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EXISTENCE OF GLOBAL AND BLOWUP SOLUTIONS FOR A SINGULAR SECOND-ORDER ODE

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Abstract. We study the existence of global solutions of a singular ordinary differential equation arising in the construction of self similar solution for a backward-forward parabolic equation. Also we present several numerical simulations and obtain an upper bound for the blowup time.

1. Introduction

The evolution equation

$$u_t = \frac{1}{2}[W'(u_x)]_x, \tag{1.1}$$

arises in the study of nonlinear elasticity and phase transitions [16, 17]). Here $W(p) = \frac{1}{2}(p^2-1)^2$ is the so called double well potential and equation (1.1) describes the L^2-gradient dynamics of the Energy

$$\frac{1}{2}\int_I W(u_x)dx, \tag{1.2}$$

where $I = (0,1)$. Both solving equation (1.1) and minimizing (1.2) are ill-posed problems. In particular, the equation (1.1) has a forward backward character depending on the sign of W'', while the functional (1.2) is non convex and admits infinitely many minimizers. Another interesting choice of the potential function is the following

$$W(p) = \frac{1}{2}\ln(1+p^2).$$

The corresponding evolution equation is the so called Perona-Malik equation [18] which has a similar (see [15]) ill-posed behavior as that of equation (1.1).

It is well known that the Perona-Malik equation admits global or not global solutions depending on the initial datum. In particular if the evolution is restricted in a bounded interval I with zero Neumann boundary conditions we have:

(1) If $|u_{0x}(x)| < 1$ for all $x \in I$ then there exists a unique global solution such that $|u_x(x,t)| < 1$ for all $t \geq 0$ and $x \in I$ (see [14]).
(2) If there exists a point $x_0 \in I$ such that $|u_{x0}(x_0)| > 1$ then then there are no global solutions (see [14] and [13]).

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In [13] the author also provided a bound for the maximum interval of time of existence.

We would like to remark that there is a large body of literature (see [2, 3], [4]-[10]) about the regularization of the ill-posed problems (1.1) and (1.2). In general the idea is to add an high order term depending on a small parameter ϵ which makes the equation well-posed (see for example [19, 3]). The further step is to study the limit of the solutions u_ε of the regularized problem [3] as $\varepsilon \to 0$.

Instead of considering a possible regularization of problem (1.1) or of (1.2), we limit our analysis to the class of self similar solutions (see [1]). In general self similar solutions are useful to understand the transient dynamics or the asymptotic behavior of solutions of partial differential equations. The idea of this work is to describe in details the global/non global existence for this class of solutions depending on the initial data. Consider the transformation

$$(u, x, t) \quad \to \quad (\hat{u}, \hat{x}, \hat{t}),$$

with

$$u = U\hat{u}, \quad x = L\hat{x}, \quad t = T\hat{t},$$

where U, L and T are abstract positive numbers (see [1] for a complete discussion). Then, by dimension Analysis, we have that the equation is invariant under the above transformation if and only if

$$\frac{U^2 T}{L^4} = 1, \quad \frac{T}{L^2} = 1,$$

this suggests that quantities $\frac{x}{t^{1/2}}$ and x are dimensionless and will play an important role in the study of the solutions. The standard procedure (see [1]) consists in looking for general Self-Similar Solutions of the form

$$\begin{aligned} u(x,t) &= t^\beta f(y), \\ y &= \frac{x}{t^\alpha}. \end{aligned} \tag{1.3}$$

We observe that

$$\begin{gathered} y_t = -\frac{\alpha}{t} y, \quad y_x = t^{-\alpha}, \\ y_{xx} = 0, \quad u_t = \beta t^{\beta-1} f(y) - \alpha t^{\beta-1} y f'(y), \\ u_x = t^{\beta-\alpha} f'(y), \quad u_{xx} = t^{\beta-2\alpha} f''(y). \end{gathered}$$

By putting the previous expression into (1.1) we obtain

$$\beta f(y) - \alpha y f'(y) = \frac{1}{2} W''(t^{\beta-\alpha} f'(y)) f''(y) t^{1-2\alpha},$$

from which, $\alpha = \beta = 1/2$. Then we obtain the equation

$$f(y) - y f'(y) = W''(f'(y)) f''(y),$$

that is,

$$f(y) - y f'(y) = 2[3(f')^2 - 1] f''(y). \tag{1.4}$$

Any linear function of the form $f(y) = ay$ satisfies the previous equation, giving rise to the well known class of affine solutions (see [13]):

$$u(x,t) = t^{1/2} f(y) = ax.$$

Then the interesting case is when

$$f(y) \not\equiv ay.$$

We first suppose that there exist C^2 solutions of (1.4) with non constant first derivative and constant second derivative:

$$f''(y) = b.$$

By substituting the expression $f(y) = f(0) + yf'(0) + \frac{1}{2}by^2$ into (1.4) we obtain the condition $b = f(0) = 0$ and we obtain again the above class of affine solutions. What it is expected is that the second derivative will play the important role for the blow-up of solution when the first derivative f' approaches the critical values $\pm\frac{1}{\sqrt{3}}$.

In the next sections we will consider the problem of finding the interval of existence of solutions of (1.4) distinguishing two cases. We will deal with solutions with infinite interval of existence $[0, +\infty)$ in section 2 with initial data satisfying

$$f(0) > 0 \text{ (resp } < 0), \quad f'(0) > \frac{1}{\sqrt{3}} \text{ (resp } < -\frac{1}{\sqrt{3}}).$$

The case of a finite interval $[0, y^*)$ of existence will be analyzed in section 3. This case corresponds to the choice $|f'(0)| < \frac{1}{\sqrt{3}}$, moreover, in the same section, we will deal with the remaining possible choices of initial data. In section 4 we provide a bound for the length of the interval $[0, y^*)$ in case of blow up of solutions. Moreover, using a method suggested in [13] for the Perona-Malik equation, we find a bound for the time interval of existence for non global solutions of (1.1). The results of sections 2-4 are illustrated by numerical experiments in section 5.

2. Global Solutions

In this section we study the case in which equation (1.4) admits global solutions $f(y)$ with $y \in [0, +\infty)$. The existence of such solutions depends on the values of $f'(0)$, moreover we will see that the second derivative is not identically zero and $f(y)$ approaches a linear solution of the form Ky in finite or infinite time.

Theorem 2.1. *Suppose that $f(0) > 0$ and $f'(0) > 1/\sqrt{3}$. Then the C^2 solutions of (1.4) are global and satisfy*

$$\lim_{y\to\infty} f'(y) = K > \frac{1}{\sqrt{3}}.$$

Moreover

$$\lim_{y\to\infty} [f(y) - yf'(y)] = 0,$$

with

$$f(y) - yf'(y) \geq 0, \quad \forall\, y \geq 0.$$

Moreover,

$$\lim_{y\to\infty} |f(y) - Ky| = 0, \quad \lim_{y\to\infty} f''(y) = 0,$$

with $f'' \not\equiv 0$ and $f''(y) \geq 0$ for all $y \geq 0$.

Proof. Using the hypothesis on $f(0)$ and $f'(0)$ we have

$$f(0) = 2[3(f'(0))^2 - 1]f''(0) > 0,$$

from which $f''(0) > 0$. For C^2 solutions we have that there exists $\delta > 0$:

$$f(y) > 0, \quad f'(y) > \frac{1}{\sqrt{3}}, \quad f''(y) > 0, \quad y \in [0, \delta),$$

from which f and f' are increasing in $[0, \delta)$. We have that

$$f(y) - yf'(y) = 2[3(f')^2 - 1]f''(y) > 0, \quad y \in (0, \delta),$$
$$[f(y) - yf'(y)]' = -yf''(y) < 0, \quad y \in (0, \delta),$$

from which $f(y) - yf'(y)$ is positive and decreasing in $(0, \delta)$ and the same is true for $f''(y)$ (since $f'(y)$ is increasing).

If $\delta = +\infty$ then

$$0 \le \lim_{y\to\infty} [f(y) - yf'(y)] = \lim_{y\to\infty} 2[3(f'(y))^2 - 1]f''(y). \tag{2.1}$$

From the continuity of $f'(y)$ we have that

$$\lim_{y\to\infty} f'(y) > \frac{1}{\sqrt{3}},$$

and as a consequence

$$\lim_{y\to\infty} f''(y) \ge 0.$$

Suppose that

$$\lim_{y\to\infty} f''(y) = K > 0,$$

then f' is increasing in $(0, \infty)$ and we have

$$\lim_{y\to\infty} 2[3(f'(y))^2 - 1] = +\infty.$$

On the other hand, since the term $f(y) - yf'(y)$ is positive and decreasing, then its limit is finite,

$$\lim_{y\to\infty} [f(y) - yf'(y)] = c \ge 0,$$

but this is a contradiction. Then

$$\lim_{y\to\infty} f''(y) = 0,$$

and, as a consequence, $\lim_{y\to\infty}[f(y) - yf'(y)] = 0$.

If $\delta \ne +\infty$ then $f(\delta) > 0$ and $f'(\delta) > 1/\sqrt{3}$ since both f and f' are increasing in $[0, \delta)$. The we must have $f''(\delta) = 0$, which means

$$f(\delta) - \delta f'(\delta) = 0.$$

Moreover,

$$([f(y) - yf'(y)]')_{y=\delta} = -\delta f''(\delta) = 0.$$

We claim that $f''(y) = 0$ in the interval $[\delta, +\infty)$. Then

$$f(y) - yf'(y) = f(\delta) - \delta f'(\delta) = 0, \quad y \in [\delta, +\infty),$$
$$f'(y) = f'(\delta), \quad y \in [\delta, +\infty),$$

from which

$$f(y) = \frac{f(\delta)}{\delta} y = f'(\delta) y, \quad y \in [\delta, +\infty).$$

To see that $f''(y)=0$ in $[\delta,+\infty)$, suppose that there exists $\nu>0$ such that

$$f''(y)>0,\quad f'(y)>\frac{1}{\sqrt{3}},\quad y\in(\delta,\nu).$$

As a consequence, the term $f(y)-yf'(y)$ is decreasing on it and negative. Then, by (1.4), we have

$$3[f'(y)]^2-1<0,\quad y\in(\delta,\nu),$$

but this is a contradiction of the fact that $f'(y)>\frac{1}{\sqrt{3}}$ on the same interval. The case $f''<0$ can be analyzed using a similar argument. □

Remark 2.2. The above proof also works for the case

$$f(0)<0,\quad f'(0)<-\frac{1}{\sqrt{3}}.$$

Since the equation is invariant for the transformation $f\to(-f)$.

3. Blow-up Solutions

We start by considering the case in which the solutions of (1.4) are not global. We prove that this is always the case in which $|f'(0)|<1/\sqrt{3}$ and this strictly depends on the blow up of the second derivative of f.

Theorem 3.1. *Suppose that*

$$f(0)>0,\quad 0<f'(0)<\frac{1}{\sqrt{3}},$$

then the non trivial C^2 solutions of equation (1.4) *are not global. In particular there exists a point $y_*>0$ such that*

$$\lim_{y\to y_*}f''(y)=-\infty.$$

Proof. By hypotheses we obtain

$$f(0)=2[3(f'(0))^2-1]f''(0),$$

from which $f''(0)<0$. By continuity there exists $\delta>0$ such that

$$f(y)>0,\quad f''(y)<0,\quad y\in(0,\delta),$$
$$f'(y)<\frac{1}{\sqrt{3}},\quad y\in(0,\delta).$$

Then both f and $f(y)-yf'(y)$ are positive and increasing and they cannot change sign at $y=\delta$. The only possibility is that $f'(\delta)=0$, from which

$$f(\delta)=-2f''(\delta)>0.$$

Then there exist $\nu>\delta$ such that

$$f(y)>0,\quad f''(y)<0,\quad y\in(\delta,\nu),$$

and, as a consequence (using that $W''(f')$ must remain negative),

$$f'(y)\in\big(-\frac{1}{\sqrt{3}},0\big),\quad y\in(\delta,\nu).$$

Then both f and f' are decreasing and the term $f(y)-yf'(y)$ is positive and increasing. We need to study these functions at $y=\nu$. We observe that $W''(f'(\nu))\in(-2,0]$ then $f''(\nu)\neq 0$ and remains negative at $y=\nu$. Then we have only three cases:

(a) $f(\nu)=0,\ f'(\nu)=-\frac{1}{\sqrt{3}}$,
(b) $f(\nu)=0,\ f'(\nu)>-\frac{1}{\sqrt{3}}$,
(c) $f(\nu)>0,\ f'(\nu)=-\frac{1}{\sqrt{3}}$.

Cases (a) and (c) are compatible if

$$\lim_{y\to\nu} f''(y)=-\infty.$$

Case (b): By hypotheses there exists $\eta>\nu$ such that

$$f(y)<0,\quad f'(y)<-\frac{1}{\sqrt{3}},\quad f''(y)<0,\quad y\in(\nu,\eta).$$

Moreover, the term $f(y)-yf'(y)$ is positive and increasing and as a consequence f'' is negative and decreasing. Then, since f and f' are negative and decreasing we must have $f'(\eta)=-1/\sqrt{3}$, and, as a consequence,

$$\lim_{y\to\eta} f''(y)=-\infty. \tag{3.1}$$

If $\eta=+\infty$, since f, f', and f'' are negative and decreasing we have that

$$\lim_{y\to\infty} f''(y)\in[-\infty,0),$$

and then there exists a point $y_0>\nu$ such that $f'(y_0)=-\frac{1}{\sqrt{3}}$, and we obtain again (3.1).

The case $\nu=+\infty$ can be analyzed using the same idea of the case $\eta=\infty$, observing that f' and f'' are negative and decreasing. $\square$

Remark 3.2. From the previous proof and the arguments used in Case 1, we can extend the statement of the previous theorem for the hypotheses

$$f(0)<0,\quad f'(0)\in(-1/\sqrt{3},0).$$

Also, it is easy to see that the same proof works for

$$f(0)<0,\quad f'(0)\in(0,1/\sqrt{3}).$$

In fact is it sufficient to invert the signs of f'' and of the term $f(y)-yf'(y)$. This means that Theorem 3.1 is true for the more general hypotheses $|f'(0)|<1/\sqrt{3}$.

We note that the following cases are outside the hypotheses of Theorems 2.1 and 3.1:

$$f(0)<0,\quad f'(0)>\frac{1}{\sqrt{3}}, \tag{3.2}$$

$$f(0)>0,\quad f'(0)<-\frac{1}{\sqrt{3}}. \tag{3.3}$$

We analyze the case (3.2) since the other case can be studied in a similar way. What we expect is both global and blow up behavior:

(1) The term $f(y)-yf'(y)$ remains negative while f' approaches $\frac{1}{\sqrt{3}}$ and $f''\to-\infty$ as y approaches some critical point y^*.
(2) The term $f(y)-yf'(y)$ approaches zero at some point y_0 while f' remains away from the critical value $\frac{1}{\sqrt{3}}$. The derivative of the solution will remain constant, that is $f'(y)=\frac{f(y_0)}{y_0}$, for $y\ge y_0$.

The two situations depend on the value of f' at zero and that of the term $f(y) - yf'(y)$ near zero. In particular near zero the derivative of the term $f(y) - yf'(y)$ satisfies

$$|[f(y) - yf'(y)]'| \approx y|f''(0)| = y\frac{|f(0)|}{W''(0)}.$$

Then, if $f'(0)$ is very close to $1/\sqrt{3}$, we expect that the term $f(y)-yf'(y)$ converge to zero faster than f' to $1/\sqrt{3}$.

From the initial data we have that $f''(0) < 0$, then by continuity we can find δ such that

$$f(y) < 0, \quad f''(y) < 0, \quad y \in [0, \delta). \tag{3.4}$$

The left hand side of (1.4) is negative at zero and it is increasing in $[0, \delta)$:

$$f(y) - yf'(y) < 0, \quad y \in [0, \delta), \tag{3.5}$$

and

$$[f(y) - yf'(y)]' = -yf''(y) > 0, \quad y \in [0, \delta). \tag{3.6}$$

As a consequence we have

$$W''(f'(y)) > 0, \quad y \in [0, \delta), \quad \text{that is} \quad f'(y) > \frac{1}{\sqrt{3}}, \quad y \in [0, \delta). \tag{3.7}$$

Since f' is decreasing, $W''(f')$ will be decreasing and f'' increasing in the same interval. We need to analyze the behavior of the solutions at $y = \delta$, there are several cases:

(A) $f(\delta) = 0$, $f''(\delta) < 0$, $f(\delta) - \delta f'(\delta) < 0$,
(B) $f(\delta) < 0$, $f''(\delta) = 0$, $f(\delta) - \delta f'(\delta) < 0$,
(C) $f(\delta) < 0$, $f''(\delta) < 0$, $f(\delta) - \delta f'(\delta) = 0$,
(D) $f(\delta) = 0$, $f''(\delta) = 0$, $f(\delta) - \delta f'(\delta) < 0$,
(E) $f(\delta) = 0$, $f''(\delta) < 0$, $f(\delta) - \delta f'(\delta) = 0$,
(F) $f(\delta) < 0$, $f''(\delta) = 0$, $f(\delta) - \delta f'(\delta) = 0$,
(G) $f(\delta) = 0$, $f''(\delta) = 0$, $f(\delta) - \delta f'(\delta) = 0$.

Case (G) has already been studied in Theorem 2.1, the solution is $f(y) = \frac{f(\delta)}{\delta} y$ for $y \geq \delta$ while the hypotheses of cases (B) and (D) are not consistent since the term $W''(f')$ is bounded then if f'' is zero also the term $f(y)-yf'(y)$ must be zero. Moreover (E) and (F) are inconsistent since they would imply that $f'(\delta) = 0$ and $f'(\delta) < 0$ respectively. We pass to analyze the others cases.

Case (C) As a consequence of the hypotheses we have $3(f'(\delta))^2 - 1 = 0$, that is $f'(\delta) = 1/\sqrt{3}$, from which

$$f(\delta) - \frac{\delta}{\sqrt{3}} = 0,$$

but this is not possible since $f(\delta)$ is negative.

Case (A) We have that

$$-\delta f'(\delta) = 2[3(f'(\delta'))^2 - 1]f''(\delta),$$

from which we have $f'(\delta) > 1/\sqrt{3}$, that is, f is increasing. Then there exists $\nu > \delta$ such that

$$f(y) > 0, \quad f''(y) < 0, \quad f(y) - yf'(y) < 0, \quad y \in (\delta, \nu)$$

from which we obtain that f is increasing and

$$f(y) > 0, \quad f'(y) > \frac{1}{\sqrt{3}}, \quad y \in (\delta, \nu).$$

Then at the point ν we have the following cases: the hypotheses

$$f''(\nu) = 0, \quad f(\nu) - \nu f'(\nu) < 0,$$

are not compatible with $f'(\nu) \geq \frac{1}{\sqrt{3}}$ and f' decreasing; while the hypotheses

$$f''(\nu) < 0, \quad f(\nu) - \nu f'(\nu) = 0,$$

are only compatible with $f'(\nu) = 1/\sqrt{3}$. From which

$$f(y) = \frac{y}{\sqrt{3}}, \quad y \geq \nu.$$

The other possibility is that

$$f''(\nu) < 0, \quad f(\nu) - \nu f'(\nu) < 0, \quad f'(\nu) = \frac{1}{\sqrt{3}}.$$

This means that

$$\lim_{y \to \nu} f''(\nu) = -\infty.$$

By these arguments we expect that for initial data satisfying (3.2) and (3.3) there are both global and no global solutions. We will illustrate these two cases in section 5 below.

4. An estimate for the blowup point y_*

Using a slight modification of a method proposed by Gobbino in [13] for the Perona-Malik equation, we derive an estimate for the maximum interval of time of existence for the solutions of (1.1), in the spatial interval I=$(0,1)$, whose initial datum $u(x,0) = v(x)$ satisfies

$$v_x(I) \cap \Big(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \Big) \neq \emptyset.$$

We observe that this is related to the finite time blowup of the the self similar solutions. In fact

$$u_x(x,t) = \Big[t^{1/2} f\Big(\frac{x}{t^{1/2}} \Big) \Big]_x = f'(y),$$

and we have observed that if for $|f'(0)| < 1/\sqrt{3}$ then there exists a point $y^* \neq +\infty$ such that $\lim_{y \to y^*} f''(y) = -\infty$.

Let ψ be a C^2 function, then (see [13]):

$$\begin{aligned} \frac{d}{dt} \int_I \psi(u_x) dx &= \int_I \psi'(u_x) u_{xt} dx = -\int_I \psi''(u_x) u_{xx} u_t dx + (\psi'(u_x) u_t)_{|\partial I} \\ &= -\int_I \psi''(u_x) W''(u_x) [u_{xx}]^2 dx, \end{aligned}$$

where we have used Neumann boundary conditions. Integrating in the time interval (t_1, t_2) we obtain

$$\int_I \psi(u_x(x,t_2)) dx + \int_{t_1}^{t_2} \int_I [\psi''(u_x) W''(u_x)] [u_{xx}]^2 \, dx \, dt = \int_I \psi(u_x(x,t_1)) dx. \quad (4.1)$$

Let $x_0 \in I$ such that $v(x_0) \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and chose

$$\psi(p) = \begin{cases} \big(|p| - \frac{1}{\sqrt{3}}\big)^4, & p \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}), \\ 0, & \text{elsewhere.} \end{cases}$$

Then we observe that $\psi(p) \geq 0$ and $\psi''(p)W''(p) \leq 0$ for all p. Using inequality (4.1) with $t_1 = 0$ and $t_2 = t$ we observe that the term on the right is positive. Since $\psi''(p)W''(p) \leq 0$ then also the first term of the left-hand side must be positive which means that for every t there exists at least one point $x_0(t) \in (-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}})$, and if u is of class C^2 then by continuity for any $t \geq 0$ there exists $x_1(t) \in I$ such that $|u_x(x_1(t), t)| = 1/\sqrt{3}$. Then by Neumann boundary conditions:

$$\begin{aligned} &|W'(u_x(x_1(t), t)) - W'(u_x(0, t))| \\ &= W'(\frac{1}{\sqrt{3}}) \leq \int_0^{x_1(t)} |[W'(u_x)]_x| dx \\ &\leq \{x_1(t)\}^{1/2} \Big\{ \int_0^{x_1(t)} \{[W'(u_x)]_x\}^2 dx \Big\}^{1/2} \leq \Big\{ \int_I [W''(u_x)u_{xx}]^2 dx \Big\}^{1/2}. \end{aligned}$$

Then by putting $\psi(p) = W(p)$ in the inequality (4.1) we obtain:

$$t[W'(1/\sqrt{3})]^2 \leq \int_0^t \int_I [W''(u_x)]^2 [u_{xx}]^2 \, dx \, dt \leq \int_I W(v_x) dx,$$

from which we have the bound of the maximum interval of existence

$$T \leq \frac{1}{[W'(\frac{1}{\sqrt{3}})]^2} \int_I W(v_x) dx.$$

Again using an argument from [13], if u is only of class C^1 we obtain the same result using the inequality

$$\int_I \psi(u_x(x,t)) dx \geq \int_I \psi(u_x(x,0)) dx.$$

For the solution of equation (1.4) we have a similar result:

Theorem 4.1. *Suppose that $f(0) > 0$ and $f'(0) = 0$. Then the point y_* of blowup for f'' satisfies*

$$y_* \leq \frac{4}{3\sqrt{3} f(0)}.$$

Remark 4.2. We observe that the hypotheses $f'(0) = 0$ is not restrictive since it well describe the case in which

$$f(0) > 0, \; f'(0) \in (0, 1/\sqrt{3}), \quad \text{or} \quad f(0) < 0, \; f'(0) \in (-1/\sqrt{3}, 0).$$

In fact, for both cases, by the proof of Theorem 3.1, there exists a point at which the derivative pass trough zero.

Proof. By the hypotheses we have

$$W'(f'(y_*)) - W'(f'(0)) = W'(f'(y_*)) = \int_0^{y^*} [W'(f'(y))]_y dy = \int_0^{y_*} [f(y) - y f'(y)] dy.$$

Since the term $f(y) - yf'(y)$ is positive and increasing in $[0, y_*)$ then we obtain $y_* f(0) \le W'(f'(y_*))$. In details:

$$y_* \le \frac{4}{3\sqrt{3} f(0)}.$$

As expected the value of y_* is proportional to $f(0)^{-1}$. We give an illustration of this result in Section 5 below. $\square$

5. Numerical Experiments

In this section we give a brief description of the results from the previous sections. In particular we consider two cases in which $f''(y)$ remains bounded and $f''(y) \to \pm\infty$ respectively. We end the article by testing the bound of the point y_* of blow up of the second derivative of the solution.

Experiment 1. We consider a first example in which $f(0) = 0.1$ and $f'(0) = 0.6 > 1/\sqrt{3}$. We represent the solution $f(y)$, together with $f'(y)$ and $f''(y)$, in figure 1 and compare it with the straight line $f'(0)y + f(0)$ in figure 2. We observe that $f \to ay + b$ with $a > f'(0)$.

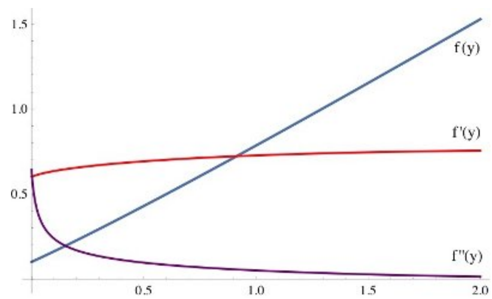


Figure 1. Experiment 1: the functions $f(y)$, $f'(y)$, and $f''(y)$.

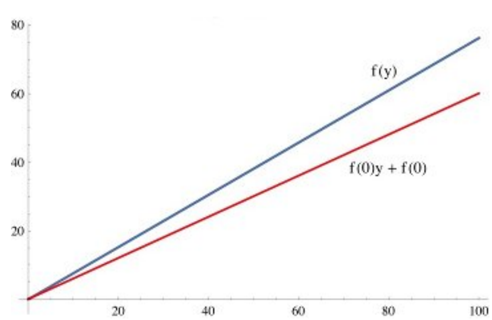


Figure 2. Experiment 1: the solution $f(y)$ and the straight line $f'(0)y + f(0)$.

By Theorem 2.1 we have that the left-hand side of (1.4) is positive and decreasing and this is represented in figure 3.

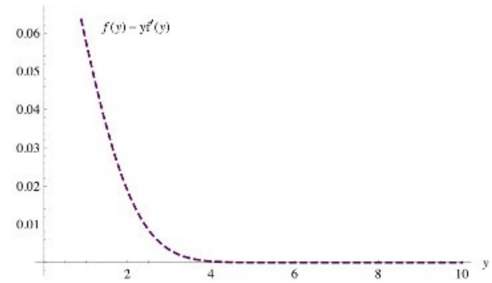


FIGURE 3. Experiment 1: the term $f(y) - yf'(y)$.

Experiment 2. A second experiment is proposed to describe the case of blow up of the second derivative. Here the initial data are:

$$f(0) = 0.1, \quad f'(0) = 0.5 < \frac{1}{\sqrt{3}}.$$

The solution $f(y)$ together with its first and second derivative is represented in figure 4 while we compare it with the straight line $f'(0)y + f(0)$ in figure 5.

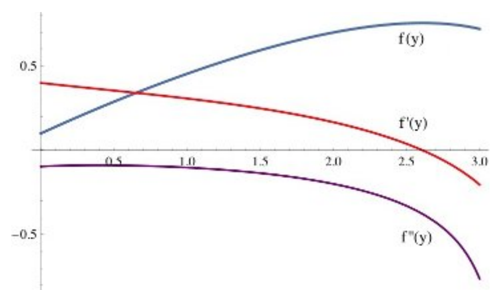


FIGURE 4. Experiment 2: The functions $f(y)$, $f'(y)$ and $f''(y)$.

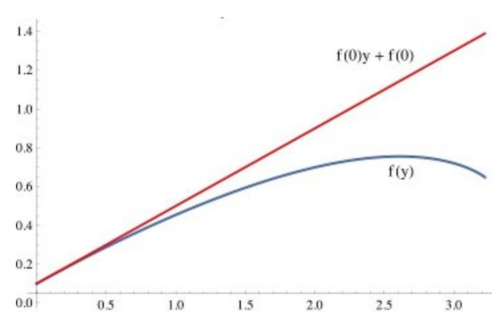


FIGURE 5. Experiment 2: the solution $f(y)$ and the straight line $f'(0)y + f(0)$.

In figure 6 below we represent the term $f(y) - yf'(y)$ that is positive and increasing as suggested by Theorem 3.1.

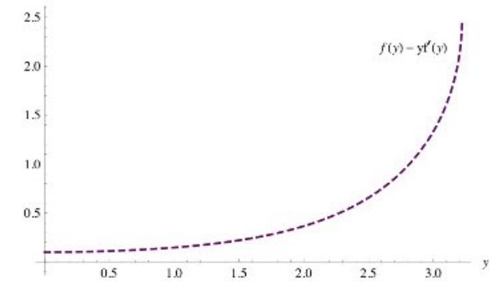


FIGURE 6. Experiment 2: The left-hand side. of the equation is positive and increasing

Finally in figure 5-5 we describe the blow up of the second derivative: the first derivative tends to the value $-\frac{1}{\sqrt{3}}$ while the second derivative diverges to $-\infty$.

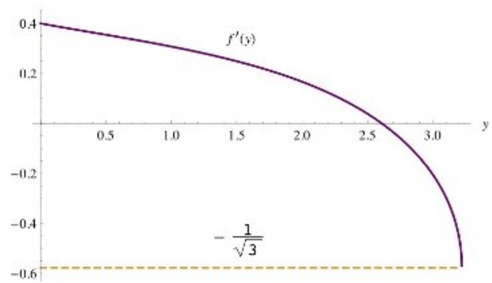


FIGURE 7. Experiment 2: The first derivative of the solutions tends to a zero of $W''(\cdot)$.

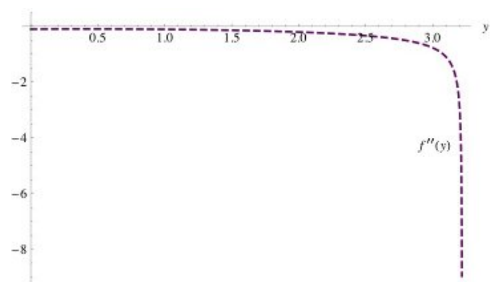


FIGURE 8. Experiment 2: The second derivative diverges to $-\infty$.

5.1. **Experiment 3.** Here we want to illustrate the cases not covered by Theorems 2.1 and 3.1; that is,

$$f(0) < 0, \quad f'(0) > \frac{1}{\sqrt{3}}.$$

We have seen in Section 3 that both global solutions and blow up are possible. If $f'(0)$ is big enough then the solution reaches a zero and the solution is global, while if not, the solution remains negative and there is the blow up of the second derivative. For a first case we have chosen

$$f(0) = -0.1, \quad f'(0) = \frac{1}{\sqrt{3}} + 0.1$$

The initial datum $f'(0)$ is very close to the critical point, then the term $f(y)-yf'(y)$ remains negative (figure 10) while f' reaches the critical point (see figure 11). This makes the second derivative diverge (figure 12)

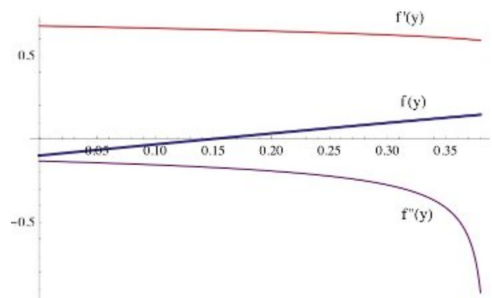


Figure 9. Experiment 3: the solution and its derivatives when $f'(0) = \frac{1}{\sqrt{3}} + 0.1$.

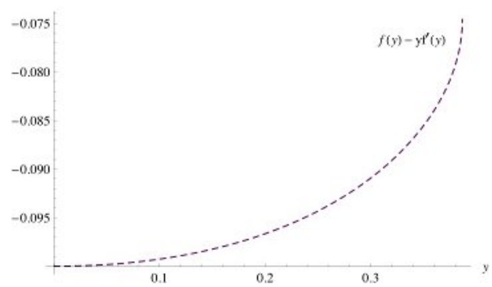


Figure 10. Experiment 3: the term $f(y) - yf'(y)$ remains negative in the interval of existence of the solution.

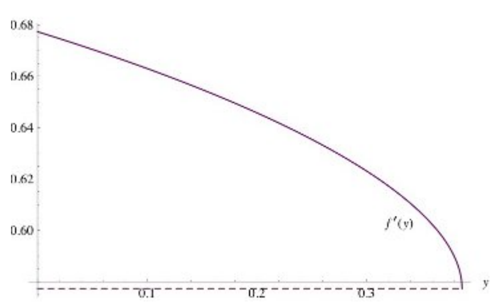


Figure 11. Experiment 3: the first derivative reaches the critical value.

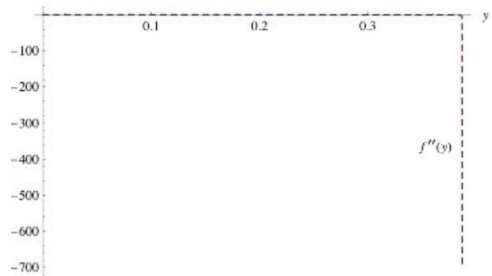


FIGURE 12. Experiment 3: the second derivative of the solution diverges to $-\infty$.

The other possibility discussed in section 3 is that if $f'(0)$ is big enough then the term $f(y) - yf'(y)$ approaches a zero at the same time that f' approaches the critical value $\frac{1}{\sqrt{3}}$. Initial data have been chosen as follows

$$f(0) = -0.1, \quad f'(0) = \frac{1}{\sqrt{3}} + 0.15$$

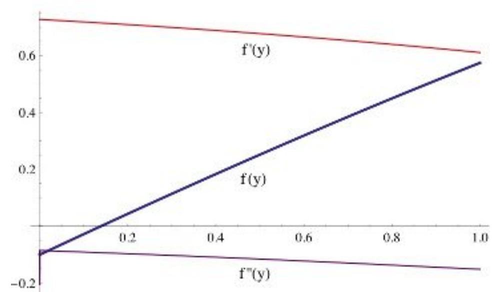


FIGURE 13. Experiment 3: the solution and its derivatives for $f'(0) = \frac{1}{\sqrt{3}} + 0.15$.

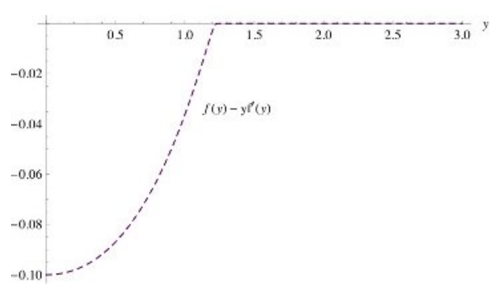


FIGURE 14. Experiment 3: the term $f(y) - yf'(y)$ is zero in the interval $[y_0, +\infty)$.

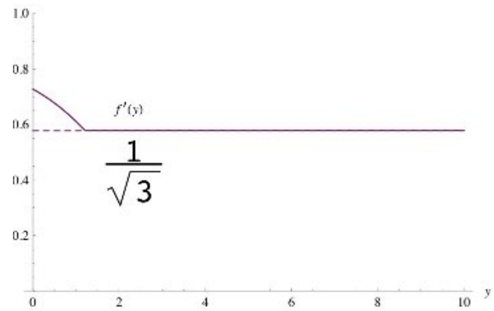
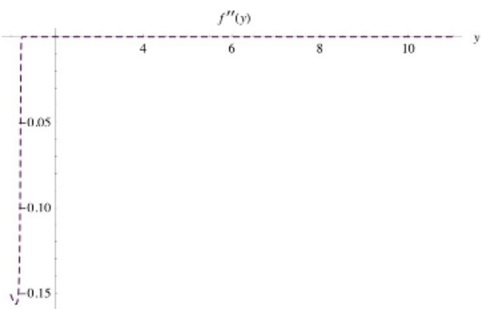


FIGURE 15. Experiment 3: the first derivative is $\frac{1}{\sqrt{3}}$ in the interval $[y_0, +\infty)$.



NTGlobal4

FIGURE 16. Experiment 3: the second derivative is zero in the interval $[y_0, +\infty)$.

5.2. **Experiment 4.** We consider a numerical experiment to illustrate the bound on the point y_* of the blow up of the second derivative of the solution. The initial data are $f(0) = 0.1$, $f'(0) = 0$. In this case the point of blow up satisfies the above bound:

$$y_* \approx 1.29 \le \frac{4}{3\sqrt{3}f(0)} \approx 1.54$$

In figure 17 we represent the solution while in figure 18 we represent the first derivative. We observe that it converges to $-\frac{1}{\sqrt{3}}$ as it approaches y_*.

Finally, in figure 19 below, we represent the second derivative of the solution close to y_*.

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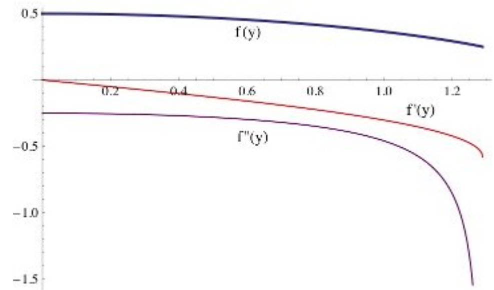
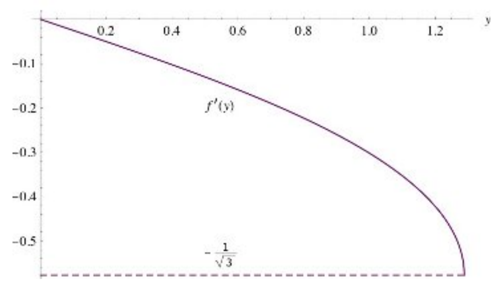


FIGURE 17. Experiment 4: the Solution f and its first and second derivative for $f(0) = 0.5$ and $f'(0) = 0$.



ExpBound2

FIGURE 18. Experiment 4: the first derivative of the solution approaches $-\frac{1}{\sqrt{3}}$ as y tends to y_*.

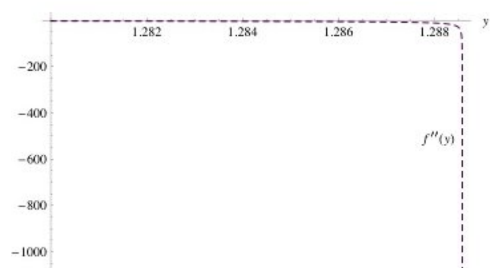


FIGURE 19. Experiment 4: the second derivative of the solution diverges to $-\infty$ as y tends to $y_* \approx 1.29$.

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