2),L^\rho(\mathbb{R}^N))}.
\end{aligned}$$

We can choose t_2 such that $t_2 > t_1$ and

$$C_2(t_2-t_1)^{p/\rho_2}2\big(2\delta\|f\|_{L^2(\mathbb{R}^N)}\big)^{p-1} \le \frac{1}{2}\,.$$

Then we have

$$\|u-v\|_{L^r(I;L^\rho(\mathbb{R}^N))} \le 0,$$

which implies $u(t)=v(t)$ on $[t_1,t_2]$. This contradicts the assumption of t_1. Therefore, $u(t)=v(t)$ for $t\in[0,T_{\max}]$. □

3. Blow up of L^2-solutions

We impose the following assumptions on the data

(H1) $f_1 \in L^1(\mathbb{R}^N)$, $\lambda_2\int_{\mathbb{R}^N} f_1(x)\,dx > 0$, for $f_2\in L^1(\mathbb{R}^N)$, $\lambda_1\int_{\mathbb{R}^N} f_2(x)\,dx<0$.

Now, we want to derive a blow-up result for (1.1)–(1.2).

Definition 3.1. Let $T>0$. We say that u is a weak solution of (1.1)–(1.2) on $[0,T)$ if $u\in C([0,T];L^p_{loc}(\mathbb{R}^N)$ and satisfies

$$\begin{aligned}
&\int_0^T\int_{\mathbb{R}^N} u(-i\partial_t\phi(x,t)-(-\Delta)^m\phi(x,t))\,dx\,dt\\
&= i\int_{\mathbb{R}^N} f(x)\phi(x,0)\,dx+\lambda\int_0^T\int_{\mathbb{R}^N}|u|^p\phi(x,t)\,dx\,dt
\end{aligned}\tag{3.1}$$

for any $\phi\in C_0^{1,\infty}((0,T)\times\mathbb{R}^N)$, $\phi\ge 0$ and satisfying $\phi(\cdot,T)=0$. Moreover, if $T=+\infty$, u is called a global weak solution for (1.1)–(1.2).

We note that an L^2-solution as in Theorem 2.4 is always a weak solution in the sense of Definition 3.1.

Theorem 3.2. *Let $1<p\le p^*=1+\frac{2m}{N}$, $\lambda\in\mathbb{C}\backslash\{0\}$ and let f satisfy (H1). Then problem (1.1)–(1.2) has no global nontrivial weak solution.*

We first prove the following lemma (see [12]).

Lemma 3.3. *Let $\psi\in L^1(\mathbb{R}^N)$ and $\int_{\mathbb{R}^N}\psi(x)\,dx<0$. Then there exists a test function $0\le\varphi\le 1$ such that*

$$\int_{\mathbb{R}^N}\psi(x)\varphi(x)\,dx<0. \tag{3.2}$$

Proof. We have

$$\int_{\mathbb{R}^N}\psi\varphi\,dx=\int_{|x|\le R}\psi\varphi\,dx+\int_{R\le|x|}\psi\varphi\,dx.$$

Take a function $\varphi=\varphi_R(x)$, $0\le\varphi_R\le 1$, such that $\varphi_R(x)\equiv 1$ for $|x|\le R$. Then

$$\int_{\mathbb{R}^N}\psi\varphi_R\,dx=\int_{|x|\le R}\psi\,dx+\int_{R\le|x|}\psi\varphi_R\,dx. \tag{3.3}$$

By the convergence of the integral $\int_{\mathbb{R}^N}|\psi|\,dx$, we have

$$\Big|\int_{R\le|x|}\psi\varphi_R\,dx\Big|\le\int_{R\le|x|}|\psi|\,dx\to 0\quad\text{as}\quad R\to+\infty.$$

After passing to the limit as $R\to+\infty$ in (3.3), we obtain

$$\lim_{R\to+\infty}\int_{\mathbb{R}^N}\psi\varphi_R\,dx=\lim_{R\to+\infty}\int_{|x|\le R}\psi\,dx=\int_{\mathbb{R}^N}\psi\,dx<0.$$

This implies the assertion of Lemma 3.3. □

Proof of Theorem 3.2. Suppose by contradiction that u is a weak global solution to (1.1)–(1.2). Let Φ be a radial, smooth and non-increasing function on $[0,+\infty)$ such that

$$\Phi(r)=\begin{cases}1, & \text{if } 0\le r\le 1,\\ \searrow & \text{if } 1\le r\le 2,\\ 0, & \text{if } r\ge 2.\end{cases}$$

Set

$$\phi_1(x):=\Phi\big(\frac{|x|}{BR}\big),\quad \phi_2(t):=\Phi\big(\frac{t}{R^{2m}}\big),$$

where $R,B>0$. We use the test function

$$\phi(x,t):=\phi_1(x)^\ell\phi_2(t)^\sigma,\quad \ell,\,\sigma\gg 1.$$

The constant $B>0$ in the definition of ϕ_1 is fixed and will be chosen later; it plays some role in the case $p=1+\frac{2m}{N}$, while in the case $p<1+\frac{2m}{N}$, we take $B=1$.

Let $Q:=[0,R^{2m})\times\mathbb{R}^N$. We consider only the case $\lambda_1>0$ and $\lambda_1\int_{\mathbb{R}^N}f_2\,dx<0$, since the other cases can be treated almost in the same way (see Remark 3.4). Set

$$I_R:=\int_Q|u|^p\phi\,dx\,dt.$$

Now, using the identity (3.1), and by taking the real part, we obtain

$$\lambda_1 I_R-\int_{\mathbb{R}^N}f_2(x)\phi(x,0)\,dx=\int_Q(\operatorname{Im}u)\partial_t\phi\,dx\,dt-\int_Q(\operatorname{Re}u)(-\Delta)^m\phi\,dx\,dt. \tag{3.4}$$

Furthermore, using the assumption (H1) on the initial condition f, and Lemma 3.3, we obtain

$$\lambda_1 I_R\le\int_Q|u|\,\phi_1^\ell|\partial_t\phi_2^\sigma|\,dx\,dt+\int_Q|u||\Delta^m\phi_1^\ell|\,\phi_2^\sigma\,dx\,dt\equiv K_1+K_2. \tag{3.5}$$

By applying ε-Young's inequality, $XY\le\varepsilon X^p+C(\varepsilon)Y^q$, for $X\ge0$, $Y\ge0$, $p+q=pq$, with $0<\varepsilon\ll1$, $C(\varepsilon)=(1/q)(p\varepsilon)^{-q/p})$, in K_1 and K_2, we obtain

$$(\lambda_1-2\varepsilon)I_R\le C(\varepsilon)\int_Q\phi_1^\ell\phi_2^{-\frac{\sigma}{p-1}}|\partial_t\phi_2^\sigma|^q\,dx\,dt+C(\varepsilon)\int_Q\phi_1^{-\frac{\ell}{p-1}}|\Delta^m\phi_1^\ell|^q\phi_2^\sigma\,dx\,dt.$$

At this stage, we pass to the new variables $s=t/R^{2m}$ and $y=x/R$, to obtain the estimate

$$(\lambda_1-2\varepsilon)I_R\le CR^{N+2m(1-q)}(\mathcal{A}+\mathcal{B}), \tag{3.6}$$

where

$$\mathcal{A}:=\int_{\Omega_1}\int_{\Omega_2}\Phi(y)^{\ell}\Phi(s)^{\sigma-q}|\Phi(s)'|^q\,dy\,ds<+\infty,$$
$$\mathcal{B}:=\int_{\Omega_1}\int_{\Omega_2}\Phi(y)^{-\frac{\ell}{p-1}}|\Delta^m(\Phi(y)^\ell)|^q\Phi(s)^\sigma\,dy\,ds<+\infty,$$
$$\Omega_1:=\{s\ge0:s\le2\},\quad \Omega_2:=\{y\in\mathbb{R}^N:|y|\le2\}.$$

Note that inequality $p\le p^*$ is equivalent to $\beta=N-\frac{2m}{p-1}\le0$. So, we have to distinguish two cases:

Case (i): $p<p^*\Longrightarrow\beta<0$. Passing to the limit in (3.6) as $R\to+\infty$, we have

$$\int_0^{+\infty}\int_{\mathbb{R}^N}|u|^p\phi\,dx\,dt=0\quad\Longrightarrow\quad u\equiv0;$$

this is a contradiction.
Case (ii): $p = p^* \implies \beta = 0$. We estimate the first term in the right hand side of inequality (3.5) using the Hölder inequality and the second term by the Young inequality as follows

$$\begin{aligned}(\lambda_1 - \varepsilon) I_R &\le C(\varepsilon) \int_Q \phi_1^{-\frac{\ell}{p-1}} |\Delta^m \phi_1^\ell|^q \phi_2^\sigma \, dx\, dt \\ &\quad + \Big(\int_{C_R} \int_{\mathbb{R}^N} |u|^p \phi \, dx\, dt\Big)^{1/p} \Big(\int_{C_R} \int_{\mathbb{R}^N} \phi_1^\ell \phi_2^{-\frac{\sigma}{p-1}} |\partial_t \phi_2^\sigma|^q \, dx\, dt\Big)^{1/q},\end{aligned}$$

where $C_R := \{t \in [0,+\infty) : R^{2m} \le t \le 2R^{2m}\}$ is the support of $\partial_t \phi_2$. Note that

$$\lim_{R\to+\infty} \int_{C_R} \int_{\mathbb{R}^N} |u|^p \phi \, dx\, dt = 0.$$

Now, introducing the new variables $s = t/R^{2m}$ and $y = x/BR$, we obtain

$$(\lambda_1 - \varepsilon) I_R \le B^{N/q} \mathcal{E} \Big(\int_{C_R} \int_{\mathbb{R}^N} |u|^p \phi \, dx\, dt\Big)^{1/p} + B^{-2m} \mathcal{B}, \tag{3.7}$$

where

$$\mathcal{E} := \Big(\int_{C_R} \int_{\Omega_2} \Phi(y)^\ell \Phi(s)^{\sigma - q} |\Phi(s)'|^q \, dy ds\Big)^{1/q} < +\infty.$$

Passing to the limit first in (3.7) as $R \to +\infty$, and then $B \to +\infty$, we get

$$\int_0^{+\infty} \int_{\mathbb{R}^N} |u|^p \phi \, dx\, dt = 0 \quad \implies \quad u \equiv 0;$$

which is a contradiction. $\square$

Remark 3.4. For the other cases, setting

$$I_R := \begin{cases} -\int_Q \lambda_1 |u|^p \phi(x,t) \, dx\, dt & \text{if } \lambda_1 < 0,\ \lambda_1 \int_{\mathbb{R}^N} f_2(x)\, dx < 0, \\ \int_Q \lambda_2 |u|^p \phi(x,t) \, dx\, dt & \text{if } \lambda_2 > 0,\ \lambda_2 \int_{\mathbb{R}^2 N} f_1(x)\, dx > 0, \\ -\int_Q \lambda_2 |u|^p \phi(x,t) \, dx\, dt & \text{if } \lambda_2 < 0,\ \lambda_2 \int_{\mathbb{R}^N} f_1(x)\, dx > 0, \end{cases}$$

we can prove the same conclusion in the same manner as above.

Next, we will mention that an L^2-solution $u \in C([0,T]; L^2(\mathbb{R}^N))$ is a weak solution in the sense of Definition 3.1.

Proposition 3.5. *Let $T > 0$. If u is an L^2-solution for problem (1.1)–(1.2) on $[0,T)$, then u is also a weak solution on $[0,T)$ in the sense of Definition 3.1.*

Proof. Let $T > 0$, $f \in L^2(\mathbb{R}^N)$ and let $u \in C([0,T); L^2(\mathbb{R}^N)) \cap L^r((0,T); L^\rho(\mathbb{R}^N))$ be a solution of (2.1). Given $\phi \in C^{1,\infty}((0,T) \times \mathbb{R}^N)$ such that $\operatorname{supp} \phi := \Omega$ is compact with $\phi(\cdot, T) = 0$. Then after multiplying (2.1) by $\phi \equiv \phi(x,t)$ and integrating over $\mathbb{R}^N$, we obtain

$$\int_\Omega u\, \phi \, dx\, dt = \int_\Omega S(t) f(x) \phi \, dx - \lambda i \int_\Omega \int_0^t S(t-s) |u(s)|^p \, ds \phi \, dx.$$

So after differentiating in time, we obtain

$$\frac{d}{dt} \int_\Omega u\, \phi \, dx\, dt = \int_\Omega \frac{d}{dt} (S(t) f(x) \phi) \, dx - \lambda i \int_\Omega \frac{d}{dt} \int_0^t S(t-s) |u(s)|^p \, ds \phi \, dx. \tag{3.8}$$

Now, using the properties of the semigroup $S(t)$ (see [1]), we have

$$\begin{aligned} \int_\Omega \frac{d}{dt}(S(t)f(x)\phi)\,dx &= i\int_\Omega A(S(t)f(x))\phi\,dx + \int_\Omega S(t)f(x)\partial_t\phi\,dx \\ &= i\int_\Omega S(t)f(x)A\phi\,dx + \int_\Omega S(t)f(x)\partial_t\phi\,dx, \end{aligned} \tag{3.9}$$

and

$$\begin{aligned} &\int_\Omega \frac{d}{dt}\int_0^t S(t-s)F(u)\,ds\phi\,dx \\ &= i\int_\Omega\int_0^t A(S(t-s)F(u))\,ds\,\phi\,dx + \int_\Omega F(u)\phi\,dx + \int_\Omega\int_0^t S(t-s)F(u)\,ds\,\partial_t\phi\,dx \\ &= i\int_\Omega\int_0^t S(t-s)F(u)\,ds\,A\phi\,dx + \int_\Omega F(u)\phi\,dx + \int_\Omega\int_0^t S(t-s)F(u)\,ds\,\partial_t\phi\,dx, \end{aligned} \tag{3.10}$$

where $F(u) := |u(t)|^p$. Thus, using (2.1), (3.9) and (3.10), we conclude that (3.8) implies

$$\frac{d}{dt}\int_\Omega u\,\phi\,dx\,dt = \int_\Omega u\,\partial_t\phi\,dx\,dt - i\int_\Omega u\,A\phi\,dx\,dt - i\lambda\int_\Omega F(u)\phi\,dx\,dt.$$

Finally, by integrating in time over $[0,T]$ and using that $\phi(\cdot,T)=0$, we complete the proof. $\square$

Let

$$\begin{aligned} T_m \equiv \sup\Big\{&T\in[0,+\infty)\,;\text{ there exists a unique solution } u \text{ to (2.1)} \\ &\text{such that } u\in C([0,T);L^2(\mathbb{R}^N))\cap L^r([0,T);L^\rho(\mathbb{R}^N))\Big\} \end{aligned}$$

be the maximal existence time of L^2-solution, where $1<p\le 1+\frac{2m}{N}$, $\rho=p+1$ and $\frac{2m}{r}=\frac{N}{2}-\frac{N}{\rho}$.

Theorem 3.6. *Let $1<p\le 1+\frac{2m}{N}$, $\lambda=\lambda_1+i\lambda_2\in\mathbb{C}\backslash\{0\}$ and $f\in L^2(\mathbb{R}^N)$. If the initial data $f=f_1+if_2$ satisfies (H1), then the life span $T_m<+\infty$ and the L^2-norm of the solution blows up at $t=T_m$.*

$$\liminf_{t\to T_m}\|u(t)\|_{L^2}=+\infty. \tag{3.11}$$

Proof. We assume the life span $T_m=+\infty$. Then u is also a global weak solution of (1.1)–(1.2) in the sense of Definition 3.1. Then we can apply Theorem 3.2 and obtain $u\equiv 0$. On the other hand, by the identity (3.1), we obtain

$$\int_{\mathbb{R}^N} f_2(x)\phi_1(x)\,dx = 0,$$

which is a contradiction. Therefore, we have $T_m<+\infty$.

Next, we show a blowup of the L^2-norm for a local solution u by using a contradiction argument again. First we assume

$$\liminf_{t\to T_m}\|u(t)\|_{L^2}<+\infty;$$

then there exists a sequence $\{t_n\}_{n\ge1}\subset[0,T_m)$ and a positive constant $M>0$ such that

$$\lim_{n\to+\infty} t_n = T_m, \tag{3.12}$$

$$\sup_{n\in\mathbb{N}} \|u(t_n)\|_{L^2} \leq M. \tag{3.13}$$

Thus for any $t_n \in \{t_n\}_{n\geq 1}$, by the estimate (3.13) and the local existence theorem, there exists a positive constant $T(M)$ independent on t_n such that we can construct a solution

$$u \in X := C([t_n, t_n + T(M)); L^2(\mathbb{R}^N)) \cap L^r([t_n, t_n + T(M)); L^\rho(\mathbb{R}^N));$$

to the integral equation (2.1). Moreover, since the limit of $\{t_n\}_{n\geq 1}$ exists, we can take $t_n \in [0, T_m)$ such that $T_m - \frac{T(M)}{3} < t_n < T_m$. For this $t_n \in [0, T_m)$, we can also construct a solution $u \in X$. But the estimate $t_n + T(M) > T_m$ is a contradiction to the definition of T_m. Therefore we obtain

$$\liminf_{t\to T_m} \|u(t)\|_{L^2} = +\infty,$$

which completes the proof. $\square$

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