*Electronic Journal of Differential Equations*, Vol. 2015 (2015), No. 297, pp. 1–12.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

# SINGULAR CRITICAL ELLIPTIC PROBLEMS WITH FRACTIONAL LAPLACIAN

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Abstract. In this article, we consider the existence of solutions of the critical problem with a Hardy term for fractional Laplacian

$$\begin{gathered}(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = u^{2_s^*-1} \quad \text{in } \Omega,\\ u > 0 \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega,\end{gathered}$$

where $\Omega \subset \mathbb{R}^N$ is a smooth bounded domain and $0 \in \Omega$, $\mu$ is a positive parameter, $N > 2s$ and $s \in (0,1)$, $2_s^* = \frac{2N}{N-2s}$ is the critical exponent. $(-\Delta)^s$ stands for the spectral fractional Laplacian. Assuming that $\Omega$ is non-contractible, we show that there exists $\mu_0 > 0$ such that $0 < \mu < \mu_0$, there exists a solution. We also discuss a similar problem for the restricted fractional Laplacian.

## 1. Introduction

In this article, we consider the existence of solutions for the critical problem with a Hardy term and fractional Laplacian

$$\begin{gathered}(-\Delta)^s u - \mu \frac{u}{|x|^{2s}} = u^{2_s^*-1} \quad \text{in } \Omega,\\ u > 0 \quad \text{in } \Omega,\\ u = 0 \quad \text{on } \partial\Omega\end{gathered} \tag{1.1}$$

in a smooth bounded domain $\Omega \subset \mathbb{R}^N$ and $0 \in \Omega$, where $\mu$ is a positive parameter, $N > 2s$ and $s \in (0,1)$, $2_s^* = \frac{2N}{N-2s}$ is the critical exponent. The operator $(-\Delta)^s$ is the spectral Laplacian defined in section 2.

In the case $s = 1$, such a problem has been extensively studied, see [8, 11, 14, 16, 21, 27] etc. It is known that problem (1.1) with $s = 1$ has no nontrivial solutions if $\mu \ge 0$ and $\Omega$ is star shaped [2]. However, the situation becomes different if the domain $\Omega$ has nontrivial topology. In [17], a nontrivial solution was found for problem (1.1) with $s = 1$ and $\mu = 0$, if $\Omega$ is an annulus. Then it was shown in [3] that there exists a nontrivial solution of (1.1) with $s = 1$ and $\mu = 0$, if $\Omega$ has nontrivial topology. If $\mu > 0$, there is a Hardy term in (1.1) with $s = 1$. In [15], it proved that problem (1.1) with $s = 1$ admits a solution in a non-contractible

2010 *Mathematics Subject Classification.* 35J20, 35J25, 35J61.
*Key words and phrases.* Fractional Laplacian; singular critical problem; non-contractible domain.

Submitted September 6, 2015. Published December 3, 2015.

domain. Since (1.1) with $s = 1$ is a critical problem, it involves the ground state solution of the problem in the whole space

$$\begin{gathered} -\Delta u - \mu \frac{u}{|x|^2} = u^{2^*-1} \quad \text{in } \mathbb{R}^N, \\ u > 0 \quad \text{in } \mathbb{R}^N. \end{gathered} \tag{1.2}$$

The ground state solutions of (1.2) are found in [25] for $\mu = 0$ and in [27] for $\mu \neq 0$.

Recently, Secchi et al [22] proved that Coron type problem admits a solution for problem (1.1) with $\mu = 0$ and the restricted fractional Laplacian; see section 2 for a definition. Similarly, the argument in [22] relies on, among other things, the explicit form of the minimizer of the problem

$$\Lambda_s = \inf_{u \in \dot{H}^s(\mathbb{R}^N), u \not\equiv 0} \frac{\int_{\mathbb{R}^N} |(-\Delta)^{s/2} u(x)|^2 \, dx}{\big( \int_{\mathbb{R}^N} |u(x)|^{2_s^*} \, dx \big)^{2/2_s^*}}, \tag{1.3}$$

where the space $\dot{H}^s(\mathbb{R}^N)$ is defined as the completion of $C_0^\infty(\mathbb{R}^N)$ under the norm

$$\|u\|^2_{\dot{H}^s(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi, \tag{1.4}$$

here $\hat{u}$ denotes the Fourier transform of $u$. In $\mathbb{R}^N$, the operator $(-\Delta)^{s/2}$, $s \in \mathbb{R}$ is defined by the Fourier transform

$$(\widehat{(-\Delta)^{s/2}} u)(\xi) = |\xi|^s \hat{u}(\xi) \tag{1.5}$$

for $u \in C_0^\infty(\mathbb{R}^N)$. Therefore, for $s > 0$, we have

$$\|(-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi. \tag{1.6}$$

For $N > 2s$, the minimizing problem $\Lambda_s$ in (1.3) is related to the fractional Sobolev embedding $\dot{H}^s(\mathbb{R}^N) \hookrightarrow L^{\frac{2N}{N-2s}}(\mathbb{R}^N)$. The continuity of this inclusion corresponds to the inequality

$$\|u\|^2_{L^{2_s^*}(\mathbb{R}^N)} \leq \Lambda_s^{-1} \|u\|^2_{\dot{H}^s(\mathbb{R}^N)}. \tag{1.7}$$

The best constant $\Lambda_s$ in (1.7) was computed in [10]. A minimizer $u$ of $\Lambda_s$ weakly solves the problem

$$(-\Delta)^s u = |u|^{2^*_{s,\alpha}-2} u \quad \text{in } \mathbb{R}^N \tag{1.8}$$

up to a multiplying constant. Using the moving plane method for integral equations, Chen et al [9] classified the solutions of an integral equation, which is related to problem (1.8). Positive regular solutions of (1.8), and then the minimizers of $\Lambda_s$ are precisely given by

$$U_\varepsilon(x) = \Big( \frac{\varepsilon}{\varepsilon^2 + |x - x_0|^2} \Big)^{\frac{N-2s}{2}} \tag{1.9}$$

for $\varepsilon > 0$ and $x_0 \in \mathbb{R}^N$.

In this paper, we consider the existence of solutions of problem (1.1) with $0 < \mu < \mu_H$ and $s \in (0,1)$ in a non-contractible domain $\Omega$, where $\mu_H$ is the best constant in the Hardy inequality. Problem (1.1) is related to the variational problem

$$\Lambda_{s,\mu} = \inf_{u \in \dot{H}^s(\mathbb{R}^N), u \not\equiv 0} \frac{\|(-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^N)} - \mu \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} \, dx}{\big( \int_{\mathbb{R}^N} |u(x)|^{2_s^*} \, dx \big)^{\frac{2}{2_s^*}}}. \tag{1.10}$$

Although minimizers of $\Lambda_{1,\mu}$ were found explicitly in [27] for $s = 1$, it is not the case for $s \in (0, 1)$. So in our argument, we need to avoid using it. Our main result for the spectral Laplacian is as follows.

**Theorem 1.1.** *Suppose $\Omega$ is not contractible. Then, there exists $0 < \mu_0 < \mu_H$ such that for each $\mu \in (0, \mu_0)$, there exists a solution of $(P_{s,\mu})$.*

For the restricted fractional Laplacian $(-\Delta_{|\Omega})^s$, we consider the problem

$$\begin{gathered} (-\Delta_{|\Omega})^s u - \mu \frac{u}{|x|^{2s}} = u^{2_s^*-1} \quad \text{in } \Omega, \\ u > 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \mathbb{R}^N \setminus \Omega. \end{gathered} \tag{1.11}$$

Similarly, we have the following result.

**Theorem 1.2.** *Suppose $\Omega$ is not contractible. Then, there exists $0 < \mu_0 < \mu_H$ such that for each $\mu \in (0, \mu_0)$, there exists a solution of* (1.11).

The article is organized as follows. After some preparations in section 2, we prove Theorems 1.1 and 1.2 in section 3.

## 2. Sobolev-Hardy inequality

In this section, we develop some properties of minimizers of $\Lambda_{s,\mu}$, and give the definition of fractional operator $(-\Delta)^s$. First, we define for each $s \geq 0$, the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \{u \in L^2(\mathbb{R}^N) : |\xi|^s \hat{u}(\xi) \in L^2(\mathbb{R}^N)\}$$

via the Fourier transform

$$\hat{u}(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-ix\cdot\xi} u(x)\, dx.$$

For $s \in (0, 1)$, it is known from [19] that there holds

$$\int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi = C_{s,N} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx\, dy, \tag{2.1}$$

where $C_{s,N}$ is a positive constant. This provides an alternative norm

$$\|u\|_{\dot{H}^s(\mathbb{R}^N)} = \Big( \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^2}{|x - y|^{N+2s}} \, dx\, dy \Big)^{1/2}$$

on $\dot{H}^s(\mathbb{R}^N)$. If $N > 2s$, the optimal constant $\mu_{s,N}$ was found in [28] for the fractional Hardy inequality

$$\mu_H \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} \, dx \leq \int_{\mathbb{R}^N} |\xi|^{2s} |\hat{u}(\xi)|^2 \, d\xi, \tag{2.2}$$

where $u \in C_0^\infty(\mathbb{R}^N)$. By a denseness argument, we have

$$\mu_H \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} \, dx \leq \int_{\mathbb{R}^N} |(-\Delta)^{s/2} u(x)|^2 \, dx \tag{2.3}$$

for $u \in \dot{H}^s(\mathbb{R}^N)$, where $\mu_H = 4^s \frac{\Gamma^2(\frac{N+2s}{4})}{\Gamma^2(\frac{N-2s}{4})}$. If $0 < \mu < \mu_H$, we may verify that

$$|u|_{\dot{H}^s(\mathbb{R}^N)} := \Big( \|(-\Delta)^{s/2} u\|^2_{L^2(\mathbb{R}^N)} - \mu \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}} \, dx \Big)^{1/2}$$

defines an equivalent norm on $\dot{H}^s(\mathbb{R}^N)$. Therefore, there exists $C > 0$ such that for any $u \in \dot{H}^s(\mathbb{R}^N)$,

$$\Big(\int_{\mathbb{R}^N} |u(x)|^{2_s^*}\,dx\Big)^{\frac{2}{2_s^*}} \le C\Big[\int_{\mathbb{R}^N} |(-\Delta)^{s/2}u(x)|^2\,dx - \mu\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^{2s}}\,dx\Big], \tag{2.4}$$

where $2_s^* = \frac{2N}{N-2s}$. Define

$$\Lambda_{s,\mu} = \inf_{u\in \dot{H}^s(\mathbb{R}^N),u\not\equiv 0} \frac{|u|^2_{\dot{H}^s(\mathbb{R}^N)}}{\big(\int_{\mathbb{R}^N} |u(x)|^{2_s^*}\,dx\big)^{\frac{2}{2_s^*}}}. \tag{2.5}$$

It is proved in [13] that $\Lambda_{s,\mu} > 0$ is achieved if $\mu \ge 0$.

We remark that any minimizer of $\Lambda_{s,\mu}$ does not change sign, and is radially symmetric. Indeed, let $u$ be a minimizer. By formula (A.11) in [23],

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2}|u||^2\,dx \le \int_{\mathbb{R}^N} |(-\Delta)^{s/2}u|^2\,dx.$$

Hence, $|u|$ is also a minimizer, and we have $u > 0$. Denote by $u^*$ the symmetric-decreasing rearrangement of $u$. By strict rearrangement inequalities in [12], $u^*$ is also a minimizer of $\Lambda_{s,\mu}$. Therefore, by strict rearrangement inequalities again, $u(x) = u^*(x - A)$ for some $A \in \mathbb{R}^N$. Moreover, any minimizer $u$ of $\Lambda_{s,\mu}$ weakly solves (1.1) with $\Omega = \mathbb{R}^N$ up to multiplying a constant. In the case $\Omega = \mathbb{R}^N$, since both $u(x)$ and $u^*(x)$ solve equation (1.1) , and (1.1) is not translation invariant, we obtain that $A = 0$, that is $u = u^*$.

Next, we define fractional Laplacians in a bounded domain. There are two types of fractional Laplacians in bounded domainds, one is the spectral fractional Laplacian, another one is the restricted fractional Laplacian.

In a bounded domain $\Omega \subset \mathbb{R}^N$, we define the spectral fractional Laplacian $(-\Delta)^s$ as follows. Let $\{\lambda_k, \varphi_k\}_{k=1}^\infty$ be the eigenvalues and corresponding eigenfunctions of the Laplacian operator $-\Delta$ in $\Omega$ with zero Dirichlet boundary values on $\partial\Omega$ normalized by $\|\varphi_k\|_{L^2(\Omega)} = 1$, i.e.

$$-\Delta\varphi_k = \lambda_k\varphi_k \quad \text{in } \Omega; \quad \varphi_k = 0 \quad \text{on } \partial\Omega.$$

For any $u \in L^2(\Omega)$, we may write

$$u = \sum_{k=1}^\infty u_k\varphi_k, \quad \text{where} \quad u_k = \int_\Omega u\varphi_k\,dx.$$

We define the space

$$H = \{u = \sum_{k=1}^\infty u_k\varphi_k \in L^2(\Omega) : \sum_{k=1}^\infty \lambda_k^{2s}u_k^2 < \infty\}, \tag{2.6}$$

which is equipped with the norm

$$\|u\|_H = \Big(\sum_{k=1}^\infty \lambda_k^{2s}u_k^2\Big)^{1/2}.$$

For any $u \in H$, the spectral fractional Laplacian $(-\Delta)^s$, is defined by

$$(-\Delta)^s u = \sum_{k=1}^\infty \lambda_k^s u_k\varphi_k. \tag{2.7}$$

The space $H$ defined in (2.6) is the interpolation space $(H_0^2(\Omega), L^2(\Omega))_{s,2}$, see [1, 18, 26]. It was shown in [18] that $(H_0^2(\Omega), L^2(\Omega))_{s,2} = H_0^s(\Omega)$ if $0 < s < 1$ and $s \neq 1/2$; while $(H_0^2(\Omega), L^2(\Omega))_{\frac{1}{2},2} = H_{00}^{1/2}(\Omega)$, where

$$H_{00}^{1/2}(\Omega) = \{u \in H^{1/2}(\Omega) : \int_\Omega \frac{u^2(x)}{d(x)}\,dx < \infty\},$$

and $d(x) = \operatorname{dist}(x, \partial\Omega)$ for all $x \in \Omega$.

Now, using the idea in [7], for any $u \in H_0^s(\Omega)$, we may define the extension $w = E_s(u)$ of $u$ as the solution $w \in H_{0,L}^1(\mathcal{C}_\Omega)$ of the problem

$$\begin{gathered} -\operatorname{div}(y^{1-2s}\nabla w) = 0, \quad \mathcal{C}_\Omega = \Omega \times (0,\infty), \\ w = 0, \quad \partial_L\mathcal{C}_\Omega = \partial\Omega \times (0,\infty), \\ w = u, \quad \Omega \times \{0\}, \end{gathered} \tag{2.8}$$

where

$$H_{0,L}^1(\mathcal{C}_\Omega) = \big\{w \in L^2(\mathcal{C}_\Omega) : w = 0 \text{ on } \partial_L\mathcal{C}_\Omega,\ \kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w|^2\,dx\,dy < \infty\big\}$$

is a Hilbert space with the norm

$$\|w\|_{H_{0,L}^1(\mathcal{C}_\Omega)}^2 = \kappa_s \int_{\mathcal{C}_\Omega} y^{1-2s}|\nabla w|^2\,dx\,dy.$$

The extension operator $E$ is an isometry between $H_0^s(\Omega)$ and $H_{0,L}^1(\mathcal{C}_\Omega)$. That is

$$\|E(u)\|_{H_{0,L}^1(\mathcal{C}_\Omega)} = \|u\|_{H_0^s(\Omega)}. \tag{2.9}$$

It was shown in [7], see also [5], that

$$-\kappa_s \lim_{y\to 0^+} y^{1-2s}\frac{\partial w}{\partial y} = (-\Delta)^s u, \tag{2.10}$$

where $(-\Delta)^s$ is the spectral fractional Laplacian. We remark that if $\Omega = \mathbb{R}^N$, for any $u \in H^s(\mathbb{R}^N)$, the extension $w = E(u)$ of $u$ is given by

$$\begin{gathered} -\operatorname{div}(y^{1-2s}\nabla w) = 0, \quad \mathbb{R}_+^{N+1}, \\ w = u, \quad \mathbb{R}^N, \end{gathered} \tag{2.11}$$

then by [7], problem (2.11) corresponds to the fractional Laplacian given by the Fourier transform in (1.5), which also satisfies (2.10). In this sense, the fractional Laplacian given in (1.5) can be approached by the spectral fractional Laplacian through extending the domain $\Omega$ in (2.8) to $\mathbb{R}^N$. In the sequel, we use the same notation $(-\Delta)^s$ to denote these two operators.

Using this sort of extension, we may reformulate the nonlocal problem (1.1) in a local way; that is,

$$\begin{gathered} -\operatorname{div}(y^{1-2s}\nabla w) = 0, \quad \mathcal{C}_\Omega, \\ w = 0, \quad \partial_L\mathcal{C}_\Omega, \\ \kappa_s y^{1-2s}\frac{\partial w}{\partial \nu} = \mu\frac{u}{|x|^{2s}} + u^{2_s^*-1} \quad \Omega \times \{0\}, \end{gathered} \tag{2.12}$$

where $\frac{\partial}{\partial\nu}$ is the outward normal derivative.

The other type of fractional Laplacian is the restricted fractional Laplacian defined by the following formula:

$$(-\Delta_{|\Omega})^s u(x) = c_{N,s}\, P.V. \int_{\mathbb{R}^N} \frac{u(x)-u(y)}{|y-x|^{N+2s}}\, dy$$

for all functions $u$ which are zero outside $\Omega$. Similarly, the extension problem is given as follows.

$$\begin{gathered} \operatorname{div}(y^{1-2\alpha}\nabla w) = 0 \quad \text{in } \mathbb{R}^{N+1}, \\ w = 0 \quad \text{in } \mathbb{R}^N \setminus \Omega, \\ y^{1-2\alpha}\frac{\partial w}{\partial \nu} = \mu \frac{u}{|x|^{2s}} + u^{2_s^*-1} \quad \text{on } \Omega. \end{gathered} \tag{2.13}$$

## 3. Existence of solutions

In this section, we first show that problem (1.1) admits a nontrivial solution in a non-contractible domain. Define the functional

$$I_\mu(u) = \frac{1}{2}\int_\Omega \big(|(-\Delta)^{s/2}u|^2 - \mu\frac{u^2}{|x|^{2s}}\big)\, dx - \frac{1}{2_s^*}\int_\Omega |u|^{2_s^*}\, dx$$

on $H_0^s(\Omega)$, where $(-\Delta)^s$ is the spectral factional Laplacian defined in (2.7), and the functional

$$J_\mu(u) = \frac{1}{2}\int_{\mathbb{R}^N} \big(|(-\Delta)^{s/2}u|^2 - \mu\frac{u^2}{|x|^{2s}}\big)\, dx - \frac{1}{2_s^*}\int_{\mathbb{R}^N} |u|^{2_s^*}\, dx$$

on $\dot{H}^s(\mathbb{R}^N)$, where $(-\Delta)^s$ is given in (1.5). It is discussed in section 2 the relation between the spectral factional Laplacian and the fractional Laplacian defined by the Fourier transform.

We consider the minimizing problem

$$c_\mu = \inf\big\{J_\mu(u) : u \in \mathcal{N}_{\mu,\mathbb{R}^N}\big\},$$

where $\mathcal{N}_{\mu,\Omega}$ denotes the Nehari manifold

$$\mathcal{N}_{\mu,\Omega} = \big\{u \in H_0^s(\Omega)\setminus\{0\} : \int_\Omega \big(|(-\Delta)^{s/2}u|^2 - \mu\frac{u^2}{|x|^{2s}}\big)\, dx = \int_\Omega |u|^{2_s^*}\, dx\big\}.$$

Obviously, we have

$$c_\mu = \inf\big\{\frac{s}{N}\int_{\mathbb{R}^N} \big(|(-\Delta)^{s/2}u|^2 - \mu\frac{u^2}{|x|^{2s}}\big)\, dx : u \in \mathcal{N}_{\mu,\mathbb{R}^N}\big\}$$

We may verify that $c_\mu = \frac{s}{N}\Lambda_{s,\mu}$. If $u$ is a minimizer of $\Lambda_{s,\mu}$, then $v = \Lambda_{s,\mu}^{\frac{N-2s}{4s}} u$ is a minimizer of $c_\mu$, and vice verse. Since minimizers of $\Lambda_{s,\mu}$ are radially symmetric, so are minimizers of $c_\mu$. The following result is proved in [20].

**Lemma 3.1.** *Let $0 < s < \frac{N}{2}$ and $\{u_n\} \subset \dot{H}^s(\mathbb{R}^N)$ be a bounded sequence such that*

$$\inf_{n\in\mathbb{N}} \|u_n\|_{L^{2_s^*}} \ge C > 0.$$

*Then up to subsequence, there exist $\{x_n\} \subset \mathbb{R}^N$, $\lambda_n \in (0,\infty)$ such that*

$$v_n \rightharpoonup v \not\equiv 0 \quad \text{in } \dot{H}^s(\mathbb{R}^N),$$

*where $v_n(x) := \lambda_n^{\frac{N-2s}{2}} u_n(x_n + \lambda_n x)$.*

Using Lemma 3.1, we have a description of $(PS)_c$ sequences. By a $(PS)_c$ sequences for $I_\mu$ we mean a sequence $\{u_n\} \subset H_0^s(\Omega)$ such that $I_\mu(u_n) \to c$ and $I'_\mu(u_n) \to 0$.

**Proposition 3.2.** *Let $\mu \in (0, \mu_H)$ and $\{u_n\} \subset \mathcal{N}_{\mu,\Omega}$ be a sequence such that for $0 < c < c_0$,*

$$\lim_{n\to\infty} I_\mu(u_n) = c, \quad \lim_{n\to\infty} I'_\mu(u_n) = 0. \tag{3.1}$$

*Suppose problem (1.1) has no nontrivial solutions. Then, there exist $\{\lambda_n\} \subset \mathbb{R}_+^N\}$ and $\{x_n\} \subset \mathbb{R}^N$ such that*

$$\lim_{n\to\infty} \lambda_n = 0, \quad \lim_{n\to\infty} x_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u_n - u^\mu_{\lambda_n,x_n}\|_{\dot{H}^s} = 0,$$

*where $u^\mu$ is a minimizer of $c_\mu$.*

*Proof.* By (3.1) and the Hardy inequality, we know that $\{u_n\}$ is bounded in $H_0^s(\Omega)$, and

$$\lim_{n\to\infty} \|u_n\|_{L^{2_s^*}(\Omega)} = \frac{cN}{s} > 0. \tag{3.2}$$

Therefore, we have

$$u_n \rightharpoonup u \quad \text{in } H_0^s(\Omega), \quad u_n \to u \quad \text{in } L^{2_s^*-1}(\Omega), \quad u_n \to u \quad \text{a.e. } \Omega.$$

By the assumption that problem (1.1) does not have nontrivial solution, we have $u = 0$. Extend $u_n$ to be zero outside $\Omega$, then the extension of $u_n$ belongs to $H^s(\mathbb{R}^N)$, see [1, Theorem 7.40]. By (3.2) and Lemma 3.1, there exist $\{x_n\} \subset \mathbb{R}^N$, $\lambda_n \in (0,\infty)$ such that

$$v_n \rightharpoonup v \not\equiv 0 \quad \text{in } \dot{H}^s(\mathbb{R}^N),$$

where $v_n(x) := \lambda_n^{\frac{N-2s}{2}} u_n(x_n + \lambda_n x)$, and $u_n$ has been extended to $\mathbb{R}^N$ by setting $u_n = 0$ outside $\Omega$. We also have $v_n \in H_0^s(\Omega_n)$, where $\Omega_n = \{x \in \mathbb{R}^N : x_n + \lambda_n x \in \Omega\}$. Moreover, $v_n$ satisfies

$$(-\Delta)^s v_n - \mu \frac{\lambda_n^{2s} v_n}{|x_n + \lambda_n x|^{2s}} - v_n^{2_s^*-1} \to 0 \tag{3.3}$$

and

$$\begin{aligned} &\frac{1}{2}\int_{\Omega_n} \big(|(-\Delta)^{s/2} v_n|^2 - \mu \frac{\lambda_n^{2s} v_n^2}{|x_n + \lambda_n x|^{2s}}\big)\,dx - \frac{1}{2_s^*}\int_{\Omega_n} |v_n|^{2_s^*}\,dx \\ &= \frac{1}{2}\int_{\Omega} \big(|(-\Delta)^{s/2} u_n|^2 - \mu \frac{u_n^2}{|x|^{2s}}\big)\,dx - \frac{1}{2_s^*}\int_{\Omega} |u_n|^{2_s^*}\,dx \to c \end{aligned} \tag{3.4}$$

as $n \to \infty$.

We may assume that $\lambda_n \to \lambda_0 \geq 0$. If $\lambda_0 > 0$, since $u_n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$, we have $v_n \rightharpoonup 0$ in $\dot{H}^s(\mathbb{R}^N)$, which is a contradiction.

We may assume, up to a sequence, $\frac{x_n}{\lambda_n} \to x_0 \in \mathbb{R}^N$ or $|\frac{x_n}{\lambda_n}| \to \infty$ as $n \to \infty$. If $|\frac{x_n}{\lambda_n}| \to \infty$ as $n \to \infty$, by (3.2), we see that the limit function $v$ satisfies

$$(-\Delta)^s v = v^{2_s^*-1} \quad \text{in } \mathbb{R}^N. \tag{3.5}$$

Thus, equations (3.3) and (3.4) imply

$$\begin{aligned} c_0 > c &= \lim_{n\to\infty}\Big\{\frac{1}{2}\int_{\Omega_n}\big(|(-\Delta)^{s/2}v_n|^2-\mu\frac{\lambda_n^{2s}v_n^2}{|x_n+\lambda_n x|^{2s}}\big)\,dx-\frac{1}{2_s^*}\int_{\Omega_n}|v_n|^{2_s^*}\,dx\Big\}\\ &= \frac{s}{N}\lim_{n\to\infty}\int_{\Omega_n}\big(|(-\Delta)^{s/2}v_n|^2-\mu\frac{\lambda_n^{2s}v_n^2}{|x_n+\lambda_n x|^{2s}}\big)\,dx\\ &\ge \frac{s}{N}\int_{\Omega_n}|(-\Delta)^{s/2}v|^2\,dx\ge c_0, \end{aligned}$$

which is impossible.

So we have $\frac{x_n}{\lambda_n}\to x_0\in\mathbb{R}^N$, which yields $\lim_{n\to\infty}|x_n|=0$. It follows that $v$ satisfies

$$(-\Delta)^s v-\mu\frac{v}{|x_0+x|^{2s}}=v^{2_s^*-1}\quad\text{in }\mathbb{R}^N.$$

By the translation, we may assume $x_0=0$. Indeed, let $\tilde v_n(x)=v_n(x-\frac{x_n}{\lambda_n})$. Then $\tilde v_n\rightharpoonup\tilde v$ in $\dot H^s(\mathbb{R}^N)$. By (3.3), $\tilde v_n$ satisfies

$$(-\Delta)^s\tilde v_n-\mu\frac{\tilde v_n}{|x|^{2s}}-\tilde v_n^{2_s^*-1}\to 0 \tag{3.6}$$

as $n\to\infty$ and $\tilde v$ satisfies

$$(-\Delta)^s\tilde v-\mu\frac{\tilde v}{|x|^{2s}}=\tilde v^{2_s^*-1}\quad\text{in }\mathbb{R}^N.$$

Now, we prove that

$$\lim_{n\to 0}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}(\tilde v_n-\tilde v)|^2\,dx=0.$$

Suppose on the contrary that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}(\tilde v_n-\tilde v)|^2\,dx>0.$$

Noting

$$\begin{aligned} &\lim_{n\to\infty}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}(\tilde v_n-\tilde v)|^2\,dx+\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\tilde v)|^2\,dx\\ &=\lim_{n\to\infty}\int_{\mathbb{R}^N}|(-\Delta)^{s/2}\tilde v_n|^2\,dx, \end{aligned}$$

and by the Brézis-Lieb lemma [6],

$$\lim_{n\to\infty}\|\tilde v_n-\tilde v\|^{2_s^*}+\|\tilde v\|^{2_s^*}=\lim_{n\to\infty}\|\tilde v_n\|^{2_s^*},$$
$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|\tilde v_n-\tilde v|^2}{|x|^{2s}}\,dx+\int_{\mathbb{R}^N}\frac{|\tilde v|^2}{|x|^{2s}}\,dx=\lim_{n\to\infty}\int_{\mathbb{R}^N}\frac{|\tilde v_n|^2}{|x|^{2s}}\,dx,$$

we find from (3.6) that

$$\lim_{n\to\infty}\int_{\mathbb{R}^N}\big(|(-\Delta)^{s/2}(\tilde v_n-\tilde v)|^2-\mu\frac{|\tilde v_n-\tilde v|^2}{|x|^{2s}}-|\tilde v_n-\tilde v|^{2_s^*}\big)\,dx=0. \tag{3.7}$$

That is, $\tilde{v}_n - \tilde{v}$ is close to the manifold $\mathcal{N}_{\mu,\mathbb{R}^N}$. Let $t_n$ be such that $t_n(\tilde{v}_n - \tilde{v}) \in \mathcal{N}_{\mu,\mathbb{R}^N}$. By (3.7), we can show that $t_n \to 1$ as $n \to \infty$. Hence, $J_\mu(\tilde{v}_n - \tilde{v}) - J_\mu(t_n(\tilde{v}_n - \tilde{v})) \to 0$ as $n \to \infty$. But $J_\mu(t_n(\tilde{v}_n - \tilde{v})) \geq c_\mu$, it yields

$$\liminf_{n\to\infty} J_\mu(\tilde{v}_n - \tilde{v}) \geq c_\mu,$$

and then

$$\liminf_{n\to\infty} J_\mu(\tilde{v}_n) \geq \liminf_{n\to\infty} J_\mu(\tilde{v}_n - \tilde{v}) + J_\mu(\tilde{v}) \geq 2c_\mu.$$

This is a contradiction. Consequently, $\tilde{v}_n \to \tilde{v}$ strongly in $\dot{H}^s(\mathbb{R}^N)$ and $J_\mu(\tilde{v}) = c_\mu$. The proof is complete. □

In the case $\mu = 0$, we have the following result, its proof is similar to that of Proposition 3.2.

**Proposition 3.3.** *Let*$\{u_n\} \subset \mathcal{N}_{0,\Omega}$ *be a sequence such that*

$$\lim_{n\to\infty} I(u_n) \leq c_0, \quad \lim_{n\to\infty} I'(u_n) = 0. \tag{3.8}$$

*Then, there exist* $\{\lambda_n\} \subset \mathbb{R}_+^N\}$ *and* $\{x_n\} \subset \mathbb{R}^N$ *such that*

$$\lim_{n\to\infty} \lambda_n = 0, \quad \lim_{n\to\infty} x_n = 0, \quad \lim_{n\to\infty} \|u_n - u^0_{\lambda_n,x_n}\|_{\dot{H}^s} = 0,$$

*where* $u^0$ *is a minimizer of* $c_0$.

For each set $A \subset \mathbb{R}^N$ and each point $x \in \mathbb{R}^N$, $d(x,A)$ denotes the distance between $x$ and $A$. For each $d > 0$, we denote $\Omega_d = \{x \in \mathbb{R}^N : d(x,\Omega) < d\}$ and $\Omega_d^i = \{x \in \Omega : d(x,\partial\Omega) > d\}$. For subsets $A, B \subset \mathbb{R}^N$, $A \cong B$ stands for that $A$ and $B$ are homotopy equivalent.

Now, we choose $d > 0$ so that $\Omega_d \cong \Omega$. Let

$$\beta(u) = \frac{\int_\Omega x|(-\Delta)^{s/2}u|^2\,dx}{\int_\Omega |(-\Delta)^{s/2}u|^2\,dx} \quad \text{for } u \in H_0^s(\Omega) \setminus \{0\}.$$

**Lemma 3.4.** *There exists* $\mu_0 \in (0,\mu_H)$ *such that for each* $\mu \in (0,\mu_0)$ *and* $u \in \mathcal{N}_{\mu,\Omega}$ *with* $I(u) < c_0$, $\beta(u) \in \Omega_d$.

*Proof.* Suppose on the contrary that there exist $\mu_n \in \mathbb{R}_+$ and $u_n \in \mathcal{N}_{\mu,\Omega}$ such that $\lim_{n\to\infty} \mu_n = 0$, $I_{\mu_n}(u_n) < c_0$ and $\beta(u_n) \notin \Omega_d$ for all $n \geq 1$. We may assume that $\beta(u_n) \to x_0 \in \mathbb{R}^N \setminus \Omega_d$. Since $\mu_n \to 0$, by the Hardy inequality,

$$\mu_n \int_\Omega \frac{u_n^2}{|x|^{2s}}\,dx \to 0$$

as $n \to \infty$. Hence,

$$\lim_{n\to\infty} \int_\Omega |(-\Delta)^{s/2}u_n|^2\,dx = \lim_{n\to\infty} \int_\Omega |u_n|^{2_s^*}\,dx,$$

$$\lim_{n\to\infty} I_{\mu_n}(u_n) \leq c_0.$$

By Proposition 3.3, there exist sequences $\{\lambda_n\}$ and $\{x_n\} \subset \mathbb{R}^N$ such that

$$\lim_{n\to\infty} \lambda_n = 0 \quad \text{and} \quad \lim_{n\to\infty} \|u_n - u^0_{\lambda_n,x_n}\|_{\dot{H}^s} = 0.$$

By the assumption, we have $\lim_{n\to\infty} x_n = x_0$. However, $x_0 \notin \Omega_d$, we have that $\lim_{n\to\infty} \|u_n - u^0_{\lambda_n,x_n}\|_{\dot{H}^s} \neq 0$. This is a contradiction. The assertion follows. □

Now, we choose $d_1 > 0$ such that $\Omega \cong \Omega^i_{d_1}$. Let

$$\lambda = \inf\{\frac{\mu}{|x|^{2s}} : x \in \Omega_d\}.$$

Let $\xi \in C^\infty(\mathbb{R}_+)$ be such that $\xi(t) = 1$ for $t \in [0, \frac{d_1}{2}]$ and $\xi(t) = 0$ for $t \in [d_1, \infty)$. For each $(\varepsilon, z) \in \mathbb{R}_+ \times \mathbb{R}^N$, we define

$$w_{\varepsilon,z}(x) = \tau_{\varepsilon,z}\xi(x - z)U_\varepsilon(x - z)$$

for $x \in \mathbb{R}^N$, where $U_\varepsilon$ is given in (1.9) and $\tau_{\varepsilon,z}$ is a positive constant such that $w_{\varepsilon,z}$ satisfying

$$\int_\Omega \left(|(-\Delta)^{s/2}w_{\varepsilon,z}|^2 - \lambda w^2_{\varepsilon,z}\right) dx = \int_\Omega |w_{\varepsilon,z}|^{2^*_s}\, dx.$$

It is proved in [4] that if $0 \in \Omega$,

$$\|\xi U_\varepsilon\|^2_{H^s_0(\Omega)} = \|U_\varepsilon\|^2_{H^s_0(\Omega)} + O(\varepsilon^{N-2s});$$

$$\|\xi U_\varepsilon\|^2_{L^2(\Omega)} = \begin{cases} C\varepsilon^{2s} + O(\varepsilon^{2s}), & \text{if } N > 4s,\\ -C\varepsilon^{2s}\log\varepsilon + O(\varepsilon^{2s}), & \text{if } N = 4s,\end{cases}$$

$$\|\xi U_\varepsilon\|^{2^*_s-1}_{L^{2^*_s-1}(\Omega)} \geq C\varepsilon^{\frac{N-2s}{2}} \quad \text{if } N > 2s.$$

Let

$$Q(u) = \frac{1}{2}\int_\Omega \left(|(-\Delta)^{s/2}u|^2 - \lambda u^2\right) dx - \frac{1}{2^*_s}\int_\Omega |u|^{2^*_s}\, dx.$$

Then, we may verify that

$$Q(w_{\varepsilon,z}) = \begin{cases} c_0 - \lambda C\varepsilon^{2s} + O(\varepsilon^{2s}), & \text{if } N > 4s,\\ c_0 + C\varepsilon^{2s}log\varepsilon + O(\varepsilon^{2s}), & \text{if } N = 4s,\end{cases}$$

for all $z \in \Omega^i_{d_1}$, it implies that for all $z \in \Omega^i_{d_1}$,

$$Q(w_{\varepsilon,z}) < c_0 \tag{3.9}$$

if $\varepsilon > 0$ small enough.

**Lemma 3.5.** *Let $\mu \in (0, \mu_0)$. Then for $\varepsilon > 0$ small, there holds*

$$\sup\{I(t_{w_{\varepsilon,z},\mu}w_{\varepsilon,z}) : z \in \Omega^i_{d_1}\} < c_0.$$

*Proof.* Let $t = t_{w_{\varepsilon,z}}$. Since $\frac{\mu}{|x|^{2s}} > \lambda$ for $x \in \Omega$,

$$\begin{aligned}\frac{t^2}{2}\int_\Omega \left(|(-\Delta)^{s/2}w_{\varepsilon,z}|^2 - \lambda w^2_{\varepsilon,z}\right) dx &> \frac{t^2}{2}\int_\Omega \left(|(-\Delta)^{s/2}w_{\varepsilon,z}|^2 - \mu\frac{w^2_{\varepsilon,z}}{|x|^{2s}}\right) dx\\ &= \frac{t^{2^*_s}}{2}\int_\Omega w^{2^*_s}_{\varepsilon,z}\, dx\\ &= \frac{t^{2^*_s}}{2}\int_\Omega \left(|(-\Delta)^{s/2}w_{\varepsilon,z}|^2 - \lambda w^2_{\varepsilon,z}\right) dx.\end{aligned}$$

Therefore, $t < 1$. It results from (3.9) that

$$\begin{aligned}I_\mu(tw_{\varepsilon,z}) &= \frac{st^2}{N}\int_\Omega \left(|(-\Delta)^{s/2}w_{\varepsilon,z}|^2 - \mu\frac{w^2_{\varepsilon,z}}{|x|^{2s}}\right) dx\\ &\leq \frac{s}{N}\int_\Omega \left(|(-\Delta)^{s/2}w_{\varepsilon,z}|^2 - \lambda w^2_{\varepsilon,z}\right) dx < c_0.\end{aligned}$$

The assertion follows. $\square$

*Proof of Theorem 1.1.* We argue by contradiction. For fixed $\mu \in (0, \mu_0)$, assume problem (1.1) has no solutions. Hence, we know from [24] that there exists a pseudo-gradient flow $\eta : [0, \infty) \times \mathcal{N}_{\mu,\Omega} \to \mathcal{N}_{\mu,\Omega}$ associated with $I_\mu$, such that the function $\eta$ satisfies for $s, t \in \mathbb{R}_+$ with $s > t$ and $u \in \mathcal{N}_{\mu,\Omega}$ that

$$I_\mu(\eta(s,u)) < I_\mu(\eta(t,u)),$$
$$\lim_{t\to\infty} I_\mu(\eta(t,u)) > -\infty \quad \text{implies that} \quad \lim_{t\to\infty} I'_\mu(\eta(t,u)) = 0.$$

For $\varepsilon$ given in Lemma 3.5, we set $\omega = \{t_{w_{\varepsilon,z},\mu} w_{\varepsilon,z} : z \in \omega^i_{d_1}\}$. By the definition of $w_{\varepsilon,z}$, we have $w_{\varepsilon,z} \in H^s_0(\Omega)$ for each $z \in \omega$. Therefore, $\omega \subset \mathcal{N}_{\mu,\Omega}$, and $\sup\{I_\mu(\eta(t,u) : t \geq 0)\}$ is bounded from below for each $u \in \omega$. By Proposition 3.2, there exist $\{(\varepsilon_t, z_t\} \subset \mathbb{R}_+ \times \Omega$ such that $\lim_{t\to\infty} z_t = 0$ and

$$\lim_{t\to\infty} \|\eta(t,u) - u^\mu_{\varepsilon_t,z_t}\|_{\dot{H}^s} = 0.$$

Since $u^\mu$ is radially symmetric, we have

$$\lim_{t\to\infty} \beta(\eta(t,u)) = 0 \in \Omega$$

for all $u \in \omega$. On the other hand, by Lemma 3.4,

$$\{\beta(\eta(t,u)) : u \in \omega\} \subset \Omega_d.$$

Since $\{\beta(\eta(0,u)) : u \in \omega\} = \Omega^i_{d_1}$, we see that $\Omega^i_{d_1}$ is contractible in $\Omega_d$. This contradicts to the assumption that $\Omega^i_{d_1} \cong \Omega \cong \Omega_d$ and that $\Omega$ is not contractible. This shows that problem (1.1) possesses a positive solution in $\mathcal{N}_{\mu,\Omega}$. $\square$

*Proof of Theorem 1.2.* We remark that Lemma 3.1 can be applied in this case. The proof of Theorem 1.2 is similar to that of Theorem 1.1 with minor changes, we omit the details. $\square$

**Acknowledgments.** This work was supported by the NNSF of China: 11271170 and 11371254, and by the GAN PO 555 program of Jiangxi.

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