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GLOBAL STABILITY OF SIR MODELS WITH NONLINEAR INCIDENCE AND DISCONTINUOUS TREATMENT

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Abstract. In this article, we study an SIR model with nonlinear incidence rate. By defining the Filippov solution for the model and constructing suitable Lyapunov functions, we show that the global dynamics are fully determined by the basic reproduction number R_0, under certain conditions on the incidence rate and treatment functions. When $R_0 \leq 1$ the disease-free equilibrium is globally asymptotically stable, and when $R_0 > 1$ the unique endemic equilibrium is globally asymptotically stable.

1. Introduction

The asymptotic behavior of SIR models with the general nonlinear incidence rate have been studied by many researchers; see for example [5, 6, 7, 9, 11, 12] and the references therein. One of the basic SIR epidemic models is described as follows:

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= \mu - f(S(t), I(t)) - \mu S(t),\\
\frac{\mathrm{d}I}{\mathrm{d}t} &= f(S(t), I(t)) - (\mu+\sigma)I(t),\\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \sigma I(t) - \mu R(t),
\end{aligned}$$

where the host population size is constant and is divided into three classes: susceptible, infectious and recovered. We denote these classes by $S(t)$, $I(t)$ and $R(t)$, respectively (that is, $S + I + R = 1$). The positive constant μ represents the birth/death rate and the positive constant σ represents the recovery rate. The function $f(S, I)$ denotes the nonlinear incidence rate. It is assumed that all newborns are susceptible, and the immunity received upon recovery is permanent. A number of works display the threshold behavior of the model. That is, for a basic reproduction number R_0 of some form, the disease dies out for $R_0 \leq 1$, whereas it is permanent for $R_0 > 1$.

Note that treatment is an essential measure for the precautions taken of some disease (e.g. measles, phthisis, influenza). In [10], the SIR model with a limited resource for treatment is studied. In [11], the SIR model with a constant removal rate

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of infective individuals is analyzed. Many classic epidemic models with treatment are modelled by continuous systems. However, it is known that the therapeutic measures vary from each period of the transmission. For example, a small number of infectives can not get treatment timely and completely during the early spread for lacking of high attention paid by the society. After a period of time, when people are aware of the seriousness, the therapeutic measures would be enhanced and improved sharply. The treatment function related to the number of the infective should be piecewise continuous. Thus, it is meaningful to introduce discontinuous treatment into classical infectious disease model to reinforce the efficacy on prevention and treatment of the epidemic.

In this article, we consider the following 2-dimensional SIR model with a class of nonlinear incidence rate of $S(t)g(I(t))$ and discontinuous treatment:

$$\begin{aligned}\frac{\mathrm{d}S}{\mathrm{d}t} &= \mu - S(t)g(I(t)) - \mu S(t),\\ \frac{\mathrm{d}I}{\mathrm{d}t} &= S(t)g(I(t)) - (\mu+\sigma)I(t) - h(I(t)),\end{aligned} \tag{1.1}$$

where the function $h(I)$ denotes the removal rate of infective individuals because of the treatment of infectives. The initial condition for system (1.1) is

$$S(0) = S_0 \geq 0, \quad I(0) = I_0 \geq 0, \tag{1.2}$$

The purpose of our study is to show the threshold behavior of system (1.1) using stability theory based on the Filippov solution [1, 2, 3, 4].

The article is organized as follows. In Section 2, some elementary assumptions on the functions g and h will be given, and the basic reproduction number R_0 is provided. After defining the Filippov solution for a given initial condition, the equilibrium points are discussed. The local and global stability of equilibrium points are analyzed in Sections 3 and 4, respectively.

2. Basic reproduction number and Equilibrium

To define the basic reproduction number R_0 and indicate the existence of equilibriums, we give some hypotheses.

- (H1) $h(I(t)) = \varphi(I)\cdot I$, where $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is piecewise continuous and monotone nondecreasing; that is φ is continuous apart from a countable number of isolated points $\{\xi_k\}$, $\varphi(\xi_k^+) > \varphi(\xi_k^-)$ holds, where $\varphi(\xi_k^-)$ and $\varphi(\xi_k^+)$ represent the left and right limits of φ at $\{\xi_k\}$, respectively, and have only finite number of discontinuous points in any compact subset of $\mathbb{R}_+$. We assume that φ is continuous for $I = 0$.
- (H2) $g(0) = 0$, $g'(I) > 0$ and $g''(I) \leq 0$ for $I \geq 0$. Furthermore, we assume that the function $\phi(I) = g(I)/I$ is bounded.

Remark 2.1. (1) We modify the definition of treatment rate to be $h(I(t)) = \varphi(I)I$, which means that the treatment rate is proportional to the number of the infectives.

(2) Because of (H2), we have that $\phi(I)$ is a monotone decreasing function on $I > 0$.

(3) By the assumptions, it is easy to find that system (1.1) always has a disease-free equilibrium point $E_0 = (1, 0)$.

Then we define the basic reproduction number R_0 for model (1.1) as

$$R_0 = \frac{g'(0)}{\mu + \sigma + \varphi(0)}.$$

We shall assume that (H1) and (H2) hold in the rest of this article. Let us recall the definition of the Filippov solution to system (1.1) with initial condition (1.2).

Definition 2.2 ([1, 2, 3, 4])**.** A vector function $(S(t), I(t))$ defined on $[0, T)$, $T \in (0, +\infty]$, is called the Filippov solution of (1.1)-(1.2), if it is absolutely continuous on any subinterval $[t_1, t_2]$ of $[0, T)$, $S(0) = S_0$, $I(0) = I_0$, and satisfies the differential inclusion

$$\begin{gathered}\frac{\mathrm{d}S}{\mathrm{d}t} = \mu - Sg(I) - \mu S,\\ \frac{\mathrm{d}I}{\mathrm{d}t} \in Sg(I) - (\mu + \sigma)I - \bar{\text{co}}[h(I)],\end{gathered} \tag{2.1}$$

for almost all of $t \in [0, T)$, where $\bar{\text{co}}[h(I)]$ is the interval $[h(I^-), h(I^+)]$, and $h(I^-)$ and $h(I^+)$ represent the left and right limits of function $h(\cdot)$ at I, respectively.

By noting that the equilibrium (S^*, I^*) of (2.1) satisfies

$$\begin{gathered}0 = \mu - S^* g(I^*) - \mu S^*,\\ 0 \in S^* g(I^*) - (\mu + \sigma)I^* - \bar{\text{co}}[h(I^*)],\end{gathered}$$

from the measurable selection theorem (see [2]), there exists a unique constant

$$\xi^* = S^* g(I^*) - (\mu + \sigma)I^* \in \bar{\text{co}}[h(I^*)]$$

such that

$$\begin{gathered}\mu - S^* g(I^*) - \mu S^* = 0,\\ S^* g(I^*) - (\mu + \sigma)I^* - \xi^* = 0.\end{gathered}$$

The following proposition shows that for positive initial values, the solution of (1.1) is positive and is bounded. $T = \infty$ implies that the solution exists globally.

Proposition 2.3. *Let $(S(t)$ and $I(t))$ be the unique solution of* (1.1)-(1.2)*. Then $S(t)$ and $I(t)$ are positive for all $t > 0$. Moreover, this solution is bounded and thus exists globally.*

Proof. By the fact $dS/(dt)|_{S=0} > 0$ and $S_0 \geq 0$, we obtain the positivity of $S(t)$. Since $\bar{\text{co}}[h(0)] = \{0\}$, and φ is continuous at $I = 0$, there exists $\delta > 0$ such that $\varphi(I)$ is continuous for $|I| < \delta$. Also, the second differential inclusion in (2.1) can be rewritten as

$$\frac{\mathrm{d}I}{\mathrm{d}t} = I\Big[\frac{Sg(I)}{I} - (\mu + \sigma) - \varphi(I)\Big], \tag{2.2}$$

for $|I| < \delta$. If $I_0 = 0$, we derive that $I(t) = 0$, for all $t \in [0, T)$. If $I_0 > 0$, then we claim $I(t) > 0$, for all $t \in [0, T)$. Otherwise, let $t_1 = \inf\{t | I(t) = 0\} \in [0, T)$. Because of the continuity of $I(t)$ on $[0, T)$, there exists $\theta > 0$ such that $t_1 - \theta > 0$ and $0 < I(t) < \delta$ for $t \in [t_1 - \theta, t_1)$. Integrating both sides of (2.2) from $t_1 - \theta$ to t_1 gives

$$0 = I(t_1) = I(t_1 - \theta)e^{\int_{t_1-\theta}^{t}[\frac{Sg(\xi)}{\xi} - (\mu+\sigma) - \varphi(\xi)]d\xi} > 0,$$

which is a contradiction. Thus, we have $I(t) > 0$ for $t \in [0, T)$.

Since semi-continuous set-valued mapping $(S, I) \to (\mu - Sg(I) - \mu S, Sg(I) - (\mu + \sigma)I - \bar{\text{co}}[h(I)])$ has compact and convex image, we have that (1.1) has a solution

$(S(t), I(t))$ defined on $[0, t_0)$, which satisfies the initial condition (1.2). From (2.1), we know that

$$\frac{d(S+I)}{dt} \in \mu - \mu S - (\mu+\sigma)I - \bar{\text{co}}[h(I)],$$

for any $\nu \in \bar{\text{co}}[h(I)]$. If $S + I > 1$, then we have $\mu - \mu(S+I) - \sigma I - \nu \leq 0$. Thus, $0 \leq S + I \leq \max\{S_0 + I_0, 1\}$, which yields the boundedness of $(S(t), I(t))$ on $[0, t_0)$. Moreover, $(S(t), I(t))$ is defined and bounded on $[0, +\infty)$. $\square$

The following result concerns the existence and uniqueness of an endemic equilibrium.

Proposition 2.4. *If $R_0 > 1$, then there exists an unique endemic equilibrium E_*.*

Proof. We look for solutions (S^*, I^*) of the differential inclusion $dS/(dI) = 0$ and $0 \in dI/(dt)$:

$$S^* = \frac{\mu}{\mu + g(I^*)},$$

$$0 \in \frac{\mu g(I^*)}{[\mu + g(I^*)]I^*} - (\mu+\sigma) \in \bar{\text{co}}[\varphi(I^*)].$$

Let

$$L(I) = \frac{\mu g(I)}{[\mu + g(I)]I} - (\mu+\sigma) \in \bar{\text{co}}[\varphi(I)].$$

If $R_0 > 1$, then $g'(0) > \mu + \sigma + \varphi(0)$ so that

$$L(0) = g'(0) - (\mu+\sigma) > \varphi(0) \geq 0.$$

A direct calculation gives

$$\begin{aligned} L'(I) &= \frac{\mu}{[\mu + g(I)]^2 I^2}[\mu g'(I)I - \mu g(I) - g^2(I)] \\ &= \frac{\mu}{[\mu + g(I)]^2 I^2}[g'(I)\mu I - \mu g'(\xi)I - g^2(I)], \end{aligned}$$

where, $\xi \in (0, I)$. Observe that $g(0) = 0$ and $g''(I) \leq 0$ for $I > 0$, we have $L'(I) < 0$. That is, $L(I)$ is monotonously decreasing on $I > 0$. Since $\varphi(I)$ is nondecreasing for $I > 0$, and $L(I) \leq 0$ for $I \leq (\mu+\sigma)[\mu + g(I)]/[\mu g(I)]$, the set $\{I | L(I) \geq \varphi(I^+), I > 0\}$ should be bounded. Let $\tilde{I} = sup\{I | L(I) \geq \varphi(I^+), I > 0\}$, then $L(\tilde{I}) \geq \varphi(\tilde{I}^-)$. We claim that $L(\tilde{I}) \in [\varphi(\tilde{I}^-), \varphi(\tilde{I}^+)]$. Suppose on the contrary, there exists a constant $\delta > 0$ such that $L(\tilde{I} + \delta) > \varphi(\tilde{I} + \delta) = \varphi((\tilde{I} + \delta)^+)$, which contradicts to the definition of $\tilde{I}$. This leads to $L(\tilde{I}) \in [\varphi(\tilde{I}^-), \varphi(\tilde{I}^+)]$ and $\tilde{I}$ is a positive solution to $L(I) \in \bar{\text{co}}[\varphi(I)]$.

Next, we prove the uniqueness of E_*. Let $I_1 = \tilde{I}$ and $I_2 \neq I_1$ is another solution to $L(I) \in \bar{\text{co}}[\varphi(I)]$. There exist $\eta_i \in \bar{\text{co}}[\varphi_i(I)]$ $(i = 1, 2)$ satisfying

$$\begin{aligned} \frac{\mu m_1}{(\mu + m_1)I_1} - (\mu+\sigma) - \eta_1 = 0, \\ \frac{\mu m_2}{(\mu + m_2)I_2} - (\mu+\sigma) - \eta_2 = 0, \end{aligned}$$

where we denote $m_i = g(I_i)$ $(i = 1, 2)$ for convenience such that

$$\mu = (\mu + \sigma + \eta_1)\big(1 + \frac{\mu}{m_1}\big)I_1,$$

$$\mu = (\mu + \sigma + \eta_2)\big(1 + \frac{\mu}{m_2}\big)I_2.$$

Thus, we have

$$\begin{aligned} 0 &= (\mu+\sigma)(I_1-I_2)+\eta_1(I_1-I_2)+I_2(\eta_1-\eta_2) \\ &\quad+\mu(\mu+\sigma)\Big(\frac{I_1}{m_1}-\frac{I_2}{m_2}\Big)+\mu\Big(\frac{\eta_1 I_1}{m_1}-\frac{\eta_2 I_2}{m_2}\Big) \\ &= [(\mu+\sigma)+\eta_1+I_2H_1+\mu(\mu+\sigma)H_2+\mu H_3](I_1-I_2), \end{aligned} \tag{2.3}$$

where

$$\begin{gathered} H_1=(\eta_1-\eta_2)/(I_1-I_2), \\ H_2=(I_1/m_1-I_2/m_2)/(I_1-I_2), \\ H_3=(\eta_1 I_1/m_1-\eta_2 I_2/m_2)/(I_1-I_2). \end{gathered}$$

Note that $H_1 \geq 0$ for the monotonicity of φ, $H_2 \geq 0$ for the monotonicity of $x/g(x)$, and $H_3 \geq 0$ for the monotonicity of $x/g(x)$ and $\varphi(I)$. This yields a contradiction for (2.3). Hence, the uniqueness of E_* is proved. ☐

3. Local stability of equilibria

In this section, we prove the following results, which guarantee the local asymptotical stability of the disease-free equilibrium, and the endemic equilibrium of (1.1).

Theorem 3.1. *If $R_0 < 1$, then the disease-free equilibrium E_0 of* (1.1) *is locally asymptotically stable.*

Proof. The Jacobian matrix of (1.1) at $E_0=(1,0)$ is

$$J_0=\begin{bmatrix} -\mu & -g'(0) \\ 0 & g'(0)-(\mu+\sigma)-\varphi(0) \end{bmatrix}.$$

Let λ_1 and λ_2 be the characteristic roots of J_0. We obtain

$$\begin{gathered} \lambda_1+\lambda_2=-\mu+(R_0-1)[\mu+\sigma+\varphi(0)]<0, \\ \lambda_1\lambda_2=-\mu(R_0-1)[\mu+\sigma+\varphi(0)]>0, \end{gathered}$$

for $R_0<1$. This proves that E_0 is locally asymptotically stable. ☐

Theorem 3.2. *If $R_0 > 1$ and φ is differential at I^*, then E_0 is unstable and E_* is locally asymptotically stable.*

Proof. The Jacobian matrix of system (1.1) at E_* is

$$J_*=\begin{bmatrix} -\mu-g(I^*) & -S^*g'(I^*) \\ g(I^*) & S^*g'(I^*)-(\mu+\sigma)-\varphi(I^*)-\varphi'(I^*)I^* \end{bmatrix}.$$

Observe that system (1.1) at E_* can be rewritten into the form

$$\begin{gathered} 0=\mu-S^*g(I^*)-\mu S^*, \\ 0=\frac{S^*g(I^*)}{I^*}-(\mu+\sigma)-\varphi(I^*). \end{gathered}$$

Let α_1 and α_2 be the characteristic roots of J_*. According to the second equality of the above system, since $g''(I)<0$, we can obtain that

$$\begin{aligned} \alpha_1+\alpha_2 &= S^*g'(I^*)-\frac{S^*g(I^*)}{I^*}-\mu-g(I^*)-\varphi'(I^*)I^* \\ &= S^*\Big[\frac{I^*g'(I^*)-g(I^*)}{I^*}\Big]-\mu-g(I^*)-\varphi'(I^*)I^* \end{aligned}$$

$$= S^*[g'(I^*) - g'(\xi)] - \mu - g(I^*) - \varphi'(I^*)I^* < 0,$$

where $\xi \in (0, I^*)$.

$$\begin{aligned}
&\alpha_1\alpha_2 \\
&= \mu \frac{S^* g(I^*)}{I^*} - \mu S^* g'(I^*) + g(I^*)[(\mu+\sigma) + \varphi'(I^*)I^* + \varphi(I^*)] + \mu\varphi'(I^*)I^* \\
&= \mu S^*[g'(\zeta) - g'(I^*)] + g(I^*)[(\mu+\sigma) + \varphi'(I^*)I^* + \varphi(I^*)] + \mu\varphi'(I^*)I^* > 0,
\end{aligned}$$

where $\zeta \in (0, I^*)$. Hence, E_* is locally asymptotically stable. □

4. Global stability of equilibria

In this section, we discuss the global stability of disease-free equilibrium E_0 and endemic equilibrium E_*. The following theorem indicates that the disease can be eradicated in the host population if $R_0 \le 1$, while $R_0 > 1$, the disease is permanent.

Theorem 4.1. *If $R_0 \le 1$, then E_0 is globally asymptotically stable.*

Proof. We move E_0 to the origin firstly by setting $x = S - 1$. Then, system (2.1) becomes

$$\begin{gathered}
\frac{dx}{dt} = -\mu x - xg(I) - g(I), \\
\frac{dI}{dt} \in xg(I) + g(I) - (\mu+\sigma)I - \bar{\text{co}}[\varphi(I)]I.
\end{gathered}$$

We now consider the Lyapunov function $V_1(x, I) = x^2/2 + I$. We can calculate that $\nabla V_1 = (x, 1)$. Then, the constant l, which means that the set $L_l = \{(x, I) \in R^1 \times R^1 | V_1(x, I) \le l\}$ can be arbitrarily large. Let

$$G(x, I) = \begin{bmatrix} -\mu x - xg(I) - g(I) \\ xg(I) + g(I) - (\mu+\sigma)I - \bar{\text{co}}[\varphi(I)]I \end{bmatrix}.$$

It is clear that semi-continuous and set-valued mapping G has compact and convex image. For each $\nu = (\nu_1, \nu_2) \in G(x, I)$, there exists a measurable function $\eta(t) \in \bar{\text{co}}[\varphi(I)]$ corresponding to $(x(t), I(t))$; that is,

$$\nu = \begin{bmatrix} -\mu x - xg(I) - g(I) \\ xg(I) + g(I) - (\mu+\sigma)I - \eta(t)I \end{bmatrix}.$$

We now calculate the derivative of $V_1(x, I)$ by

$$\begin{aligned}
\langle \nabla V_1, \nu \rangle &= -\mu x^2 - x^2 g(I) + g(I) - (\mu+\sigma)I - \eta(t)I \\
&= -\mu x^2 - x^2 g(I) + \Big[\frac{g(I)}{I} - (\mu+\sigma) - \eta(t)\Big]I.
\end{aligned}$$

It follows from Remark 2.1 that

$$g(I)/I \le \lim_{I \to 0^+} g(I)/I = g'(0).$$

Using $\eta(t) > \varphi(0)$, we have

$$\langle \nabla V_1, \nu \rangle \le -\mu x^2 - x^2 g(I) + [\mu + \sigma + \varphi(0)](R_0 - 1)I.$$

Since $R_0 \le 1$, we claim $\langle \nabla V_1, \nu \rangle \le 0$. Observe that if $R_0 < 1$, then $Z_{V_1} = \{(x, I) \in R^1 \times R^1 | \langle \nabla V_1, \nu \rangle = 0\} = \{(0, 0)\}$. That is, if $R_0 = 1$, then $Z_{V_1} = \{(0, 0)\} \cup \{(0, I) | \eta(t) = \varphi(0), I \ne 0\}$. In addition, if $x = x(t) \equiv 0$, then $I = I(t) \equiv 0$. Thus, $\{(0, 0)\}$ is the largest weak invariant subset of $\bar{Z}_{V_1} \cap L_l$ for all $l > 0$, which leads to the global asymptotical stability of $\{(0,0)\}$ due to the invariance theorem (see [2]), and thus E_0 is globally asymptotical stable. $\square$

Theorem 4.2. *If $R_0 > 1$, then the unique endemic equilibrium E_* is globally asymptotical stable.*

Proof. Denote
$$\eta^* = S^* g(I^*)/I^* - (\mu+\sigma) \in \bar{\text{co}}[\varphi(I^*)].$$
We consider
$$V_2 = S - \int_\varepsilon^S S^*/\tau d\tau + I - I^* \ln I,$$
where ε is a small unspecified parameter. Note that V_2 is well-defined and continuous for all $S > \varepsilon$ and $I > 0$, and $\nabla V_2 = (1 - S^*/S, 1 - I^*/I)$. Let
$$F(S,I) = \begin{bmatrix} \mu - Sg(I) - \mu S \\ Sg(I) - (\mu+\sigma)I - \bar{\text{co}}[\varphi(I)]I \end{bmatrix}.$$
It is easy to check that F is a semi-continuous and set-valued mapping, which has compact and convex image. Thus, for any $\omega = (\omega_1, \omega_2) \in F(S,I)$, there exists a measurable function $\eta(t) \in \bar{\text{co}}[\varphi(I)]$ such that
$$\omega = \begin{bmatrix} \mu - Sg(I) - \mu S \\ Sg(I) - (\mu+\sigma)I - \eta(t)I \end{bmatrix}.$$
Thus, it follows that
$$\begin{aligned}
\langle \nabla V_2, \omega \rangle &= \mu - Sg(I) - \mu S - \mu \frac{S^*}{S} + S^* g(I) + \mu S^* \\
&\quad + \Big[Sg(I) - (\mu+\sigma+\eta)I - Sg(I)\frac{I^*}{I} + (\mu+\sigma+\eta)I^* \Big] \\
&= [\mu S^* + S^* g(I^*)] - \mu S - \Big[\mu S^* + S^* g(I^*)\Big]\frac{S^*}{S} + S^* g(I) + \mu S^* \\
&\quad - \Big[\frac{S^* g(I^*)}{I^*} - \eta^* + \eta\Big] I - Sg(I)\frac{I^*}{I} + \Big[\frac{S^* g(I^*)}{I^*} - \eta^* + \eta\Big] I^* \\
&= \mu S^* \Big(1 - \frac{S}{S^*} - \frac{S^*}{S} + 1\Big) + 2S^* g(I^*) - S^* g(I^*)\frac{S^*}{S} + S^* g(I) \\
&\quad - S^* g(I^*)\frac{I}{I^*} + (\eta^* - \eta)I - Sg(I)\frac{I^*}{I} - (\eta^* - \eta)I^* \\
&= \mu S^* \Big(1 - \frac{S}{S^*}\Big)\Big(1 - \frac{S^*}{S}\Big) + 2S^* g(I^*) - S^* g(I^*)\frac{S^*}{S} - Sg(I)\frac{I^*}{I} \\
&\quad - S^* g(I^*)\frac{g(I^*)}{g(I)}\frac{I}{I^*} + S^* g(I) - S^* g(I^*)\frac{I}{I^*} + S^* g(I^*)\frac{g(I^*)}{g(I)}\frac{I}{I^*} \\
&\quad + (\eta^* - \eta)(I - I^*) \\
&= \mu S^* (1 - \frac{S}{S^*})(1 - \frac{S^*}{S}) + S^* g(I^*)\Big[3 - \frac{S^*}{S} - \frac{Sg(I)}{S^* g(I^*)}\frac{I^*}{I} - \frac{g(I^*)}{g(I)}\frac{I}{I^*}\Big] \\
&\quad + S^* g(I^*)\Big[-1 - \frac{I}{I^*} + \frac{g(I)}{g(I^*)} + \frac{g(I^*)}{g(I)}\frac{I}{I^*}\Big] + (\eta^* - \eta)(I - I^*) \\
&= \mu S^* \Big(1 - \frac{S}{S^*}\Big)\Big(1 - \frac{S^*}{S}\Big) + S^* g(I^*)\Big[3 - \frac{S^*}{S} - \frac{Sg(I)}{S^* g(I^*)}\frac{I^*}{I} - \frac{g(I^*)}{g(I)}\frac{I}{I^*}\Big] \\
&\quad + S^* g(I^*)\Big[\frac{I}{I^*} - \frac{g(I)}{g(I^*)}\Big]\Big[\frac{g(I^*)}{g(I)} - 1\Big] + (\eta^* - \eta)(I - I^*).
\end{aligned}$$

One can see that

$$\big(1-\frac{S}{S^*}\big)\big(1-\frac{S^*}{S}\big)\le 0.$$

Since the arithmetic mean is greater than or equal to the geometric mean, we have

$$\frac{S^*}{S}+\frac{Sg(I)}{S^*g(I^*)}\frac{I^*}{I}+\frac{g(I^*)}{g(I)}\frac{I}{I^*}\ge 3.$$

The monotonicity of $g(I)$ and $\phi(I)$ (see Remark 2.1) gives

$$\frac{I}{I^*}\le\frac{g(I)}{g(I^*)}\le 1 \text{ for } 0<I<I^*,\quad 1\le\frac{g(I)}{g(I^*)}\le\frac{I}{I^*} \text{ for } I\ge I^*.$$

Then we have

$$\big[\frac{I}{I^*}-\frac{g(I)}{g(I^*)}\big]\big[\frac{g(I^*)}{g(I)}-1\big]\le 0,$$

for all $I>0$. Since $(\eta^*-\eta)(I-I^*)\le 0$ for the monotony of φ, we obtain $\frac{dV_2}{dt}\big|_{(2.1)}\le 0$.

Clearly, we see that $\frac{dV_2}{dt}=0$ holds only when $S=S^*$, $I=I^*$ and that E_* is the only equilibrium of that system. This implies the globally asymptotical stability of E_*. □

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